REALIZATIONS OF INVOLUTIVE AUTOMORPHISMS
σ AND G* OF EXCEPTIONAL LINEAR LIE GROUPS
G, PART III, G = E₈

By
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M. Berger [1] classified involutive automorphisms σ of simple Lie algebras g and determined the type of the subalgebras g* of fixed points. In the preceding papers [Y1], [Y2], we found involutive automorphisms σ and realized the subgroups G* of fixed points explicitly for the connected exceptional universal linear Lie groups G of type G₂, F₄, E₆ and E₇. In this paper we consider the case of type E₈. Our results are as follows.

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<th>G</th>
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<td>(SL(2, R) x E₇(-24))/Z₂×2</td>
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<td>Sso*(16) x 2</td>
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This paper is a continuation of [Y1], [Y2] and we use the same notations as them. So the numbering of sections and theorems starts from 5.1 and 5.1.1.

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respectively.

Groups $E_8$

5.1. The complex Lie algebra $e_8^C$

In the 248-dimensional $C$-vector space

$$e_8^C = e_8^C \oplus e_7^C \oplus C \oplus C \oplus C \oplus C,$$

we define the Lie bracket $[R_1, R_2]$ by

$$[(\Phi_1, P_1, Q_1, r_1, s_1, t_1), (\Phi_2, P_2, Q_2, r_2, s_2, t_2)] = (\Phi, P, Q, r, s, t)$$

where

\[
\begin{align*}
\Phi &= [\Phi_1, \Phi_2] + P_1 \times Q_2 - P_2 \times Q_1, \\
P &= \Phi_1 P_2 - \Phi_2 P_1 + r_1 P_2 - r_2 P_1 + s_1 Q_2 - s_2 Q_1, \\
Q &= \Phi_2 Q_1 - \Phi_1 Q_2 + r_2 Q_1 - r_1 Q_2 + t_1 P_2 - t_2 P_1, \\
r &= -\frac{1}{8} \{P_1, Q_1\} + \frac{1}{8} \{P_2, Q_2\} + s_1 t_2 - s_2 t_1, \\
s &= \frac{1}{4} \{P_1, P_2\} + 2 r_1 s_2 - 2 r_2 s_1, \\
t &= -\frac{1}{4} \{Q_1, Q_2\} - 2 r_1 t_2 + 2 r_2 t_1,
\end{align*}
\]

then $e_8^C$ becomes a simple $C$-Lie algebra of type $E_8$ [16], [17]. For $R \in e_8^C$, $\text{ad} R (\langle \text{ad} R \rangle R_1 = [R, R_1], R_1 \in e_8^C)$ is denoted by $\Theta(R)$. Hereafter we often identify $R$ and $\Theta(R)$, $e_8^C$ and $\text{Der}_C(e_8^C) = \{\Theta(R) | R \in e_8^C\}$, respectively. The group $E_8^C$ is defined to be the automorphism group of the Lie algebra $e_8^C$:

$$E_8^C = \{ \alpha \in \text{Isoc}(e_8^C) | \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2] \}.$$

5.2. Involutions of the Lie group $E_8^C$

We arrange here main involutions used in this chapter $E_8$. We define $C$-linear transformations $\gamma, \sigma, \lambda, \nu, \omega$ of $e_8^C$ by

$$\begin{align*}
\gamma(\Phi, P, Q, r, s, t) &= (\gamma \Phi, \gamma P, \gamma Q, r, s, t), \\
\sigma(\Phi, P, Q, r, s, t) &= (\sigma \Phi, \sigma P, \sigma Q, r, s, t), \\
\lambda(\Phi, P, Q, r, s, t) &= (\lambda \Phi, \lambda P, \lambda Q, r, s, t)
\end{align*}$$

where $\gamma, \sigma, \lambda$ of right sides are the same ones as $\gamma \in G_2^C \subset F_4^C \subset E_6^C \subset E_7^C, \sigma \in F_4^C \subset E_6^C \subset E_7^C, \lambda \in E_7^C$, respectively.
Realizations of involutive automorphisms

\[\nu(\Phi, P, Q, r, s, t) = (-\Phi, -P, -Q, r, s, t),\]
\[\omega(\Phi, P, Q, r, s, t) = (\Phi, -P, -Q, -r, -s, -t).\]

Then \(\gamma, \sigma, \lambda, \nu, \omega \in E_8^\sigma\) and \(\gamma^2 = \sigma^2 = \nu^2 = 1, \lambda^2 = \omega^2 = \nu\). (\(\nu\) is nothing but the element \(-1\) of the center of \(E_7^\sigma\) (see Proposition 5.4.1), however we use \(\nu\) in \(E_8^\sigma\) to avoid confusions because \(-1\) has many different meanings. Note that \(\nu = \phi(-E), \omega = \phi(\lambda)\) using \(\phi\) of Proposition 5.4.2). We put

\[\hat{\lambda} = \lambda \omega = \omega \lambda.\]

Then \(\hat{\lambda} \in E_8^\sigma\) and \(\hat{\lambda}^2 = 1\). The complex conjugation in \(E_8^\sigma\) is denoted by \(\tau:\)

\[\tau(\Phi, P, Q, r, s, t) = (\tau \Phi, \tau P, \tau Q, \tau r, \tau s, \tau t).\]

These transformations \(\gamma, \sigma, \nu, \lambda, \tau\) of \(E_8^\sigma\) induce involutive automorphisms \(\tilde{\gamma}, \tilde{\sigma}, \tilde{\nu}, \tilde{\lambda}, \tilde{\tau}\) of \(E_8^\sigma\):

\[\tilde{\gamma}(\alpha) = \gamma \alpha \gamma, \quad \tilde{\sigma}(\alpha) = \sigma \alpha \sigma, \quad \tilde{\nu}(\alpha) = \nu \alpha \nu, \quad \alpha \in E_8^\sigma.\]

5.3. Lie groups of type \(E_8\)

We define \(R\)-Lie algebras \(e_{8(24)}^c, e_{8(-24)}^c\) by

\[e_{8(24)}^c = e_{(7,7)}^c \oplus \mathbb{R}^8 \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \quad (e_{(7,7)}^c \cong (e_7^c)^\gamma),\]
\[e_{8(-24)}^c = e_{(7,25)}^c \oplus \mathbb{R}^8 \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \quad (e_{(7,25)}^c \cong (e_7^c)^\gamma)\]

(Theorem 4.5.2) with the Lie brackets as \(e_8^c\), respectively. The connected linear Lie groups of type \(E_8\) are obtained as

\[E_8^c = \{\alpha \in \text{Iso}(e_8^c) | \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2]\},\]
\[E_{8(24)} = \{\alpha \in \text{Iso}(e_{8(24)}^c) | \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2], \langle \alpha R_1, \alpha R_2 \rangle = \langle R_1, R_2 \rangle\},\]
\[E_{8(-24)} = \{\alpha \in \text{Iso}(e_{8(-24)}^c) | \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2]\},\]
\[E_{8(24)} = \{\alpha \in \text{Iso}(e_{8(-24)}^c) | \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2]\}\]

here \(\langle R_1, R_2 \rangle = -(1/15)B_8(\tau R_1, R_2)\) where \(B_8\) is the Killing form of \(e_8^c\) [18], [21]. \(E_8^c, E_8\) are simply connected (see Appendix).

**Lemma 5.3.1.** \((e_8^c)^\gamma \cong e_{8(24)}^c, (e_8^c)^\tau \cong e_{8(-24)}^c)\).

**Theorem 5.3.2.** \((E_8^c)^{\gamma} = E_8, (E_8^c)^{\tau} = E_{8(24)}, (E_8^c)^{\gamma} = E_{8(-24)}\).

**Proof.** As for \(E_{8(24)}, E_{8(-24)}\), these are direct results of Lemma 5.3.1. \(E_8\) is nothing but its definition.
Note that $\gamma \in G \subset F \subset E \subset E \subset E \subset E \subset E$ (see Proposition 5.4.1) $= E_8$. $\sigma \in F \subset E \subset E \subset E \subset E$, $\nu \in E \subset E \subset E$, $\lambda \in E_8$. We know that the simply connected compact Lie group $E_8$ has two classes of involutive elements up to conjugation.

**Lemma 5.3.3.** For $\sigma_1, \sigma_2 \in E_8$ such that $\sigma_1^2 = \sigma_2^2 = 1$, we have

$$\sigma_1 \sim \sigma_2 \iff (E_8)^{\sigma_1} \cong (E_8)^{\sigma_2} \iff (E_8)^{\sigma_1} \cong (E_8)^{\sigma_2},$$

moreover if and only if $\dim(e_8)^{\sigma_1} = \dim(e_8)^{\sigma_2}$.

**Proof** is due to [19].

**Proposition 5.3.4.** (1) $\lambda, \nu, \gamma, \omega \sigma$ are conjugate in $E_8$ with one another.

(2) $\lambda \gamma$ and $\nu \gamma$ are conjugate in $E_8$.

**Proof.** We can easily calculate $\dim(e_8)^{\lambda} = \dim(e_8)^{\nu} = \dim(e_8)^{\gamma} = 136$ and $\dim(e_8)^{\lambda \gamma} = \dim(e_8)^{\nu \gamma} = 120$, hence Proposition 5.3.4 follows from Lemma 5.3.3.

**Remark.** The author can not find any element $\delta \in E_8$ which gives the conjugation: $\delta \lambda = \nu \delta$ etc..

### 5.4. Subgroups of type $A_1 \oplus E_7$ of Lie groups of type $E_8$

We consider a subgroup $(E_8^C)_{1,1,-,1}$ of $E_8^C$:

$$(E_8^C)_{1,1,-,1} = \{ \alpha \in E_8^C | \alpha_1 = 1, \alpha_1^- = 1^-, \alpha_1 = 1 \}$$

where $1 = (0, 0, 0, 1, 0, 0), 1^- = (0, 0, 0, 0, 1, 0), 1 = (0, 0, 0, 0, 0) \in e_8^C$.

**Proposition 5.4.1.** $(E_8^C)_{1,1,-,1} \cong E_7^C$.

**Proof** ([17]). For $\beta \in E_7^C$ we correspond $\alpha \in E_8^C$,

$$\alpha(\Phi, P, Q, r, s, t) = (\beta \Phi \beta^{-1}, \beta P, \beta Q, r, s, t)$$

for $(\Phi, P, Q, r, s, t) \in e_8^C$. It is easy to verify $\alpha(\in (E_8^C)_{1,1,-,1}$. Conversely let $\alpha \in (E_8^C)_{1,1,-,1}$. From the conditions $\alpha_1 = 1, \alpha_1^- = 1^-, \alpha_1 = 1$, $\alpha$ has the form

$$\alpha = \begin{pmatrix}
\beta_1 & \beta_{12} & \beta_{13} & 0 & 0 & 0 \\
\beta_{12} & \beta_2 & \beta_{23} & 0 & 0 & 0 \\
\beta_{13} & \beta_{23} & \beta_3 & 0 & 0 & 0 \\
a_1 & b_1 & c_1 & 1 & 0 & 0 \\
a_2 & b_2 & c_2 & 0 & 1 & 0 \\
a_3 & b_3 & c_3 & 0 & 0 & 1 \\
\end{pmatrix},$$

where

$$\beta_1 \in \text{Hom}(e_7^C, e_7^C), \quad \beta_2, \beta_3, \beta_{23} \in \text{Hom}(\mathbb{P}^C, \mathbb{P}^C),$$

$$\beta_{12}, \beta_{13} \in \text{Hom}(e_7^C, \mathbb{P}^C), \quad \beta_{12}, \beta_{13} \in \text{Hom}(\mathbb{P}^C, e_7^C),$$

$$a_1 \in \text{Hom}(e_7^C, C), \quad b_1, c_1 \in \text{Hom}(\mathbb{P}^C, C).$$
From $[a\Phi, 1]=a[\Phi, 1]=0$ we have $\beta_2=\beta_5=0$, $a_2=a_5=0$, and from $[a\Phi, 1^-]=a[\Phi, 1^-]=0$ we have $a_1=0$. Put $P^+=(0, P, 0, 0, 0, Q, 0, 0, 0)$, $Q^-(0, 0, Q, 0, 0, 0, 0)$.

From $[aP^-, 1]=aP^-$ we have $\beta_1=\beta_3=0$, $b_1=b_3=0$. Similarly from $[aQ^-, 1]=aQ^-$ we have $\beta_1=\beta_3=0$, $c_1=c_3=c_5=0$. Operate $\alpha$ on $[P^-, Q^-]=(P\times Q, 0, 0, -(1/8)\{P, Q\}, 0, 0)$, then

$$\beta_1(P\times Q)=\beta_1P\times \beta_4Q, \quad \{\beta_2P, \beta_4Q\} = \{P, Q\}. \quad (i)$$

Again operate $\alpha$ on $[P^-, Q^-]=(1/4)\{P, Q\}1$, then

$$\{\beta_2P, \beta_4Q\} = \{P, Q\}. \quad (ii)$$

Operate $\alpha$ on $[\Phi, P^-]=(\Phi P)^-$, then

$$\{\beta_2\Phi, \beta_4P\} = \beta_2(\Phi P). \quad (iii)$$

From (i), (ii) we have $\{\beta_2P, \beta_4Q\} = \{\beta_2P, \beta_4Q\}$ for all $P, Q \in \mathfrak{e}$, hence $\beta_2=\beta_2$ (put $\beta=\beta$). In (iii) put $\beta^{-1}P$ instead of $P$, then $\beta_1\Phi=\beta\Phi\beta^{-1}$. Therefore, from (1) we have $\beta(P\times Q)\beta^{-1}=\beta_1(P\times Q)=\beta P\times \beta Q$. Thus $\beta \in \mathfrak{e}$.

**Proposition 5.4.2.** $(E_8^0)^*$ has a subgroup $\phi(SL(2, C))$ which is isomorphic to the group $SL(2, C)$. Where $\phi(A), A=\bigl(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\bigr) \in SL(2, C)$, is the $C$-linear transformation of $e^c_{ij}$ defined by

$$\phi\bigl(\bigl(\begin{smallmatrix} a \\ b \\ c \\ d \end{smallmatrix}\bigr)\bigr)=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a1 & c1 & 0 & 0 & 0 \\ 0 & b1 & d1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+2bc-ab & cd \\ 0 & 0 & 0 & -2ac & a^2-c^2 \\ 0 & 0 & 0 & 2bd & -b^2 & d^2 \end{bmatrix}.$$

**Proof.** For $A=\bigl(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\bigr)=exp\bigl(\begin{smallmatrix} r & s \\ t & -r \end{smallmatrix}\bigr)$, we have

$$\phi(A)=exp\biggl(\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & r1 & s1 & 0 & 0 \\ 0 & t1 & -r1 & 0 & 0 \\ 0 & 0 & 0 & -t & s \\ 0 & 0 & 0 & 2s & 2r \\ 0 & 0 & 0 & 2t & -2r \end{bmatrix}\biggr).$$

$=exp(\Theta(0, 0, 0, r, s, t))$.

**Lemma 5.4.3.** $\phi: SL(2, C) \rightarrow (E_8^0)^*$ of Proposition 5.4.2 satisfies

$$\tau \phi(A) \tau = \phi(\tau A), \quad \omega \phi(A) \omega^{-1} = \phi(\omega^{-1} A).$$
\[
\lambda \phi(A) \lambda^{-1} = \sigma \phi(A) \sigma = \tau \phi(A) \tau = \phi(A).
\]

**Theorem 5.4.4.** \((E^C_8)^{\gamma} \cong (SL(2, C) \times E_7^C) / \mathbb{Z}_2, \mathbb{Z}_2 = \{ (E, 1), (-E, -1) \} \).  

**Proof.** We define a mapping \( \phi : SL(2, C) \times E_7^C \to (E^C_8)^{\gamma} \) by
\[
\phi(A, \beta) = (\phi(A) \beta).
\]
Obviously \( \phi(A, \beta) \in (E^C_8)^{\gamma} \). Since \( \phi(A) \), \( A \in SL(2, C) \) and \( \beta \in E_7^C \) are commutative, \( \phi \) is a homomorphism. \( \ker \phi = \{ (E, 1), (-E, -1) \} = \mathbb{Z}_2 \). Since \( (E^C_8)^{\gamma} \) is connected (Lemma 0.7) and \( \dim_c(sl(2, C) \oplus e^7_c) = 3 + 133 = 136 = \dim_c(e^8_c) \gamma \) (note that \( e^8_c = \{ (0, 0, r, s, t) \in e^7_c | (0, 0, r, s, t) \in C \} \), \( \phi \) is onto. Thus we have the required isomorphism.

**Theorem 5.4.5.** (1) \( (E_8)^{\gamma} \cong (SU(2) \times E_7) / \mathbb{Z}_2 \cong ((\tau \lambda \nu)^{\gamma} \sim (E^{\gamma}_{8(-34)})^{\gamma} \).

(2) \( (E^{\gamma}_{8(-34)})^{\gamma} \cong (SU(2) \times E^{\gamma}_{7(-3)}) / \mathbb{Z}_2 \cong ((\tau \lambda \nu)^{\gamma} \sim (E^{\gamma}_{8(3)})^{\gamma} \).

**Proof.** (1) Let \( \alpha \in (E_8)^{\gamma} = ((E^C_8)^{\gamma})^{\gamma} = (\tau \lambda)^{\gamma} \). By Theorem 5.4.4, there exist \( A \in SL(2, C), \beta \in E_7^C \) such that \( \alpha = \phi(A) \beta \). From the condition \( \tau \lambda \alpha = \alpha \tau \lambda \), we have \( \phi(\tau A^{-1}) \tau \lambda \beta \tau = \phi(A) \beta \) (Lemma 5.4.3).

\[
\begin{cases}
\tau A^{-1} = A & \text{or} & \tau A^{-1} = -A \\
\tau \lambda \beta \tau = \beta & \text{or} & \tau \lambda \beta \tau = \nu \beta.
\end{cases}
\]

The latter case is impossible because \( (\tau A)A = -E \) is false. In the first case, \( (\tau A)A = E \), that is, \( A \in SU(2) \). For \( \beta \in E_7^C \), \( \tau \lambda \beta \tau = \beta \) is \( \tau \lambda \beta \lambda^{-1} \tau = \beta \), hence \( \beta \in (E_7^C)^{\lambda \lambda} = E_7 \) (Theorem 4.3.2). Thus \( (E_8)^{\gamma} = \phi(SU(2) \times E_7) \cong (SU(2) \times E_7) / \mathbb{Z}_2 \).

\[
E^{\gamma}_{8(-34)} = (E^C_8)^{\gamma} \cong (E^C_8)^{\tau \lambda \nu}.
\]

In fact, since \( \lambda \sim \nu \) under some \( \delta \in E_8 : \delta \lambda = \nu \delta, \delta \tau \lambda = \tau \delta \) (Proposition 5.3.4), \( (E^C_8)^{\gamma} \cong (E^C_8)^{\tau \lambda} \) gives an isomorphism. Now \( (E^{\gamma}_{8(-34)})^{\gamma} \sim (\tau \lambda \nu)^{\gamma} = (\tau \lambda)^{\gamma} \).

(2) \( E^{\gamma}_{8(-34)} = (E^C_8)^{\gamma} \cong (E^C_8)^{\tau \lambda \nu} \).

In fact, since \( \lambda \sim \gamma \) under some \( \delta \in E_8 : \delta \lambda = \gamma \delta, \delta \tau \lambda = \tau \delta \) (Proposition 5.3.4), \( (E^C_8)^{\gamma} \cong (E^C_8)^{\tau \lambda \nu} \) gives an isomorphism. Let \( \alpha \in (E^C_8)^{\tau \lambda \nu} = (\tau \lambda \gamma), \alpha = \phi(A) \beta, A \in SL(2, C), \beta \in E_7^C \). From \( \tau \lambda \alpha = \alpha \tau \lambda \), we have \( \phi(\tau A^{-1}) \tau \lambda \beta \gamma \lambda \tau = \phi(A) \beta \) (Lemma 5.4.3).

\[
\begin{cases}
\tau A^{-1} = A & \text{or} & \tau A^{-1} = -A \\
\tau \lambda \beta \gamma \lambda \tau = \beta & \text{or} & \tau \lambda \beta \gamma \lambda \tau = \nu \beta.
\end{cases}
\]

As similar to (1), the latter case is impossible. In the first case, \( A \in SU(2) \) and \( \beta \in (E_7^C)^{\tau \lambda \nu} = (E_7^C)^{\tau \lambda} = E^{\gamma}_{7(-3)} \) (Theorem 4.3.2). Thus \( (E^{\gamma}_{8(-34)})^{\gamma} \sim (\tau \lambda)^{\gamma} \cong (SU(2) \times E_7)^{\gamma} \).
In fact, since $\tilde{\gamma} \sim \nu \tau$ under some $\delta \in E_8$: $\delta \tilde{\gamma} = \nu \tau \delta$, $\delta \tau \tilde{\gamma} = \tau \delta \tilde{\gamma}$ (Proposition 5.3.4), $(E_8^c)^{\gamma} \cong (E_8^c)^{\nu \tau}$ gives an isomorphism. Now $(E_8^c)^{\gamma} \sim (\tau \tilde{\gamma} \nu \tau)^{\gamma} = (\tau \tilde{\gamma} \nu)^{\gamma}$.

**Theorem 5.4.6.** (1) $(E_8^c)^{\gamma} \cong (SL(2, R) \times E_{7(7)}) / Z_2 \times 2$.

(2) $(E_8^c)^{\gamma \nu} \cong (SL(2, R) \times E_{7(7)}) / Z_2 \times 2$.

**Proof.** (1) Let $\alpha \in (E_8^c)^{\gamma} = ((E_8^c)^{\gamma})^{\nu} = (\tau \gamma)^{\nu}$, $\alpha = \phi(A)\beta$, $A \in SL(2, C)$, $\beta \in E_7^c$.

From $\tau \gamma = \alpha \gamma$, we have $\phi(\tau \alpha)\gamma \beta \tau = \phi(A)\beta$ (Lemma 5.4.3). Hence

$$\begin{cases} \tau A = A & \text{or} \quad \tau A = -A, \\ \tau \beta \gamma \tau = \beta & \text{or} \quad \tau \beta \gamma \tau = -A \beta. \end{cases}$$

In the first case, $A \in SL(2, R)$, $\beta \in (E_7^c)^{\gamma} = E_{7(7)}$ (Theorem 4.3.2). In the latter case, $A = (il) B$, $B \in SL(2, R)$, $\beta = \nu \beta$, $\beta \in E_7^c$, where $\ddot{\tau}$ is one defined in 4.2.

Thus $(E_8^c)^{\gamma} \cong (SL(2, R) \times E_{7(7)}) \cup (il) SL(2, R) \times E_{7(7)}) / Z_2 = (SL(2, R) \times E_{7(7)}) / Z_2 \times 2$. (The explicit form of $\phi(i, \iota)$ is

$$\phi(i, \iota)(\Phi, P, Q, r, s, t) = (i\Phi^{-1}, i\nu P, -i\nu Q, r, -s, -t)).$$

(2) Let $\alpha \in (E_8^c)^{\gamma} = ((E_8^c)^{\gamma})^{\nu} = (\tau \gamma)^{\nu}$, $\alpha = \phi_A(\beta)$, $A \in SL(2, C)$, $\beta \in E_7^c$. From $\tau \gamma = \alpha \gamma$, we have $\phi(\tau \alpha)\gamma \beta \tau = \phi(A)\beta$ (Lemma 5.4.3). As similar to (1), $(E_8^c)^{\gamma \nu} \cong (SL(2, R) \times E_{7(7)}) \cup (il) SL(2, R) \times E_{7(7)}) / Z_2 = (SL(2, R) \times E_{7(7)}) / Z_2 \times 2$.

### 5.5. Subgroups of type $D_8$ of Lie groups of type $E_8$

We define an $R$-algebraic homomorphism $l : M(8, C) \to M(16, R)$ by

$$l((x + y i)l) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}, \quad x, y \in R$$

and $l : M(8, C^c) \to M(16, C)$ is its complexification. These $l$ satisfy

$$l(l(X)) = l(X)^*$$

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Proposition 5.5.1. (1) $l(\mathfrak{so}(8, C^c)) = \{B \in \mathfrak{so}(16, C) | JB = BJ\}$,

$$l(\mathfrak{so}(8, C^c)) = \{B \in \mathfrak{so}(16, C) | JB = -BJ\}.$$

(2) Any element $B \in \mathfrak{so}(16, C)$ is uniquely expressed by

$$B = l(D) + l(S)I, \quad D = l(D) + l(S)I + l(i c E),$$

$D \in \mathfrak{so}(8, C^c), S \in \mathfrak{so}(8, C^c), c \in C.$
Proof. (1) If $D \in \mathfrak{u}(8, C^c)$, then $\mathcal{I}(D) = \mathcal{I}(D)J$, $\mathcal{I}(D) = \mathcal{I}(D^*) = -\mathcal{I}(D)$. Conversely, for $B \in \mathfrak{so}(16, C)$ such that $J B = B J$, put $B = \mathcal{I}(D)$, $D \in M(8, C^c)$. Then $\mathcal{I}(D) = -B = \mathcal{I}(D) = \mathcal{I}(D^*)$, hence $D^* = -D$. Next, for $S \in \mathfrak{so}(8, C^c)$, $\mathcal{I}(S) I = \mathcal{I}(S) J I = -\mathcal{I}(S) I J$, $\mathcal{I}(S) I = \mathcal{I}(S^*) = -\mathcal{I}(S) I$. Conversely, for $B \in \mathfrak{so}(16, C)$ such that $J B = -B J$, consider $B I$. Then $J B I = B I J$, hence we can put $B I = \mathcal{I}(S)$, $S \in M(8, C^c)$. Then $B = \mathcal{I}(S) I$, $I B = \mathcal{I}(S^*) = \mathcal{I}(S) I$, hence $-S = I S$, that is, $S \in \mathfrak{z}(8, C^c)$.

(2) $B = (B - J B J)/2 + (B + J B J)/2$ and use the above (1).

We consider Lie subalgebras $\mathfrak{so}(4, 12)$, $\mathfrak{so}(8, 8)$, $\mathfrak{so}^*(16)$ (which are isomorphic to the ordinary ones) of $\mathfrak{so}(16, C)$ as

$\mathfrak{so}(4, 12) = \{ B \in \mathfrak{so}(16, C) | \mathcal{I}(I_2) B \mathcal{I}(I_2) = B \}$,

$\mathfrak{so}(8, 8) = \{ B \in \mathfrak{so}(16, C) | \mathcal{I}(I_2') B \mathcal{I}(I_2') = B \}$,

$\mathfrak{so}^*(16) = \{ B \in \mathfrak{so}(16, C) | J B = (\mathcal{I} B) J \}$

where $I_2' = \operatorname{diag}(1, 1, 1, 1, -1, -1, -1, -1)$, and corresponding the above we use the following notations.

$\mathfrak{su}(2, 6) = \{ D \in \mathfrak{su}(8, C^c) | I_4 (\mathcal{I} D) I_4 = D \}$,

$\mathfrak{S}(2, 6, C) = \{ S \in \mathfrak{S}(8, C^c) | I_4 (\mathcal{I} S) I_4 = S \}$,

$\mathfrak{su}(4, 4) = \{ D \in \mathfrak{su}(8, C^c) | I_4 (\mathcal{I} D) I_4 = D \}$,

$\mathfrak{S}(4, 4, C) = \{ S \in \mathfrak{S}(8, C^c) | I_4 (\mathcal{I} S) I_4 = S \}$.

From Proposition 5.5.1, we have easily the following.

**Proposition 5.5.2.** Any elements $B \in \mathfrak{so}(16)$, $\mathfrak{so}(4, 12)$, $\mathfrak{so}(8, 8)$, $\mathfrak{so}^*(16)$ are, respectively, expressed by $B = \mathcal{I}(D) + \mathcal{I}(S) I + \mathcal{I}(\mathfrak{e} E)$,

(1) case $\mathfrak{so}(16)$, $D \in \mathfrak{su}(8)$, $S \in \mathfrak{S}(8, C)$, $c \in \mathbb{R}$,

(2) case $\mathfrak{so}(4, 12)$, $D \in \mathfrak{su}(2, 6)$, $S \in \mathfrak{S}(2, 6, C)$, $c \in \mathbb{R}$,

(3) case $\mathfrak{so}(8, 8)$, $D \in \mathfrak{su}(4, 4)$, $S \in \mathfrak{S}(4, 4, C)$, $c \in \mathbb{R}$,

(4) case $\mathfrak{so}^*(16)$, $D \in \mathfrak{su}(8)$, $S \in \mathfrak{S}(8, C)$, $c \in \mathbb{R}$.

Recall the $C$-linear isomorphism $\chi: \mathfrak{z}^c = \mathfrak{z}^c \oplus \mathfrak{z}^c \oplus C \to \mathfrak{S}(8, C^c)$,

$$\chi(X, Y, \xi, \gamma) = \left( k \begin{pmatrix} gX & -\xi Y \\ 2 & 2 \end{pmatrix} \right) J + i \left( k \begin{pmatrix} gY & -\gamma E \\ 2 & 2 \end{pmatrix} \right)$$

which is used to define the homomorphism $\phi: SU(8, C^c) \to \mathfrak{e}^c \chi$, $\phi(A) P = \chi^{-1}(A(\chi P)^t A)$, $P \in \mathfrak{z}^c$ (Theorem 4.5.3).

**Lemma 5.5.3.** For $S \in \mathfrak{S}(8, C^c)$, we have
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\[ \gamma^{-1}S = \gamma^{-1}(I_5S)I_2, \quad \tau^{-1}S = \tau^{-1}(I_5\tau S)I_2, \quad \sigma^{-1}S = \sigma^{-1}(I_5\sigma S)I_2. \]

**Proof.** \( \gamma^{-1}S = \gamma^{-1}(X, Y, \xi, \eta) = (\gamma X, \gamma Y, \xi, \eta) \)

\[
= k \left( g(\gamma X) - \frac{\xi}{2} E \right) J + ik \left( gY - \frac{\eta}{2} E \right) J \]

\[
= k \left( I_5(gX)I_5 - \frac{\xi}{2} E \right) J + ik \left( I_5(gY)I_5 - \frac{\eta}{2} E \right) J \]

\[
= I_5 \left( k \left( gX - \frac{\xi}{2} E \right) J + ik \left( gY - \frac{\eta}{2} E \right) J \right) J \]

\[
= I_5 \gamma(X, Y, \xi, \eta)J_2 = I_5S. \]

Using \( g(\tau X) = I_5(\tau(gX))I_5, \ g(\sigma X) = I_5'(gX)I_5', \ I_5' = \text{diag}(1, 1, -1, -1) \), other formulae are similarly obtained.

**Lemma 5.5.4.** For \( S_1, S_2 \in \mathcal{S}(8, C^c) \), we have

1. \( \text{tr}(S_1S_2 - S_2S_1) = 4i(\chi^{-1}S_1, \chi^{-1}S_2). \)
2. \( \phi_\delta(S_1S_2 - S_2S_1 - \frac{1}{8} \text{tr}(S_1S_2 - S_2S_1)E) = 4(\lambda\gamma^{-1}S_1 \times \chi^{-1}S_2 - \lambda\gamma^{-1}S_2 \times \chi^{-1}S_1) \)

where \( \phi_\delta : \mathcal{H}(4, H^c) \to (e_8^c)^{\chi'} \) is one defined in 3.5.

**Proof.** is in [21, Proposition 6].

**Theorem 5.5.5.** \( (e_8^c)^{\chi'} = \mathfrak{so}(16, C). \)

**Proof.** \( (e_8^c)^{\chi'} = \{ \Theta \in \text{Der}_c(e_8^c) | \lambda\gamma\Theta = \Theta \lambda\gamma \} \)

\[
= \{ \Theta(\Phi, \lambda\gamma Q, Q, 0, s, -s) | \Phi \in (e_8^c)^{\chi'}, Q \in \mathbb{P}^c, s \in C \}. \]

We define a mapping \( \zeta : \mathfrak{so}(16, C) \to (e_8^c)^{\chi'} \) by

\[
\zeta(l(D) + l(S)I + l(icE)) = \Theta(\phi_\phi(D), 2\lambda\gamma S_1, 2X^{-1}S, 0, 2c, -2c) \]

where \( D \in \mathfrak{su}(8, C^c), S \in \mathcal{S}(8, C^c), c \in C \) (Proposition 5.5.1) and \( \phi_\phi : \mathfrak{su}(8, C^c) \to (e_8^c)^{\chi'}, \lambda : \mathbb{P}^c \to \mathcal{S}(8, C^c) \) are ones defined in the beginning. Clearly \( \zeta \) is bijective.

We have to prove that \( \zeta \) is a homomorphism, that is,

\[
\zeta[X, Y] = \zeta[X, \zeta(Y)], \quad X, Y = \xi(D), \xi(S)I, \xi(icE). \]

For example, to prove \( \zeta[l(S_1)I, l(S_2)I] = [\zeta(l(S_1)I), \zeta(l(S_2)I)] \), we use Lemma 5.5.4. The details of calculations are in [21].

**Theorem 5.5.6.**

1. \( (e_8(16))^{\chi'} = \mathfrak{so}(16) \equiv (e_8)^{\chi'} \).
2. \( (e_8(22))^{\chi'} \equiv \mathfrak{so}(4, 12). \)
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(3) \((e_8(c))^{\gamma}\sim((e_8(c)^{r\sigma})))^{\gamma}\equiv \xi_0(8, 8).

(4) \((e_8(c))^{\gamma}\sim((e_8(c)^{r\sigma})))^{\gamma}\equiv \xi_0(8, 16)\equiv((e_8(c)^{r\sigma})))^{\gamma}\sim((e_8(c)^{r\sigma})))^{\gamma}.

**Proof.** (1) Let \(\Theta \in (e_8(c))^{\gamma}\equiv \xi_0(8, C)^\gamma\). By Theorem 5.5.5, there exist \(D \in \xi_0(8, C)^\gamma\), \(S \in \xi(8, C)^\gamma\), \(c \in C\) such that

\[ \Theta = \Theta(\phi_8(D), 2\lambda \chi^{-1} S, 2\chi^{-1} S, 0, 2c, -2c). \]

From the condition \(\tau \gamma \Theta(R) = \Theta(R), R \in e_8,\) that is, \(\Theta(\tau \gamma R) = \Theta(R)\), we have

\[ \tau \gamma \phi_8(D) \gamma = \phi_8(D), \quad \tau \gamma \chi^{-1} S = \chi^{-1} S, \quad \tau c = c. \]

Hence \(D \equiv \xi_0(8)(\) Theorem 4.5.5), \(\chi^{-1}(\tau S) = \chi^{-1} S\) (Lemma 5.5.3), \(\tau S = S\), that is, \(S \equiv \xi(8, C)\) and \(c \in R\). Therefore, \(\Theta \in \xi((\xi_0(8)) = \xi(8, C))I + l(iR E) = \xi(8, 16)\) (Proposition 5.5.2. (1)). Thus \((e_8(c))^{\gamma}\equiv \xi_0(8, 16). \quad (e_8(c))^{\gamma} = ((e_8(c)^{r\sigma})))^{\gamma} = ((e_8(c)^{r\sigma})))^{\gamma}.

(2) Let \(\Theta \in (e_8(-2\lambda))^{\gamma} = ((e_8(c)^{r\sigma})))^{\gamma}, \Theta = \Theta(\phi_8(D), 2\lambda \chi^{-1} S, 2\chi^{-1} S, 0, 2c, -2c)\) as in (1). From \(\tau \Theta = \Theta \tau\), we have

\[ \tau \gamma \phi_8(D) \gamma = \phi_8(D), \quad \tau \gamma \chi^{-1} S = \chi^{-1} S, \quad \tau c = c. \]

Hence \(D \equiv \xi_0(2, 6)(\) Theorem 4.5.5), \(\chi^{-1}(\tau S) = \chi^{-1} S\) (Lemma 5.5.3), \(I_8(\tau S)I_8 \equiv S\), that is, \(S \equiv \xi(2, 6, C)\) and \(c \in R\). Therefore, from Proposition 5.5.2. (2), we have \((e_8(-2\lambda))^{\gamma}\equiv \xi_0(4, 12).

(3) \(E_8(c) = (E_8(c)^{r\sigma})))^{\gamma} = (E_8(c)^{r\sigma})))^{\gamma}\)

because \(\gamma \sim \sigma\gamma\) under \(\delta \in F\) (Proposition 2.2.3) \(\subset E_8 \subset E_8 \subset E_8\): \(\delta \gamma = \sigma \gamma \delta, \delta \tau = \tau \delta\).

Now let \(\Theta \in ((e_8(c)^{r\sigma})))^{\gamma}, \Theta = \Theta(\phi_8(D), 2\lambda \chi^{-1} S, 2\chi^{-1} S, 0, 2c, -2c)\) as in (1). From \(\tau \sigma \gamma \Theta = \Theta \tau \sigma \gamma\), we have

\[ \tau \sigma \gamma \phi_8(D) \gamma = \phi_8(D), \quad \tau \sigma \gamma \chi^{-1} S = \chi^{-1} S, \quad \tau c = c. \]

Hence \(D \equiv \xi_0(4, 4)(\) Theorem 4.5.7), \(\chi^{-1}(I_8(\tau S)I_8) = \chi^{-1} S\) (Lemma 5.5.3), \(I_8(\tau S)I_8 \equiv S\), that is, \(S \equiv \xi(4, 4, C)\) and \(c \in R\). Therefore, from Proposition 5.5.2. (3), we have \((e_8(c))^{\gamma}\sim ((e_8(c)^{r\sigma})))^{\gamma}\equiv \xi_0(8, 8).

(4) \(E_8(c) = (E_8(c)^{r\sigma})))^{\gamma} = (E_8(c)^{r\sigma})))^{\gamma}\)

In fact, consider \(\delta: e_8 \to e_8\),

\[ \delta(\Phi, P, Q, r, s, t) = (\Phi, iP, -iQ, r, -s, -t) \]

(\(\delta\) is \(\phi(iI)\) of Proposition 5.4.2), then \(\delta \in E_8, \delta \gamma = \gamma \delta, \delta \tau = \tau \delta, \) and \((E_8(c)^{r\sigma})))^{\gamma} = \alpha \to \delta \alpha \delta^{-1} \in (E_8(c)^{r\sigma})))^{\gamma}\) gives an isomorphism. Now let \(\Theta \in ((e_8(c)^{r\sigma})))^{\gamma}, \Theta = \Theta(\phi_8(D), 2\lambda \chi^{-1} S, 2\chi^{-1} S, 0, 2c, -2c)\) as in (1). From \(\tau \sigma \gamma \Theta = \Theta \tau \sigma \gamma\), we have

\[ \tau \gamma \phi_8(D) \gamma = \phi_8(D), \quad -\tau \gamma \chi^{-1} S = \chi^{-1} S, \quad \tau c = c. \]
Hence $D \subseteq \mathfrak{su}(8)$ (Theorem 4.5.5), $-\chi^\prime(\tau S) = \chi^\prime S$ (Lemma 5.5.3), $-\tau S = S$, that is, $S \in i\mathbb{Z}(8, C)$ and $c \in R$. Therefore, from Proposition 5.5.2.(4), we have $(\epsilon_{8\mathbb{C}(-4)})^\chi \sim (\epsilon_{8\mathbb{C}})^{\chi^\prime}$ (Theorem 5.4.5.(1)) = $(\epsilon_{8\mathbb{C}})^{\chi^\prime}$.

**Theorem 5.5.7.**  
(1) $(E_8\mathbb{C})^{\chi^\prime} \cong S(16, C) (= \text{Spin}(16, C)/\mathbb{Z}_2 \text{ and not } SO(16, C))$
(2) $(E_8)^{\chi'} \cong S(16) (= \text{Spin}(16)/\mathbb{Z}_2 \text{ and not } SO(16))$

**Proof.** is in [21]. The outline of the proof is as follows. The group $(E_8\mathbb{C})^{\chi'}$ is connected (Lemma 0.7) and the center $z((E_8\mathbb{C})^{\chi'})$ is $Z_2 = \{1, \chi\}$, hence $(E_8\mathbb{C})^{\chi'}$ is isomorphic to one of $SO(16, C)$ or $S(16, C)$. $(E_8\mathbb{C})^{\chi'}$ has the 128-dimensional irreducible $C$-representation $(\epsilon_{8\mathbb{C}})^{\chi^\prime}$, however $SO(16, C)$ has no 128-dimensional irreducible $C$-representation. Therefore $(E_8\mathbb{C})^{\chi'}$ must be $S(16, C)$. As for $(E_8)^{\chi'}$, the argument is similar to $(E_8\mathbb{C})^{\chi'}$.

According to Theorem 5.5.6, we use the following notations.

$\text{So}^*(16) = ((E_8\mathbb{C})^{\chi^\prime})^{\chi^\prime} \sim (E_{8\mathbb{C}(-4)})^{\chi^\prime}$, 
$\text{So}^*(16) = (\text{So}^*(16)\partial)\partial$, 
$\text{So}(4, 12) = ((E_8\mathbb{C})^{\chi^\prime})^{\chi^\prime} \sim (E_{8\mathbb{C}(-4)})^{\chi^\prime}$, 
$\text{So}(8, 8) = ((E_8\mathbb{C})^{\chi^\prime})^{\chi^\prime} \sim (E_{8\mathbb{C}(-4)})^{\chi^\prime}$, 
$\text{So}(8, 8) = (\text{So}(8, 8)\partial)\partial$.

**Theorem 5.5.8.** $(E_{8\mathbb{C}(-4)})^{\chi^\prime} \sim (\tau^{\lambda_2})^{\chi^\prime} = \text{So}^*(16) = \text{So}^*(16) \times 2 \cong (\tau^{\lambda_2})^{\chi^\prime} \sim (E_{8\mathbb{C}(-4)})^{\chi^\prime}$.

The Cartan decomposition of $\text{So}^*(16)$ is

$\text{So}^*(16) \sim ((\text{So}(2) \times SU(8))/Z_4) \times \mathbb{R}^{\mathfrak{g}}$

where $Z_4 = \{(E, E), (E, -E), (-E, iE), (-E, -iE)\}$.

**Proof.** The maximal compact subgroup $\text{So}^*(16)_K$ of $\text{So}^*(16)$ is

$\text{So}^*(16)_K = \text{So}^*(16) \cap E_{8\mathbb{C}} = ((E_8\mathbb{C})^{\chi^\prime})^{\chi^\prime} = ((\tau^{\lambda_2})^{\chi^\prime})^{\chi^\prime} = (\tau^{\lambda_2})^{\chi^\prime}$

$=((\tau^{\lambda_2})^{\chi^\prime})^{\chi^\prime} = (\text{So}(2) \times E_{8\mathbb{C}})/Z_4 \cong (\text{So}(2) \times E_{8\mathbb{C}})/Z_4$

$=((\tau^{\lambda_2})^{\chi^\prime})^{\chi^\prime} = (\text{So}(2) \times E_{8\mathbb{C}})/Z_4$.

Let $\alpha \in \text{So}^*(16)_K = ((E_8\mathbb{C})^{\chi^\prime})^{\chi^\prime}$. By Theorem 5.4.5, there exist $A \in SU(2)$, $\beta \in E$, such that $\alpha = \phi(A)\beta$. From the condition $\lambda_2 \alpha = \alpha \lambda_2$, we have $\phi(\alpha^{-1})\lambda_2 \beta \beta \lambda = \phi(A)\beta$ (Lemma 5.4.3). Hence

$\left\{ \begin{array}{l} \alpha^{-1} = A \\
\lambda_2 \beta \beta \lambda = \beta \end{array} \right. \text{ or } \left\{ \begin{array}{l} \alpha^{-1} = -A \\
\lambda_2 \beta \beta \lambda = \beta \end{array} \right.$

In the first case, $A \in SO(2)$, $\beta \in (E_8\mathbb{C})^{\chi^\prime}$ ($\text{So}(2) = \text{SU}(2)/Z_4 \cong SU(8)/Z_2$ (where $Z_4 = \{E, -E\}$) (Theorem 4.5.5). In the latter case, $A = (iI)B$, $B \in SO(2)$, $\beta = i\beta$, $\beta' \in (E_8\mathbb{C})^{\chi^\prime}$. Thus $\text{So}^*(16)_K \cong (\text{So}(2) \times (E_{8\mathbb{C}}^{\chi^\prime}) \cup (iI)\text{SO}(2) \times \tau(E_{8\mathbb{C}}^{\chi^\prime}) \cup \mathbb{Z}_4$ (where $Z_4 = \{(E, 1), (-E, 1)\}$)}
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\[-1\})) = (SO(2) \times (E, i))^\mathbb{Z}_2 \times 2 \cong (SO(2) \times SU(8))/\mathbb{Z}_2 \times 2 \quad \text{(where } \mathbb{Z}_2 = \{(E, E), \quad (-E, iE/Z_2)\} = (SO(2) \times SU(8))/\mathbb{Z}_2 \times 2. \) 

(As for the explicit form of \(\phi(I, \ell)\), see Theorem 5.4.6). \((E_{-\ell})^{\mathbb{Z}_2} \cong (\ell_\mathbb{Z})^{\mathbb{Z}_2}\) (Theorem 5.5.6.(4)) = \((\ell_\mathbb{Z})^{\mathbb{Z}_2}\).

**Theorem 5.5.9.** \((E_{-\ell})^{\mathbb{Z}_2} \cong (\ell_\mathbb{Z})^{\mathbb{Z}_2}\) is connected. The Cartan decomposition of \(So(4, 12)\) is

\[
So(4, 12) \cong (SU(2) \times SU(2) \times Spin(12))/\mathbb{Z}_2 \times \mathbb{Z}_2
\]

where \(Z_2 \times Z_2 = \{(E, E, 1), \quad (E, -E, -\sigma)\} \times \{(E, E, 1), \quad (-E, -E, \sigma)\} = \{(E, E, 1), \quad (-E, -E, -\sigma)\} \times \{(E, E, 1), \quad (-E, -E, -\sigma)\} = \{(E, E, 1), \quad (-E, -E, -\sigma)\} \times \{(E, E, 1), \quad (-E, -E, -\sigma)\}.

**Proof.**

\(E_{(E, \ell)} = (E, (E, \ell))^{\mathbb{Z}_2} \cong (E, (E, \ell))^{\mathbb{Z}_2}.

In fact, since \(\ell \sim \nu \sigma\) under some \(\delta \in E_\delta: \delta \ell = \nu \sigma \delta, \delta \ell = \tau \ell \delta \) (Proposition 5.3.4), \((E, (E, \ell))^{\mathbb{Z}_2} \cong (E, (E, \ell))^{\mathbb{Z}_2}\) gives an isomorphism. Consider the subgroup \([(E, (E, \ell)^{\mathbb{Z}_2})^{\mathbb{Z}_2}] = (\tau \ell \sigma)^{\mathbb{Z}_2}\) of \((\tau \ell \sigma)^{\mathbb{Z}_2}\). Since \(\dim((E, (E, \ell)^{\mathbb{Z}_2})^{\mathbb{Z}_2}) = 120\), the type of \((\tau \ell \sigma)^{\mathbb{Z}_2}\) must be \(D_6\). Moreover, as is shown in the following, the type of the maximal compact subgroup of \((\tau \ell \sigma)^{\mathbb{Z}_2}\) is \(D_6 \oplus D_6\), hence the type of the Lie algebra of \((\tau \ell \sigma)^{\mathbb{Z}_2}\) must be \(D_6\). Hence we put here \(So(4, 12) = (\tau \ell \sigma)^{\mathbb{Z}_2}\). Now the maximal compact subgroup \(So(4, 12)_K\) of \(So(4, 12)\) is

\[
So(4, 12)_K = So(4, 12) \cap E_\delta = ((E, (E, \ell))^{\mathbb{Z}_2}) \cong ((\ell \sigma)^{\mathbb{Z}_2}) \cong ((\ell \sigma)^{\mathbb{Z}_2})^{\mathbb{Z}_2}.
\]

Let \(\alpha \in So(4, 12)_K\). By Theorem 5.4.5, there exist \(A \in SU(2), \quad \beta \in E_\ell\), such that \(\alpha = \phi(A)\beta\). From the condition \(\sigma \alpha = \sigma \beta\), we have \(\phi(A)\beta \sigma = \phi(A)\beta\) (Lemma 5.4.3). Hence \(\sigma \beta = \beta\), that is, \(\beta \in (E, (E, \ell))^{\mathbb{Z}_2} \cong (SU(2) \times Spin(12))/\mathbb{Z}_2\) (where \(Z_2 = \{(E, 1), \quad (-E, -\sigma)\} \) (Theorem 4.6.14). Thus \((\tau \ell \sigma)^{\mathbb{Z}_2} \cong (SU(2) \times (E, (E, \ell))^{\mathbb{Z}_2}) \) (where \(Z_2 = \{(E, 1), \quad (-E, -\sigma)\} \) \(\cong (SU(2) \times Spin(12)) / Z_2 = Z_2 / Z_2\) (where \(Z_2 = \{(E, 1), \quad (-E, -\sigma)\} \) \(\cong (SU(2) \times Spin(12)) / Z_2 = Z_2 / Z_2\).

**Remark.** The maximal compact subgroup \(So(4, 12)_K\) of \(So(4, 12)\) is \(\cong (\tau \ell)^{\mathbb{Z}_2}\) is \(\cong (\tau \ell)^{\mathbb{Z}_2}\).

The author cannot give any isomorphism between \((\tau \ell)^{\mathbb{Z}_2}\) and \((\tau \ell)^{\mathbb{Z}_2}\) directly.

Before determine the maximal compact subgroup \(So(8, 8)_K\) of \(So(8, 8)\), recall the construction of the spinor group \(Spin(n)\) using the Clifford algebra \(C(R^n)\) [15]. Let \(C(R^n)\) be the Clifford algebra generated by \(e_1, \cdots, e_n\) with relations \(e_i^2 = -1\). Let \(R^n\) be the \(R\)-vector space spanned by \(e_1, \cdots, e_n\) and put \(S^{n-1} = \{a \in R^n | a^2 = -1\}\). Now the spinor group \(Spin(n)\) is defined by...
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\[ \text{Spin}(n) = \{a_1 \cdots a_m | a_i \in S^{n-1}, m = 1, 2, \cdots \}. \]

We use here the notation \( \text{Spin}(n) = \text{Spin}(e_1, \cdots, e_n) \). In the case of \( n \equiv 0 \pmod{4} \), the center \( z(\text{Spin}(n)) \) of \( \text{Spin}(n) \) is \( \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, -1\} \times \{1, e_1 \cdots e_n\} = \{1, -1, e_1 \cdots e_n, -e_1 \cdots e_n\} \) and we know

\[ \text{Spin}(n)/\{1, -1\} = \text{SO}(n), \quad \text{Spin}(n)/\{1, \cdots e_n\} = \text{Ss}(n). \]

**Theorem 5.5.10.** \( (E_n)_{\tau} \cong (\tau)_{\tau} = \text{So}(8, 8) = \text{So}(8, 8) \times 2. \) The Cartan decomposition of \( \text{Ss}(8, 8) \) is

\[ \text{So}(8, 8) \cong ((\text{Spin}(8) \times \text{Spin}(8))/([\mathbb{Z}_2 \times \mathbb{Z}_2] \times 2) \times \mathbb{R}^{44} \]

where \( \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, 1\}, (-1, -1) \times \{1, 1\}, (e_1 \cdots e_8, e_9 \cdots e_{16}) = \{(1, 1), (-1, -1), (e_1 \cdots e_8, -e_9 \cdots e_{16})\}. \)

**Proof.** Consider the group \( (E_n)_{\tau} \cong (\tau)_{\tau} \approx E_n \) (Theorem 5.5.6. (2)). J. Sekiguchi [20] shows that \( (\tau)_{\tau}/[\text{Spin}(8)_{\tau}] \) is simply connected and the fundamental group of \( (\tau)_{\tau}/[\text{Spin}(8)_{\tau}] \) is \( \mathbb{Z}_2 \), hence \( \text{Ss}(8, 8) = (\tau)_{\tau} \) must have two connected components. The maximal compact subgroup \( \text{So}(8, 8)_{K} \) of \( \text{So}(8, 8) \) is

\[ \text{So}(8, 8)_{K} = \text{So}(8, 8) \cap E_8 = ((E_8)_{\tau} \cong \text{Ss}(8)) \approx \text{Ss}(16)^*, \]

where \( \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, 1\}, (-1, -1) \times \{1, 1\}, \{e_1 \cdots e_8, e_9 \cdots e_{16}\} = \{(1, 1), (-1, -1), (e_1 \cdots e_8, -e_9 \cdots e_{16})\}. \)

We shall find a subgroup of type \( D_4 \oplus D_4 \) in \( \text{Ss}(16) \). In the spinor group \( \text{Spin}(16) = \text{Spin}(e_1, \cdots, e_{16}) \), consider two subgroups \( \text{Spin}(8) = \text{Spin}(e_1, \cdots, e_8), \text{Spin}(8) = \text{Spin}(e_9, \cdots, e_{16}) \). We define a mapping

\[ \text{Spin}(8) \times \text{Spin}(8) \xrightarrow{\phi} \text{Spin}(16) \xrightarrow{\pi} \text{Ss}(16) \]

by \( \phi(\alpha, \beta) = \alpha \beta \) and let \( \pi \) be the natural projection. Then the image of \( \pi \phi \) is isomorphic to the group \( (\text{Spin}(8) \times \text{Spin}(8))/([\mathbb{Z}_2 \times \mathbb{Z}_2]). \)

**Remark.** The author can not find any element of \( \text{So}(8, 8) \) which does not be contained in \( \text{Ss}(8, 8) \), and can not realize the subgroup \( (\text{Spin}(8) \times \text{Spin}(8))/([\mathbb{Z}_2 \times \mathbb{Z}_2]) \) in \( \text{So}(8, 8) \) concretely.

**Appendix**

The Cartan decompositions of the exceptional linear Lie groups of type \( E_8 \) are given as follows.

\[ E_8 : \text{simply connected compact Lie group of type } E_8, \]

\[ E_8 = E_8 \times \mathbb{R}^{44}. \]
\[ E_{6(8)} \cong Ss(16) \times R^{13}, \]
\[ E_{8(-24)} \cong (SU(2) \times E_7)/Z_4 \times R^{112}. \]

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