NON-NEGATIVELY CURVED C-TOTALLY REAL SUBMANIFOLDS IN A SASAKIAN MANIFOLD

By

Masumi KAMEDA

Dedicated to Professor Y. Tashiro on his 60th birthday

§ 0. Introduction.

Several authors have investigated minimal totally real submanifolds in a complex space form and obtained many interesting results. Recently F. Urbano [6] and Y. Ohnita [4] have studied pinching problems on their curvatures and stated some theorems.

On the other hand, in a $(2n+1)$-dimensional Sasakian space form of constant $\phi$-sectional curvature $c(>3)$, if a submanifold $M$ is perpendicular to the structure vector field, then $M$ is said to be C-totally real. For such a submanifold $M$, it is well-known that if the mean curvature vector field of $M$ is parallel, then $M$ is minimal. S. Yamaguchi, M. Kon and T. Ikawa [8] obtained that if the squared length of the second fundamental form of $M$ is less than $n(n+1)(c+3)/4(2n-1)$, then $M$ is totally geodesic. Furthermore, D. E. Blair and K. Ogiue [2] proved that if the sectional curvature of $M$ is a greater than $(n-2)(c+3)/4(2n-1)$, then $M$ is totally geodesic.

In this paper, we consider a curvature-invariant C-totally real submanifold $M$ in a Sasakian manifold with $\gamma$-parallel mean curvature vector field. Then $M$ is not necessary minimal. Making use of methods of [3] and [4], we prove that if the sectional curvature of $M$ is positive, then $M$ is totally geodesic.

In Sec. 1, we recall the differential operators on the unit sphere bundle of a Riemannian manifold. Sec. 2 is devoted to stating about fundamental formulas on a C-totally real submanifold in a Sasakian manifold. In Sec. 3, we prove Theorems and Corollaries. Throughout this paper all manifolds are always $C^\infty$, oriented, connected and complete. The author wishes to thank Professor S. Yamaguchi for his help.

§ 1. A differential operator defined by A. Gray.

Let $M$ be an $n$-dimensional Riemannian manifold and $\Gamma(M)$ the Lie algebra
of vector fields on \(M\). Denote by \(\langle \cdot, \cdot \rangle, \mathcal{F}, \text{and } R_x^y := [\mathcal{F}_x, \mathcal{F}_y] - \mathcal{F}[x, y] = X, Y \in \Gamma(M)\) the metric tensor of \(M\), the Riemannian connection on \(M\) and the curvature tensor of \(M\), respectively. The Ricci tensor \(\rho\) of \(M\) is given by

\[
\rho_{xy} := \sum_{a=1}^n \langle R_{ea}xy, e_a \rangle \quad \text{for } X, Y \in \Gamma(M),
\]

where \(\{e_1, \ldots, e_n\}\) is an arbitrary local orthonormal frame field. For \(m \in M\) we denote by \(M_m\) the tangent space to \(M\) at \(m\). Then we write \(R_{wxyz}\) in place of \(\langle R_{wxy}, z \rangle\) for \(w, x, y, z \in M_m\) and shall sometimes use such expressions as \(R_{xay}^z\) instead of \(R_{xay}^z\).

Now we define the unit sphere bundle \(S(M)\) of \(M\) by

\[
S(M) = \{(m, x) : m \in M, x \in M_m, \langle x, x \rangle = 1\}.
\]

For any unit vector \(x\) in a fibre \(S_m\) we take an orthonormal basis \(\{e_1, \ldots, e_n\}\) of \(M_m\) such that \(x = e_1\). Denote by \(\{y_2, \ldots, y_n\}\) the corresponding system of normal coordinates defined on a neighborhood of \((m, x)\) in \(S_m\).

**Lemma A [3].** Let \(F : S_m \to \mathbb{R}\) be a function. Then we have

\[
\frac{\partial^{a_2 + \cdots + a_n} F}{\partial y_2^{a_2} \cdots \partial y_n^{a_n}}(m, x) = \frac{\partial^{a_2 + \cdots + a_n}}{\partial u_2^{a_2} \cdots \partial u_n^{a_n}} F((\cos r)x + (\sin r)\sum_{i=2}^n u_i e_i)(0),
\]

where we have set \(r^2 = \sum_{i=2}^n u_i^2\).

Next we lift the frame \(\{e_1, \ldots, e_n\}\) to an orthonormal basis \(\{f_1, \ldots, f_n; g_2, \ldots, g_n\}\) of the tangent space \(S(M)_{(m, x)}\), where we require that \(f_1, \ldots, f_n\) are horizontal and \(g_2, \ldots, g_n\) are vertical. Denote by \(\{x_1, \ldots, x_n; y_2, \ldots, y_n\}\) the corresponding normal coordinate system on a neighborhood of \((m, x)\) in \(S(M)\). We define a second-order linear differential operator \(L(\lambda, \mu)\) by

\[
L(\lambda, \mu)(m, x) := \sum_{a=1}^n \frac{\partial^2}{\partial x_a^2} - \lambda \sum_{a,b=2}^n p_{a\beta} \frac{\partial^2}{\partial y_a \partial y_\beta} \mu \sum_{a=2}^n q_a \frac{\partial}{\partial y_a},
\]

where \(p_{a\beta}(m, x) := R_{a2\beta}, q_a(m, x) := \rho_{a2}\) and \(\lambda, \mu\) are constants to be chosen later. This definition is independent of the choice of normal coordinates at \((m, x)\). Hence \(L(\lambda, \mu)(m, x)\) is well-defined. Here we note that the sign of the second term in the right hand side is minus because of the definition on curvature tensor.

For a compact Riemannian manifold \(M\), we define an inner product \(\langle \cdot, \cdot \rangle\) on the space of functions by \(\langle f, g \rangle := \int_M fg \, d\mu\). Then the differential operator \(L(\lambda, \mu)\) is self-adjoint with respect to \(\langle \cdot, \cdot \rangle\) provided that \(\lambda = -\mu\) (cf. [3]).

If \(f\) is a real-valued function on \(S(M)\), we denote by \(\text{grad}^m f\) and \(\text{grad}^n f\) the
vertical and horizontal components of grad $f$ respectively.

**LEMMA B** [3]. In a compact Riemannian manifold $M$, we have

$$\int_{S(M)} [f L(\lambda, -\lambda)(f)(m, x) + |\text{grad}^h f|^2(m, x) + \lambda K_e(\text{grad}^h f)(x)] e^1 = 0,$$

where the letter $K$ indicates the sectional curvature of $M$.

§ 2. Fundamental formulas.

Let $M$ be a submanifold of a Riemannian manifold $N$. We denote by the same $\langle \, , \rangle$ the Riemannian metrics of $M$ and $N$, and by $\overline{\nabla}$ (resp. $\nabla$) the Riemannian connection of $N$ (resp. $M$) respectively. In the sequel the letters $W, X, Y$ and $Z$ (resp. $V$) will always denote any vector fields tangent (resp. normal) to $M$. Then the Gauss and Weingarten formulas are respectively given by

(2.1) \[ \overline{\nabla}_X Y = \nabla_X Y + B(X, Y), \]

(2.2) \[ \overline{\nabla}_X V = -A_X + D_X V, \]

where $B$ (resp. $A$) and $D$ are the second fundamental form (resp. shape operator) and the normal connection of $M$ respectively. Then first and second covariant derivatives of $B$ are respectively defined by

(2.3) \[ (\overline{\nabla}_X B)(Y, Z) = D_X B(Y, Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z), \]

(2.4) \[ (\overline{\nabla}_W B)(Y, Z) = D_W (\overline{\nabla}_X B)(Y, Z) - (\overline{\nabla}_{W X} B)(Y, Z) \]

\[ - (\overline{\nabla}_X B)(\nabla_W Y, Z) - (\overline{\nabla}_X B)(Y, \nabla_W Z) \]

Denoting by $\overline{\mathbf{R}}$ the Riemannian curvature tensor of $N$ and putting as $(\overline{\mathbf{R}}_{W X} Y)^n$ the normal part of $\overline{\mathbf{R}}_{W X} Y$, we have the equation of Codazzi:

(2.5) \[ (\overline{\mathbf{R}}_{W X} Y)^n = (\overline{\nabla}_W B)(X, Y) - (\overline{\nabla}_X B)(W, Y). \]

If $(\overline{\mathbf{R}}_{W X} Y)^n$ vanishes identically, then we call such a submanifold $M$ curvature-invariant.

From (2.4), the formula of Ricci with respect to the second covariant derivative of $B$ is given by

(2.6) \[ (\overline{\nabla}_W^2 B)(Y, Z) - (\overline{\nabla}_W^2 B)(Y, Z) \]

\[ = R^0_{W X} B(X, Z) - B(\overline{\mathbf{R}}_{W X} Y, Z) - B(Y, \overline{\mathbf{R}}_{W X} Z), \]
where \( R_{wX}^w := [D_w, D_X] + D_{[w, x]} \) indicates the normal curvature tensor of \( M \).

From now on let \( M \) be an \( n \)-dimensional C-totally real submanifold in a \((2n+1)\)-dimensional Sasakian manifold \( N \) with structure \((\phi, \xi, \eta)\). Then it is shown that ([7], [8], [9], [11])

\[
\langle B(Y, Z), \xi \rangle = 0,
\]

\[
D_X\phi Y = -\langle X, Y \rangle \xi + \phi p_X Y,
\]

\[
\langle R_{wX}^w \phi Y, \phi Z \rangle = \langle R_{wx}^w Y, Z \rangle - \langle W, Z \rangle \langle X, Y \rangle + \langle W, Y \rangle \langle X, Z \rangle,
\]

\[
\langle (\tilde{r} X B)(Y, Z), \xi \rangle = -\langle B(Y, Z), \phi X \rangle.
\]

For such a C-totally real submanifold \( M \), we state the definitions as follows:

**DEFINITION** [11]. We say that the mean curvature vector field of \( M \) is \( \eta \)-parallel if

\[
\sum_{a=1}^{n} \langle (\tilde{r} w B)(e_a, e_a), \phi X \rangle = 0.
\]

We say that the second fundamental form of \( M \) is \( \eta \)-parallel if

\[
\langle \tilde{r} w B(Y, Z), \phi X \rangle = 0.
\]

If \( M \) has \( \eta \)-parallel mean curvature vector field, then the equations (2.8) and (2.10) yield

\[
\sum_{a=1}^{n} \langle (\tilde{r} w X B)^2(e_a, e_a), \phi Y \rangle = -\sum_{a=1}^{n} \left[ \langle (\tilde{r} X B)(e_a, e_a), DWY \rangle + 2\langle (\tilde{r} X B)(\tilde{r} we_a, e_a), \phi Y \rangle \right]
\]

\[
= -\sum_{a=1}^{n} \left[ -\langle W, Y \rangle \langle B(e_a, e_a), \phi X \rangle + 2\langle \tilde{r} X B(\tilde{r} we_a, e_a), \phi Y \rangle \right].
\]

Taking the normal coordinate system, we can state the following.

**LEMMA 2.1.** If \( M \) has \( \eta \)-parallel mean curvature vector field, then we have

\[
\sum_{a=1}^{n} \langle (\tilde{r} w X B)(e_a, e_a), \phi Y \rangle = -\sum_{a=1}^{n} \langle W, Y \rangle \langle B(e_a, e_a), \phi X \rangle.
\]

**§ 3. C-totally real submanifolds.**

Throughout this section let \( M \) be an \( n \)-dimensional curvature-invariant C-totally real submanifold in a \((2n+1)\)-dimensional Sasakian manifold. We denote the components of the second fundamental form \( B \) by
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\[ h_{\alpha\beta} = \langle B(e_{\alpha}, e_{\beta}), \phi e_{\gamma} \rangle \quad \text{for} \quad 1 \leq \alpha, \beta, \gamma \leq n. \]

As \( M \) is C-totally real, we find that \( h \) is symmetric, i.e.,

\[ h_{\alpha\beta} = h_{\beta\alpha} = h_{\alpha\beta} \quad \text{for} \quad 1 \leq \alpha, \beta, \gamma \leq n. \]

The components of first and second covariant derivatives of \( B \) with respect to \( \phi \Gamma(M) \) are respectively expressed as

\[ (\nabla_{\alpha} h)_{\beta\gamma} = \langle \nabla_{\alpha} B(e_{\beta}, e_{\gamma}), \phi e_{\delta} \rangle \quad \text{for} \quad 1 \leq \alpha, \beta, \gamma, \delta \leq n, \]

\[ (\nabla_{\alpha}^{2} h)_{\beta\gamma} = \langle \nabla_{\alpha} \nabla_{\beta} B(e_{\gamma}, e_{\delta}), \phi e_{\epsilon} \rangle \quad \text{for} \quad 1 \leq \alpha, \beta, \gamma, \delta, \epsilon \leq n. \]

Since \( M \) is curvature-invariant, then, from (2.5) and (3.3), we find that \( \nabla h \) is symmetric with respect to \( \phi \Gamma(M) \), i.e.,

\[ (\nabla_{\alpha} h)_{\beta\gamma} = (\nabla_{\beta} h)_{\alpha\gamma} \quad \text{for} \quad 1 \leq \alpha, \beta, \gamma \leq n. \]

We consider a function \( f \) on \( S(M) \) defined by \( f(m, x) = h_{xxx} \) for any point \( (m, x) \in S(M) \) and then prove the following Lemma to use later.

**Lemma 3.1.** Let \( M \) be an \( n \)-dimensional curvature-invariant C-totally real submanifold in a \((2n+1)\)-dimensional Sasakian manifold \( N \). If \( M \) has \( \eta \)-parallel mean curvature vector field, then we have \( L(1/3, -1/3)(f) = 0. \)

**Proof.** We take any point \( (m, x) \) of \( S(M) \). For each \( \alpha, 1 \leq \alpha \leq n \), let \( \gamma_{a}(s) \) be a geodesic in \( M \) such that \( \gamma_{a}(0) = m \) and \( \gamma_{a}'(0) = e_{\alpha} \). Then we denote a vector field by parallel translating of \( x \) along \( \gamma_{a} \) as the same letter \( x \). By virtue of (2.7) — (2.10), we obtain

\[ \left( \frac{\partial^{2} f}{\partial x_{\alpha}^{2}} \right)(m, x) = \langle \phi x, D_{\alpha} \nabla_{\alpha} B(x, x) \rangle + \langle D_{\alpha} \phi x, (\nabla_{\alpha} B)(x, x) \rangle \quad \text{at} \quad m \]

\[ = \langle \phi x, (\nabla_{\alpha} B)(x, x) \rangle + x_{\alpha} \langle \phi e_{\alpha}, B(x, x) \rangle \quad \text{at} \quad m \]

\[ = (\nabla_{\alpha}^{2} h)_{xxx} + x_{\alpha} h_{xxx}, \]

where we have put \( x_{\alpha} := \langle e_{\alpha}, x \rangle \), which implies

\[ \sum_{a=1}^{n} \left( \frac{\partial^{2} f}{\partial x_{a}^{2}} \right)(m, x) = \sum_{a=1}^{n} (\nabla_{\alpha}^{2} h)_{xxx} + h_{xxx}. \]

From (2.6), (2.9), (3.2) and (3.5), we can verify
\[(p_{xx}^2 h)_{zzz} = (p_{xx}^2 h)_{zzz} \]
\[= \langle \phi x, (p_{xx}^2 B)(x, e_a) \rangle + \langle \phi x, R_{xx}^B(x, e_a) \rangle \]
\[\quad - \langle \phi x, B(R_{xx}x, e_a) \rangle - \langle \phi x, B(x, R_{xx}e_a) \rangle \quad \text{at } m \]
\[= \langle \phi x, (p_{xx}^2 B)(e_a, e_a) \rangle - \langle B(x, e_a), R_{xx}^B \rangle \]
\[\quad - \langle B(x, e_a), R_{xx}x \rangle - \langle B(x, x), R_{xx}e_a \rangle \quad \text{at } m \]
\[= (p_{xx}^2 h)_{zzz} + \sum_{j=1}^{n} \left[ -2 h_{\beta z} R_{zzz} - h_{\beta z} R_{zzz} \right. \]
\[\quad + \left. \phi \right|_{\beta z} \] from which follows that
\[(3.7) \quad \sum_{j=1}^{n} (p_{xx}^2 h)_{zzz} = \sum_{j=1}^{n} \left[ (p_{xx}^2 h)_{zzz} - 2 \sum_{j=1}^{n} h_{\beta z} R_{zzz} + h_{xx} \rho_{xx} + h_{aoz} \right] - h_{zzz}.
Thus it is shown from (3.6) and (3.7) that
\[(3.8) \quad \sum_{\alpha=1}^{n} \left( \frac{\partial^2 f}{\partial x_{\alpha}^2} \right)(m, x) = \sum_{\alpha=1}^{n} \left[ (p_{xx}^2 h)_{\alpha z} - 2 \sum_{j=1}^{n} R_{\alpha z} h_{\alpha z} + \rho_{z} h_{\alpha z} + h_{aoz} \right].
From the definition of \( f \), we have
\[(3.9) \quad f ((\cos r)x + (\sin r)y)_{\alpha} = (\cos r)^{2} h_{xx} + 3(\cos r)^{3} (\sin r) \sum_{j=1}^{n} u_{i, h_{ij}} \]
\[\quad + 3(\cos r) (\sin r) \sum_{j=1}^{n} u_{i, h_{ij}} + (\sin r)^{2} \sum_{j=1}^{n} u_{i, h_{ij}} + (\sin r)^{3} \sum_{j=1}^{n} u_{i, h_{ij}} \]
\[= (\cos r)^{3} h_{xx} + 3(\cos r)^{2} (\sin r) \sum_{j=1}^{n} u_{i, h_{ij}} \]
\[\quad + (\cos r) (\sin r)^{2} \sum_{j=1}^{n} (3 h_{ij} - h_{zzz}) u_{i} \]
\[\quad + 6(\cos r) (\sin r)^{2} \sum_{j=1}^{n} u_{i, h_{ij}} + (\sin r)^{3} \sum_{j=1}^{n} u_{i, h_{ij}} \]
because of \( r^2 = \sum_{j=1}^{n} u_{i}^{2} \). Applying Lemma A to (3.9), we find
\[(3.10) \quad \frac{\partial f}{\partial y_{\alpha}}(m, x) = 3 h_{aoz} \quad \text{for } 2 \leq \alpha \leq n, \]
\[(3.11) \quad \frac{\partial^2 f}{\partial y_{\alpha} \partial y_{\beta}}(m, x) = -3 h_{zzz} + 6 h_{aoz} \quad \text{for } 2 \leq \alpha, \beta \leq n. \]
We see from (3.8), (3.10) and (3.11) that
Non-negatively curved C-totally real submanifolds \[ L(1/3, -1/3) (f) (m, x) = \sum_{a=1}^{n} \left[ (p_{a2}^2 h)_{a22} + h_{a22} \right]. \]

On the other hand, the equation (2.13) is rewritten as
\[ \sum_{\alpha=1}^{n} (p_{\beta \gamma}^2 h)_{\alpha \gamma} = - \sum_{\alpha=1}^{n} \delta_{\beta \gamma} h_{a22} \quad \text{for } 1 \leq \beta, \gamma, \delta \leq n. \]

Combining (3.12) with (3.13), we have \[ L(1/3, -1/3) (f) (m, x) = 0. \]

**THEOREM 3.1.** Let \( M \) be an \( n \)-dimensional compact curvature-invariant C-totally real submanifold in a \((2n+1)\)-dimensional Sasakian manifold with \( \eta \)-parallel mean curvature vector field. If the sectional curvature of \( M \) is positive, then \( M \) is totally geodesic.

**PROOF.** As \( M \) has positive sectional curvature, \( L(1/3, -1/3) \) is elliptic. From the above hypothesis we have \( L(1/3, -1/3) (f) = 0 \). By maximum principle [10], \( f \) is constant on \( S(M) \). Since \( f \) is an odd function, it must be zero. Thus \( M \) is totally geodesic.

**COROLLARY 3.2.** Let \( M \) be an \( n \)-dimensional compact C-totally real submanifold in a \((2n+1)\)-dimensional Sasakian space form with \( \eta \)-parallel mean curvature vector field. If the sectional curvature of \( M \) is positive, then \( M \) is totally geodesic.

**PROOF.** If the \( \phi \)-sectional curvature of Sasakian space form \( N \) is denoted by \( c \), then the Riemannian curvature tensor \( R \) of \( N \) restricted to \( M \) is given by
\[ R_{\phi \gamma} = \frac{c+3}{3} [\langle Y, X \rangle W - \langle Y, W \rangle X], \]
which means clearly that \( M \) is curvature-invariant. By Theorem 3.1, \( M \) is totally geodesic.

**REMARK 1.** If the normal connection of \( M \) is flat, then, from (2.9), \( M \) is of constant curvature 1, so that we have the same result as those in Theorem 3.1 or Corollary 3.2.

**REMARK 2.** As a Corollary of Theorem 3.1, we can state the Blair-Ogiue's Theorem in the introduction of this paper.

**THEOREM 3.3.** Let \( M \) be an \( n \)-dimensional compact curvature-invariant C-totally real submanifold in a \((2n+1)\)-dimensional Sasakian manifold with \( \eta \)-parallel mean curvature vector field. If the sectional curvature of \( M \) is non-negative, then \( M \) has \( \eta \)-parallel second fundamental form.
PROOF. By use of Lemma 3.1, we have \( L(1/3, -1/3)(f) = 0 \). Applying Lemma B, we find that \( \text{grad}^\kappa f \) must be identically zero. From (3.2) and (3.5), the fact that \( \text{grad}^\kappa f = 0 \) is equivalent to saying that the second fundamental form is \( \eta \)-parallel.

COROLLARY 3.4. Let \( M \) be an \( n \)-dimensional compact \( C \)-totally real submanifold in a \((2n+1)\)-dimensional Sasakian space form with \( \eta \)-parallel mean curvature vector field. If the sectional curvature of \( M \) is non-negative, then \( M \) has \( \eta \)-parallel second fundamental form.

References