ON A RELATION BETWEEN THE TOTAL CURVATURE AND THE MEASURE OF RAYS

Dedicated to Professor I. Mogi on his 60th birthday

By

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§ 0. Introduction.

Let $X$ be a 2-dimensional manifold, then we say that $X$ is finitely connected if the fundamental group $\pi_1(X)$ is finitely generated. If $X$ is noncompact and finitely connected, then it is homeomorphic to a compact surface with a finite number of points removed. Let $M$ be a 2-dimensional finitely connected complete noncompact Riemannian manifold without boundary. The Euler characteristic of $M$, $\chi(M)$, equals the Euler characteristic of the associated compact surface minus the number of points removed. A geodesic $\gamma:[0, \infty) \to M$ is called a ray when any subarc of $\gamma$ is the shortest connection between its end points. And all geodesics are assumed to be parametrized by arc length. Let $T_pM$ be the tangent space of $M$ at $p$ and $S_pM$ be the unit circle of $T_pM$ centered at the origin. $S_pM$ may be regarded as a standard unit circle $S^1$ from the Euclidean metric on $T_pM$. Hence we can consider the Riemannian measure on $S_pM$. Let $A(p)$ be the subset of $S_pM$ consisting of vectors $v$ in $S_pM$ such that the geodesic $\gamma_v: [0, \infty) \to M, \gamma_v(t) = \exp_p tv$, is a ray, where $\exp_p$ is the exponential map of $M$.

Recently, Maeda has proved in [4] the following theorem with interest in a problem whether less curvedness of a Riemannian manifold in some sense implies the existence of rays on it in large quantities or not when the manifold is non-negatively curved:

Theorem ([4]). Let $M$ be a 2-dimensional complete Riemannian manifold with nonnegative Gaussian curvature $K \equiv 0$ diffeomorphic to a Euclidean plane. If $\int_M K dv < 2\pi$, then for any point $p$ in $M$ such that $\#A(p) \equiv 2$, we have

$$\text{measure } A(p) \equiv 2\pi - \int_M K dv.$$

Here the total curvature $\int_M K dv$ of a noncompact Riemannian manifold $M$ is by

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definition the limit of a sequence \( \left\{ \int_{V_n} G \, dv \right\}_{n \in \mathbb{N}} \) which does not depend on the choice of a sequence of compact domains \( \{V_n\}_{n \in \mathbb{N}} \) such that \( V_n \subset V_{n+1} \) and \( \bigcup_{n} V_n = \mathcal{M} \). And we admit \( +\infty \) and \( -\infty \) to be the value of a total curvature. Hence the total curvature always exists if the Gaussian curvature is nonpositive or nonnegative. Moreover, we know that if there exists the total curvature of a complete finitely connected surface \( \mathcal{M} \), the following well know inequality of Cohn-Vossen holds ([3]):

\[
\int_\mathcal{M} G \, dv \leq 2\pi \chi(\mathcal{M}).
\]

The aim of this note is to give a relation between the total curvature and the measure of rays, the abundance of rays, on a 2-dimensional complete finitely connected Riemannian manifold \( \mathcal{M} \). We shall prove the following theorem:

**Theorem 1.** Let \( \mathcal{M} \) be a 2-dimensional finitely connected complete noncompact Riemannian manifold with nonpositive Gaussian curvature \( G \). If \( \int_\mathcal{M} G \, dv > 2\pi(\chi(\mathcal{M}) - 1) \), then we have

\[
\text{measure } A(p) \leq 2\pi \chi(\mathcal{M}) - \int_\mathcal{M} G \, dv \quad \text{for any point } p \in \mathcal{M}.
\]

And from the proof we can get the following theorem which includes Maeda's result:

**Theorem 2.** Let \( \mathcal{M} \) be a 2-dimensional complete Riemannian manifold homeomorphic to a Euclidean plane. If \( \int_\mathcal{M} G^+ \, dv < 2\pi \), then we have

\[
\text{measure } A(p) \geq 2\pi - \int_\mathcal{M} G^+ \, dv \quad \text{for any point } p \in \mathcal{M},
\]

where \( G^+ = (|G| + G)/2 \).

We remark that the right quantity of the inequality in Theorem 1 is not guaranteed to be bounded above by \( 2\pi \). The assumption, \( \int_\mathcal{M} G \, dv > 2\pi(\chi(\mathcal{M}) - 1) \), is put for the inequality to have geometric meaning. The assumption, \( \int_\mathcal{M} G^+ \, dv < 2\pi \), in Theorem 2 is put by the same reason.

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§ 1. **Preliminaries.**

In this section, we shall introduce the various terminologies which follow [2], [3] and modifications of Shiohama [5]. Hereafter \( \mathcal{M} \) always denotes a 2-dimensional
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finately connected complete noncompact Riemannian manifold without boundary
unless otherwise mentioned. Now let $M$ be homeomorphic to $M_0\setminus\{p_1, p_2, \ldots, p_n\}$
under a homeomorphism $f$, where $M_0$ is a compact surface and $p_1, p_2, \ldots, p_n$ are
points of $M$.

Definition 1. An open set $U$ in $M$ is called an open tube if $U$ is homeomorphic
to $S^1 \times (0, \infty)$ and the boundary of $U(:=\partial U)$ is homeomorphic to $S^1$. And a closed
set of $M$ is called a tube or an $R_0$-tube if it is homeomorphic to $S^0 \times [0, \infty)$ and its
boundary is a noncontractible simply closed geodesic polygon $R_0$. It is written as
$U(R_0)$.

Now, for each point $p_j, j=1, 2, \ldots, n$, we can choose mutually disjoint open neighbour-
hood $U_j$ of $p_j$ in $M_0$ such that $U_j := f^{-1}(\bar{U}_j \setminus \{p_j\})$ is a tube.

Let $U(R_0)$ be a given tube of $M$ and let $\rho_{U(R_0)}$ be the distance function on
$U(R_0)$, that is, for any points $p, q \in U(R_0), \rho_{U(R_0)}(p, q)$ is defined to be the infimum
of the lengths of all piecewise smooth curves joining $p$ and $q$ in $U(R_0)$. Then the
function $X_{U(R_0)} : [0, \infty) \to \mathbb{R}$ is defined as follows; $X_{U(R_0)}(t)$ is the infimum
of the lengths of all piecewise smooth noncontractible closed curves $R$ in $U(R_0)$ which
satisfies $\rho_{U(R_0)}(R, R_0) \leq t$. It is easily seen that the function $X_{U(R_0)}$ is Lipschitz
continuous. We shall classify tubes by making use of $X_{U(R_0)}$ in accordance with
[2]. The following three cases may occur for $R_0$-tubes;

Case 1. $X_{U(R_0)}$ does not attain $\inf \{X_{U(R_0)}(s) : s \geq 0\},$
Case 2. $X_{U(R_0)}$ attains $\inf \{X_{U(R_0)}(s) : s \geq 0\}$ for any subtube $U(R)$ in $U(R_0)$,
Case 3. $X_{U(R_0)}$ attains $\inf \{X_{U(R_0)}(s) : s \geq 0\}$ but $X_{U(R_0)}$ does not attain $\inf \{X_{U(R_0)}(s) : s \geq 0\}$ for some subtube $U(R)$ in $U(R_0)$.

Definition 2. An $R_0$-tube $U(R_0)$ is said to be contracting, expanding or bulging
if the function $X_{U(R_0)}$ satisfies Case 1, Case 2 or Case 3, respectively.

According to this definition, a bulging tube is essentially a contracting tube. Hence
we have only to consider the contracting or expanding tubes. And note that subtubes of a contracting (expanding) tubes are also contracting (expanding).

Definition 3. Let $U(R_0)$ be a given tube and $R$ be a noncontractible simply
closed geodesic polygon in $U(R_0)$. If all vertical angles of $R$ which are measured
in $U(R)$ are less (more) than $\pi$, then the geodesic polygon $R$ is said to be convex
(concave).

Definition 4. Let an $R_0$-tube $U(R_0)$ and a nonnegative number $t$ be arbitrarily
given. If a noncontractible closed curve $R(t)$ in $U(R_0)$ satisfies following two con-
ditions, then $R(t)$ is called the solution of Minimal Problem (or simply M.P.) for
Definition 5. Let the following objects be arbitrarily given; a nonnegative number \( t \), a tube \( U(R_0) \) and a ray \( \gamma : [0, \infty) \to M \) such that \( \gamma((a, \infty)) \subset U(R_0) \) and \( \gamma(a) \in R_0 (a > 0) \). If a noncontractible closed curve \( R(t) \) in \( U(R_0) \) which passes through \( \gamma(a + t) \) satisfies \( L(R(t)) = Y_{U(R_0)}(t) \), then \( R(t) \) is called the solution of Minimal Problem along \( \gamma \) (or simply \( \gamma \)-M.P.) for \( U(R_0) \) and \( t \). Here the function \( Y_{U(R_0)} : [0, \infty) \to R \) is defined as follows; \( Y_{U(R_0)} \) is the infimum of the lengths of piecewise smooth noncontractible closed curves \( R \) in \( U(R_0) \) which pass through \( \gamma(a + t) \).

As is seen in [2] and [3], two kinds of solutions surely exist and they satisfy the following facts;

Fact 1. Let \( U(R_0) \) be a contracting tube. Then the solution of M.P. \( R(t) \) for \( U(R_0) \) and \( t \geq 0 \) is either a closed geodesic or a convex geodesic loop. Hence the distance between \( R(t) \) and \( R_0 \) is equal to the distance between the vertex of \( R(t) \) and \( R_0 \) if \( R(t) \) is a convex geodesic loop. The solution of \( \gamma \)-M.P. for \( U(R_0) \) and \( t \geq 0 \) is either a closed geodesic or a geodesic loop.

Fact 2. Let \( U(R_0) \) be an expanding tube. Then the solution of M.P. \( R(t) \) for \( U(R_0) \) and \( t \geq 0 \) is either a closed geodesic or a concave geodesic polygon. And for some \( t_0 \geq 0 \), \( R(t_0) \) is the shortest noncontractible closed curve in \( U(R(t_0)) \). The solution of \( \gamma \)-M.P. for \( U(R_0) \) and \( t \) is either a closed geodesic or a geodesic polygon whose vertical angles except for the vertical angle at \( \gamma \cap R_0 \) measured in \( U(R_0) \) are more than \( \pi \).

For the solution of \( \gamma \)-M.P. we can not get the general information about the vertical angle which is on \( \gamma \). See Cohn-Vossen ([3]), Busemann ([2]) and Bleecker ([1]) for more details of the properties on the solution of M.P.

§ 2. Construction of an expanding filtration.

Throughout this section, let \( p \) be an arbitrarily fixed point of \( M \). And let \( \mathbb{N} \) denote the set of natural numbers. It is our purpose in this section to construct a family of compact domains \( \{ V_j \}_{j \in \mathbb{N}} \) with properties (1), (2) and (3);

1. \( V_j \ni p \).
2. \( V_j \subset V_{j+1} \) and \( \bigcup_{j=1}^{\infty} V_j = M \).
3. \( \partial V_j \) is a closed geodesic or a geodesic polygon which intersects any ray emanating from \( p \) at most once.

Lemma 1. If \( U(R_0) \) is a contracting tube which does not contain the point \( p \),
then there exist noncontractible closed curves $R_j, j \in \mathcal{N}$, in $U(R_0)$ such that

1. $R_j$ is either a closed geodesic or a convex geodesic loop whose vertex lies on a fixed ray,
2. $\lim_{j \to \infty} \rho_{U(R_0)}(R_0, R_j) = \infty$,
3. $R_j$ intersects any ray with at most one point.

**Proof.** Let $C_0$ be the length of $R_0$ and let $\gamma$ be a ray emanating from $p$ and diverging in $U(R_0)$. Set $X(t) := X_{U(R_0)}(t)$ and $Y(t) := Y_{U(R_0)}(t)$. Then we know the existence of a number $t_j \in (C_0 + j, \infty)$ with $Y(t_j) < Y(0) \leq C_0$. In fact, the contracting condition implies the existence of a number $s_j \in (C_0 + j, \infty)$ with $X(s_j) < X(0) \leq C_0$, $X(s_j) = L(\bar{R}(s_j))$, and $\rho_{U(R_0)}(R_0, \bar{R}(s_j)) = s_j$, where $\bar{R}(s_j)$ is the solution of M.P. for $U(R_0)$ and $s_j$. Let $t_j$ be the number with $y(t_j) := \bar{R}(s_j) \cap \gamma$. Then we can get the following relations: $t_j > C_0 + j$ and $Y(t_j) \leq X(s_j) < X(0) \leq Y(0) \leq C_0$. Hence $t_j$ is a required number.

Now let $R_j := R(t_j)$ be the solution of $\gamma$-M.P. for $U(R_0)$ and $t_j$, then $R_j$ satisfies $\rho_{U(R_0)}(R_0, R_j) > j$. This implies $R_j \cap R_0 = \emptyset$. Hence $R_j$ is either a closed geodesic or a geodesic loop. Let $s_j' \in (t_j, \infty)$ be the number such that $X(s_j') < X(t_j)$. Such a number surely exists from the contracting condition. And putting $y(t_j') := \bar{R}(s_j') \cap \gamma$, we have $X(t_j') \leq X(s_j') < X(t_j) \leq Y(t_j)$. Therefore there exists a number $u_j \in (t_j, t_j')$ such that $Y$ is decreasing at $u_j$, $R(u_j)$ must not be a concave geodesic loop. Set newly $R_j := R(u_j)$, then $R_j$ satisfies (1) and (2). Moreover it can be easily proved that any ray which is divergent in $U(R_0)$ never intersects $R_j$ twice because of their minimality.

Lemna 2. If $U(R_0)$ is an expanding tube which does not contain the point $p$, then there exist noncontractible closed curves $R_j, j \in \mathcal{N}$, in $U(R_0)$ such that

1. $R_j$ is either a closed geodesic or a concave geodesic polygon,
2. $\lim_{j \to \infty} \rho_{U(R_0)}(R_0, R_j) = \infty$,
3. $R_j$ intersects any ray with at most one point.

**Proof.** From Fact 2, we know the existence of the shortest noncontractible closed curve $R_t$ in $U(R_t)$ which is either a closed geodesic or a convex geodesic polygon in $U(R_t)$. Let $\sigma$ be any ray emanating from $p$ and diverging in $U(R_t)$. Then $\sigma$ does not meet $R_t$ at more than one point. In fact if $R_t$ is a closed geodesic, then our assertion is trivial because of the minimality of $R_t$ and $\sigma$. Hence we may assume that $R_t$ is a concave geodesic polygon. Let $q_1 := \sigma(t_1)$ and $q_2 := \sigma(t_2)$, $t_1 < t_2$, be the first point of intersection and the second point of intersection of $\sigma$ and $R_t$, respectively. Then $\sigma([t_1, t_2])$ is contained in $U(R_t)$ because of the concavity of $R_t$. Let $R_t'$ be a new noncontractible geodesic polygon which is gotten by exchanging
the subarc of \( R \) between \( q_1 \) and \( q_2 \) for \( \sigma \mid [t_1, t_2] \). The \( R'_i \) is contained in \( U(R_i) \) and has the same length as that of \( R \) because of the minimality of \( \sigma \) and \( R_i \). Since \( R'_i \) has a vertex at \( q_i \), we can get a shorter noncontractible curve in \( U(R_i) \) by exchanging a subarc of \( R'_i \) for a minimal geodesic in a neighbourhood of \( q_i \). This contradicts the shortness of \( R \) in \( U(R_i) \). Consequently, \( \sigma \) does not meet \( R_i \) at more than one point. For \( j \geq 2 \), let \( R'_j \) be a noncontractible geodesic polygon such that \( \rho_{U(R_i)}(R_i, R'_j) > j \) and let \( R_j \) be the shortest noncontractible closed curve in \( U(R'_j) \). Then we can see that \( R_j \) satisfies (1), (2) and (3).

Since \( M \) is finitely connected, \( M \setminus K \) can be represented to a union of \( n \) tubes \( U_{\alpha}, \alpha = 1, 2, \ldots, n \), for a large compact set \( K \) whose boundary consists of \( n \) geodesic polygons each of which may be considered such as an \( R_\alpha \) in the preceding Lemmas. Thus Lemma 1 and Lemma 2 imply the existence of noncontractible closed curves \( R_{j\alpha}, j \in \mathbb{N} \), in each \( U_{\alpha} \). Let \( V_j \) be the compact domains in \( M \) bounded by \( \bigcup_{\alpha} R_{j\alpha} \). Then we have

1. \( V_i \ni p, V_j \subset V_{j+1} \) and \( \bigcup_{j=1}^{\infty} V_j = M \).
2. \( \partial V_j \cap U_{\alpha} (= R_{j\alpha}) \) is a noncontractible geodesic polygon in \( U_{\alpha} \) which does not intersect any ray at more than one point.

The proof of our theorem is achieved by constructing such a special family of compact domains that are chosen by taking into account of the position of rays emanating from \( p \). \( F(p) \) is by definition the set of all points on rays emanating from \( p \). And set \( D(p) := M \setminus F(p) \). \( F(p) \) is a closed set which is homeomorphic to a closed set of \( T_p M \) under \( \exp_p \). Hence \( F(p) \) contains no handles on it. To compute the total curvature of \( M \), we must compute that of \( V_j \). And it is a sum of those of \( F(p) \cap V_j \) and \( \text{cl} \ D(p) \cap V_j \), where \( \text{cl} \ D(p) \) denotes the closure of \( D(p) \). It is difficult to compute the total curvature of \( \text{cl} \ D(p) \cap V_j \) because of the existence of handles. However, we can get an information about the total curvature of \( \text{cl} \ D(p) \). Hence we must take into account of the position of rays emanating from \( p \) to relate the total curvature of \( F(p) \cap V_j \) and that of \( D(p) \setminus \text{int} V_j \). Namely we need the following lemma.

**Lemma 3.** There exists a family of compact domains \( \{V_j\}_{j \in \mathbb{N}} \) in \( M \) which satisfies the above properties (1), (2) and the following properties; For each \( \alpha \)

(a) if \( U_{\alpha} \) is expanding, then there is no vertices of \( \partial V_j \cap U_{\alpha} \) on the rays which are boundaries of \( D(p) \),
(b) if \( U_{\alpha} \) is contracting and if \( \text{int}(F(p) \cap U_{\alpha}) \) is not empty, then there is no vertices of \( \partial V_j \cap U_{\alpha} \) on the rays which are boundaries of \( D(p) \),
(c) if \( U_{\alpha} \) is contracting and if \( \text{int}(F(p) \cap U_{\alpha}) \) is empty, then the vertex of \( \partial V_j \cap U_{\alpha} \) lies on a ray which is a boundary of \( D(p) \) if the vertex exists.
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Proof. (a) In the case of $U_a$ being expanding, the construction follows from Lemma 2. Take $R_o$ and $R'_j$ in the proof of Lemma 2 so that their vertices do not lie on the rays which are boundaries of $D(p)$. This is possible because the rays which are boundaries of $D(p)$ are measure zero. Since $R_j$ is a solution of M.P. for $U(R'_j), R_j$ is either a closed geodesic or a concave geodesic polygon whose vertices are on those of $R'_j$. In this way, we can get a family $(R_j)_{j \in \mathbb{N}}$ of closed geodesics or concave geodesic polygons without their vertices on the rays which are boundaries of $D(p)$.

(b) and (c). In the case of $U_a$ being contracting, the construction follows from Lemma 1. Take a ray $y$ which passes through the interior of $F(p) \cap U_a$ if it is not empty and take a ray which is a boundary of $D(p) \cap U_a$ if int $(F(p) \cap U_a)$ is empty. And applying Lemma 1, we get a family $(R_j)_{j \in \mathbb{N}}$ of closed geodesics or convex geodesic loops which has the desired properties.

§ 3. Proof of Theorems.

Let $(V_j)_{j \in \mathbb{N}}$ be the family of compact domains obtained in Lemma 3. Let $\bar{D}$ be one of the connected components of $D(p) \setminus V_i$. And let $\sigma$ and $\tau$ be the rays which are boundaries of $\bar{D}$. Let $D$ be one of the connected components of $D(p) \cap V_i$, and set $F : = F' \cup \{p\}$ and $F' : = F \cap V_i$, where $F'$ is one of the connected components $F(p) \setminus \{p\}$. Let $\psi^+$ and $\psi^-$ be the vertical angles of $\partial \bar{D}$ at $\partial V_i \cap \sigma$ and $\partial V_i \cap \tau$, respectively. And let $\psi^0$ be a vertical angle of $\partial V_i \cap \partial \bar{D}$ measured in $\partial \bar{D}$.

Under these notations, we can prove the following Lemma by following Maeda [4].

Lemma 4. The following inequality holds good:

$$\int_D G \, dv \geq \psi^+ + \psi^- - \sum_{\partial V_i \cap \partial \bar{D}} (\pi - \psi^0),$$

where the summation is taken over all vertices of $\partial V_i \cap \partial \bar{D}$. And the equality holds if the Gaussian curvature $G$ is nonpositive.

Proof. Let $E_j : = \partial V_j \cap \partial D$. $A^+_{j\theta}$ and $A^-_{j\theta}$ are by definition the set of all initial tangent vectors $\xi(0)$ of the shortest geodesic $\gamma$ connecting between $p$ and $q$ of $E_j$ which satisfy $\xi(\gamma(0), \theta(0)) \leq \theta$ and $\xi(\gamma(t), \theta(t)) \leq \theta$, respectively. And let $A^+_{j\tau}$ and $A^-_{j\tau}$ be the set of all unit vectors $v$ which satisfy $\xi(v, \theta(0)) \leq \theta$ and $\xi(v, \theta(t)) \leq \theta$, respectively. Moreover define the number $\theta(j)$ for each natural number $j$ by

$$\theta(j) : = \inf \{ \theta \in \mathbb{R} : \text{there exists a geodesic } \gamma \text{ in } G(p,q)$$

such that $\gamma(0) \in A^+_{j\theta} \cup A^-_{j\theta}$ for any $q \in E_j \}$. 


Here $G(p,q)$ denotes the set of all the shortest geodesic connections from $p$ to $q$.  We assert that $\theta(j)$ tends to zero as $j$ goes to infinity.  In fact, if $\theta(j)$ does not tend to zero, then there is a constant $C_o>0$ and a subsequence $\{j_i\} \subset \{j\}$ such that $\theta(j_i) \geq C_o$ for any $j_i \in \{j\}$.  Hence for any $j_i$ there is a point $q_{j_i}$ in $E_{j_i}$ and $\overline{y}_{j_i} \in G(p,q_{j_i})$ such that $\overline{y}_{j_i}(0)$ does not belong to $A_{\delta_y,1/2}\cup A_{\delta_y,1/2}$.  From the sequence $\{\overline{y}_{j_i}(0)\}$, we can choose a convergent subsequence $\{\overline{y}_{j_i}(0)\}$.  Let $v_o \in S_p M$ be the limit vector of $\{\overline{y}_{j_i}(0)\}$, then from the construction $v_o$ is not contained in $A_{\delta_y,1/2}\cup A_{\delta_y,1/2}$ and the geodesic $\gamma_o : [0, \infty) \to M$ defined by $\gamma_o(t) : = \exp_p tv_o$ is a ray.  This contradicts the fact that $\gamma_o$ belongs to the domain which no ray passes through.  Let the set $E_1$ and $E_2$ be defined as follows:

$$E_1 : = \{q \in E_f ; \text{ there exists a geodesic } \gamma \in G(p,q) \text{ such that } \gamma(0) \in A_{\delta_y,1/2}\}.$$  

$$E_2 : = \{q \in E_f ; \text{ there exists a geodesic } \gamma \in G(p,q) \text{ such that } \gamma(0) \in A_{\delta_y,1/2}.\}$$

Then it is easily seen that $E_1 \cup E_2$ and $E_1 \setminus E_2$ are non-empty closed sets in $E_f$ from the connectivity of the cut point.  The connectivity of $E_f$ implies the existence of a point $q \in E_1 \cap E_2$ such that the initial vectors $\overline{y}_1(0)$ and $\overline{y}_2(0)$ of minimal geodesics between $p$ and $q$ which belong to $A_{\delta_y,1}$ and $A_{\delta_y,1}$ respectively.  Therefore $\gamma_1$ tends to $\sigma$ and $\gamma_2$ tends to $\tau$ as $j$ goes to infinity.  Let $\overline{D}_f$ be the subset of $\overline{D}$ bounded by $\gamma_1$, $\gamma_2$ and $\partial V_i$.  Then we can get the following inequality from Theorem of Gauss-Bonnet,

$$\int_{\partial \overline{D}_f} G \, dv = \lim_{j \to \infty} \int_{\partial \overline{D}_f} G \, dv$$

$$= \lim_{j \to \infty} [2\pi \chi(\overline{D}_f) - (\pi - \Psi_j) - (\pi - \Psi_j) - (\pi - \Psi_j) - \sum_{\overline{D}_f} (\pi - \theta^{\partial \overline{D}_f})$$

$$\geq \Psi_j^+ + \Psi_j^- - \pi - \sum_{\overline{D}_f} (\pi - \theta^{\partial \overline{D}_f}),$$

where $\Psi_j$ and $\Psi_j$ are the vertical angles of $\partial \overline{D}_f$ at $\partial V_1 \cap \gamma_1$ and $\partial V_i \cap \gamma_2$, respectively.  And $\theta^{\partial \overline{D}_f}$ is a vertical angle of $\partial V_i \cap \partial \overline{D}_f$ measured in $\partial \overline{D}_f$ and $\varphi_j = \theta^i(\gamma_j(t_j), \gamma_j(t_j))$, where $t_j$ is the distance from $p$ to $q_j$.  Thus the inequality is verified.

Next, consider the case that the Gaussian curvature of $M$ is nonpositive.  Let $p_j : = \partial V_i \cap \gamma_j$ and $r_j : = \partial V_i \cap \gamma_j$.  And let $(p_j, q_j, r_j)$ be the geodesic triangle determined by the three shortest geodesic segments.  Let $c : [0,1] \to M$ be the shortest geodesic segment with $c(0) = p_j$ and $c(1) = r_j$.  Since $(p_j, q_j, r_j)$ is contractible, we can consider the homotopy $H : [0,1] \times [0,1] \to M$ such that for any $s \in [0,1], \ H(0,s) = c(s), \ H(1,s) = q_j$ and $H([0,1],s) = \text{the shortest geodesic segment between } c(s) \text{ and } q_j$.  Let $(\tilde{p}_j, \tilde{q}_j, \tilde{r}_j)$ be a lift of $(p_j, q_j, r_j)$ in the universal Riemannian covering space $\tilde{M}$ of $M$ which is gotten by making use of the homotopy $H$ and let $\tilde{\gamma}_j$ be the vertical angle of $(\tilde{p}_j, \tilde{q}_j, \tilde{r}_j)$ at $\tilde{q}_j$.  Then from the construction we have $\varphi_j = \tilde{\gamma}_j$ and it is seen that $\varphi_j(\tilde{p}_j, \tilde{q}_j) \to \infty, \varphi_j(\tilde{r}_j, \tilde{q}_j) \to \infty$ and $\varphi_j(\tilde{p}_j, \tilde{r}_j) = C$ as $j \to \infty$, where $C$ is a constant.
Hence making use of the law of cosines, we can see that \( \varphi_j \to 0 \) as \( j \to \infty \). Therefore the equality holds when Gaussian curvature of \( M \) is nonpositive.

Hereafter let \( D^1 \) and \( D^2 \) be connected components of \( D(p) \setminus V \) and \( D(p) \cap V \), respectively. And let \( F^{1*} \) be a connected component of \( (F(p) \cap V) \setminus \{p\} \) and set \( F^2 := F^{1*} \cup \{p\} \). Then we can get the following Proposition which implies our theorem.

**Proposition 5.** The following inequality holds good;

\[
\text{measure } A(p) \geq 2\pi \chi(M) - \int_{\partial V \setminus \{p\}} G \, dv
\]

at any point \( p \) of \( M \). And the equality holds when Gaussian curvature of \( M \) is nonpositive.

**Proof.** Let \( F^1 \) be the one such that \( \text{int} \, F^1 \neq \emptyset \). Since \( E^1 \) is diffeomorphic to a polygon in \( T_p M \), we have

\[
\int_{E^1} G \, dv = a^* + (\Psi^1)^* + (\Psi^1)^{-} - \pi + \sum_{a \in \partial E^1} (\pi - \theta^a) \quad \cdots (\ast)
\]

where \( a^* \) is a vertical of \( E^1 \) at \( p \), \( (\Psi^1)^* \) and \( (\Psi^1)^{-} \) are the vertical angles of \( E^1 \) formed with \( \partial V \), and the rays which are the boundaries of \( F^1 \) and \( \theta^a \) is a vertical angle of \( \partial V \cap F^1 \) measured in \( M \setminus V \). From our construction, there is no vertex of \( \partial V \) on the rays which are the boundaries of \( F^2 \)'s. Hence we can get the following inequality by using (\ast), Lemma 4 and the fact that vertically opposite angles are identical;

\[
\int_{V^1} G \, dv = \sum_{a \in \partial E^1} \int_{E^1} G \, dv + \sum_{a \in \partial E^2} \int_{E^2} G \, dv
\]

\[
= \sum_{a \in \partial E^1} \left[ a^* + (\Psi^1)^* + (\Psi^1)^{-} - \pi + \sum_{a \in \partial E^1} (\pi - \theta^a) \right] + \sum_{a \in \partial E^2} \int_{E^2} G \, dv
\]

\[
\leq \text{measure } A(p) + \sum_{a \in \partial V \setminus \{p\}} (\pi - \theta^a) + \int_{\partial V \setminus \{p\}} G \, dv.
\]

On the other hand, we have

\[
\int_{V^1} G \, dv = 2\pi \chi(M) + \sum_{a \in \partial V} (\pi - \theta^a).
\]

Hence we get the desired inequality and the equality holds when Gaussian curvature of \( M \) is nonpositive.

Now Theorem 1 and 2 are the direct consequences of Proposition 5.
References


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