CLASS GROUPS OF GROUP RINGS WHOSE COEFFICIENTS ARE ALGEBRAIC INTEGERS

By

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Let $R$ be the ring of integers of an algebraic number field $k$. Let $A$ be an $R$-order in a finite dimensional semisimple $k$-algebra $A$. We mean by the class group of $A$ the class group defined by using locally free left $A$-modules and denote it by $C(A)$. We define $D(A)$ to be the kernel of the natural surjection $C(A)\to C(Q)$, where $Q$ is a maximal $R$-order in $A$ containing $A$, and denote by $d(A)$ the order of $D(A)$. $C(Q)$ is isomorphic to a (narrow) ideal class group of the center of $A$, which is a product of the ideal class groups of algebraic number fields with modulus some real infinite primes. Hence, in a sense, we may concentrate on $D(A)$.

Let $G$ be a finite group and let $RG$ be the group ring of $G$ with coefficients in $R$. Then $RG$ can be regarded as an $R$-order in the semisimple $k$-algebra $kG$. We define $T(RG)$ to be the kernel of the natural surjection $C(RG)\to G(R)\oplus C(RG/(\Sigma_\alpha))$, where $\Sigma_\alpha=\sum_{g\in G} g\in RG$, and denote by $t(RG)$ the order of $T(RG)$. Then $T(RG)\cong \text{Ker}(D(RG)\to D(RG/(\Sigma_\alpha)))$. Throughout this paper, $C_n$ denotes the cyclic group of order $n$ and $p$ stands for a rational prime.

Much investigation has been done on $D(ZG)$ and $T(ZG)$ (cf. [8]), but the results seem to depend on the speciality of $Z$.

The purpose of this paper is to study $D(RG)$ for the case where $R\neq Z$. In §1 we give some basic results on $D(RG)$ and $T(RG)$. In §2~§4 we assume that $R$ is the ring of integers in a quadratic field. We first give some results on $D(RC_p)$, and next examine the structure of $D(RC_p)$.

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§1.

For a ring $S$, $U(S)$ denotes its unit group. For an abelian group $A$ and a positive integer $q$, $A^{(q)}$ denotes the $q$-part of $A$ and $A^{(q)}$ denotes the maximal
subgroup of $A$ whose order is coprime to $q$. In the case where $G = C_n$, we
denote $\Sigma_n$ instead of $\Sigma^n$. Let $k$ be an algebraic number field and let $R$ be the
ring of integers of $k$. Let $\Phi_n(X)$ be the cyclotomic polynomial of degree $n$.
Write $R[X]/(\Phi_n(X)) = R[\zeta_n]$ (resp. $k[X]/(\Phi_n(X)) = k[\zeta_n]$) where $\zeta_n$
denotes the class of $X$ in $R[X]/(\Phi_n(X))$ (resp. $k[X]/(\Phi_n(X))$).

**Proposition 1.1.** $d(R_{\chi_p}) = |T(R_{\chi_p})|^{e \cdot p^{r(e)} \prod_{i=1}^s d(R[\zeta_p])}$ for some integer $f(e) \geq 0.$

**Proof.** Let $e \geq 1$. From the pullback diagrams

$$
\begin{array}{ccc}
R_{\chi_{p+1}} & \longrightarrow & R[X]/(\Phi_{p+1}(X)) = R[\zeta_{p+1}] \\
\downarrow & & \downarrow \\
R_{\chi_p} & \longrightarrow & (R/pR)_{\chi_p}
\end{array}
$$

we have an exact sequence

$$
0 \longrightarrow K \longrightarrow D(R_{\chi_{p+1}}) \longrightarrow D(R_{\chi_p}) \oplus D(R[\zeta_{p+1}]) \longrightarrow 0
$$

and a commutative diagram with exact rows

$$
\begin{array}{ccc}
U(R_{\chi_p}) \oplus U(R[\zeta_{p+1}]) & \longrightarrow & U((R/pR)_{\chi_p}) \longrightarrow K \longrightarrow 0 \\
\downarrow \phi' & & \downarrow \phi \\
U(R) \oplus U(R[\zeta_p]) & \longrightarrow & U(R/pR) \longrightarrow T(R_{\chi_p}) \longrightarrow 0,
\end{array}
$$

where the vertical maps are induced by the norm maps. Since $\phi$ is bijective on
the $p'$-parts and Coker $\phi'$ is a $p$-group, we see that $K^{(p')} \cong T(R_{\chi_p})^{(p')}$. Hence,
by induction on $e$, we have the equality as desired.

**Corollary 1.2.** Suppose that $p$ is unramified in $R$. Then

i) $D(R_{\chi_p}) (= T(R_{\chi_p}))$ is a $p'$-group.

ii) If $d(R_{\chi_p}) = 1$, then $D(RP)$ is a $p$-group for every $p$-group $P$.

**Proof.** i) Since $p$ is unramified in $R$, $U(R/pR)$ is a $p'$-group and $R[\zeta_{p,i}]$
is a Dedekind domain for every $i \geq 1$. The assertion follows from these facts.

ii) If $d(R_{\chi_p}) = 1$, then $D(R_{\chi_p})$ is a $p$-group by (1.1). Then, by the induction
theorem of Artin ([1, §1]), we see that $D(RP)$ is a $p$-group for every $p$-group $P$.

**Proposition 1.3.** i) $T(R_{\chi_n}) \cong \bigoplus_{p \mid n} T(R_{\chi_{p^n}})$ where $p \nmid n$ for each $p | n$.

ii) There is an exact sequence

$$
0 \longrightarrow P_e \longrightarrow T(R_{\chi_p}) \longrightarrow T(R_{\chi_p}) \longrightarrow 0,
$$
where $P_e$ is a $p$-group whose exponent divides $p^{e-1}$ (resp. $p^e$) if $p$ is unramified in $R$ (resp. ramified in $R$).

iii) Let $G$ be a finite group of order $n$. If $p | t(RG)$, then $p | n$ or $p | t(RC_p)$ for some prime factor $q$ of $n$.

**Proof.**

i) Let $\mathcal{M} = R \oplus \mathcal{O}$ be a maximal $R$-order in $kC_n = k \oplus kC_n/(\Sigma_n)$ containing $RC_n$. By ([2, Theorem 1]), we have

$$D(RC_n) \cong \prod_{p \mid n} U(\mathcal{O}_p) / U(\mathcal{M}) \prod_{p \mid n} U(R_pC_n),$$

where $\mathcal{M}_p = Z_p \otimes \mathcal{M}$ and $R_p = Z_p \otimes R$. Since $R_p$ can be embedded in $\mathcal{M}_p$, the map $U(\mathcal{M}_p) = U(R_p) \times U(\mathcal{M}_p) \rightarrow U(\mathcal{M}_p)$; $(x, y) \mapsto yx^{-1}$, induces an isomorphism

$$D(RC_n) \cong \prod_{p \mid n} U(\mathcal{O}_p) / U(\mathcal{M}) \prod_{p \mid n} u(R_pC_n),$$

where $u(R_pC_n) = \{ x | (1, x) \in U(R_pC_n) \subseteq U(R_p) \times U(R_pC_n/(\Sigma_n)) \}$. On the other hand, we have

$$D(RC_n/(\Sigma_n)) \cong \prod_{p \mid n} U(\mathcal{M}_p) / U(\mathcal{M}) \prod_{p \mid n} U(R_pC_n/(\Sigma_n)).$$

Hence we get

$$T(RC_n) \cong \frac{U(\mathcal{O}_p) \prod_{p \mid n} U(R_pC_n/(\Sigma_n))}{U(\mathcal{O}_p) \prod_{p \mid n} u(R_pC_n)}.$$
The natural surjection $C_{p^e} \rightarrow C_{p^e}/C_{p^{e-1}} \cong \mathbb{Z}_p$ induces the surjection $T(\mathcal{R}C_{p^e}) \rightarrow T(\mathcal{R}C_p)$; $(x, y) \rightarrow x$ where $(x, y) \in U(\mathcal{O}_1, p)(\bigoplus_{i=0}^{p-2} \mathcal{O}_i, p)$. Set $P_e = \text{Ker}(T(\mathcal{R}C_{p^e}) \rightarrow T(\mathcal{R}C_p))$. Each $\alpha \in P_e$ is represented by an element $(x, y) \in U(\mathcal{R}C_{p^e}/(\Sigma_{p^e}))$ such that $x = uv$ for some $u \in U(\mathcal{O}_1)$ and $(1, v) \in U(\mathcal{R}C_p)$. Let $f(\delta) = \sum_{i=0}^{p-2} b_i \delta^i = (x, y) \in U(\mathcal{R}C_{p^e}/(\Sigma_{p^e}))$, where $b_i \in \mathcal{R}_p$ and $\delta$ denotes the image of a generator $\sigma$ of $C_{p^e}$ in $\mathcal{R}_p C_{p^e}/(\Sigma_{p^e})$, and let $f(a) = \sum_{i=0}^{p-2} b_i \sigma^i \in \mathcal{R}_p C_{p^e}$. Then $x = f(\zeta_p) = \sum_{i=0}^{p-2} b_i \equiv uv \equiv u \pmod{\zeta_p - 1} \mathcal{O}_1, p)$, and so we see that $f(1) = \sum_{i=0}^{p-2} b_i \in U(\mathcal{R}_p)$. Hence $f(\alpha) \in U(\mathcal{R}_p C_{p^e})$. Then $\alpha = (x, y) = (f(1), f(1))$, because $(x^{-1}f(1), y^{-1}f(1)) \in u(\mathcal{R}_p C_{p^e})$.

Thus we know that

$$P_e \subseteq N = \left\{ \rho_x = (x, x) \in T(\mathcal{R}C_{p^e}) \mid x \in U(\mathcal{R}_p), x \equiv u \pmod{\zeta_p - 1} \mathcal{O}_1, p) \right\}.$$ 

It is easily verified that

$$\rho^e R \oplus \rho^{e-1}(\zeta_p - 1)R[\zeta_p] \oplus \rho^{e-2}(\zeta_p - 1)R[\zeta_p, p] \oplus \cdots \oplus (\zeta_p - 1)R[\zeta_p, p^{e-1}] \subseteq T(\mathcal{R}C_{p^e}).$$

Let $\rho_x \in N$ and $x \equiv u \pmod{\zeta_p - 1} \mathcal{O}_1, p)$, $u \in U(\mathcal{O}_1)$. If $p$ is unramified in $\mathcal{R}$, then $\mathcal{O}_1 = \mathcal{R}[\zeta_p^{e-1}]$ and $u^{-1}x \equiv 1 + (\zeta_p - 1) \mathcal{O}_1, p)$, and hence $(u^{-1}x)^{p^{e-1}} \equiv 1 + (\zeta_p - 1) \mathcal{O}_1, p)$.

By force of (*), we know that $\rho_x^{p^{e-1}} \equiv 1 \in T(\mathcal{R}C_{p^e})$. Thus we see that $\exp(P_x) \mid p^{e-1}$.

Even if $p$ is ramified in $\mathcal{R}$, $(u^{-1}x)^p \equiv 1 + (\zeta_p - 1) \mathcal{R}_p[\zeta_p]$, and so we have $\exp(P_x) \mid p^e$.

iii) By the induction theorem of Artin ([1, § 1]), we have that $T(\mathcal{R}G) \cong \sum_{\mathcal{O}} T(\mathcal{R}G(a))$, where $C$ ranges over all cyclic subgroups of $G$. The result follows from i) and ii).

**Remark 1.4.** By force of (*) above, if $p$ is unramified in $\mathcal{R}$, we can see that the exponent of $D(\mathcal{R}C_{p^e})$ divides $p^{e-1}$. Further assume that $\mathcal{R}$ is the ring of integers of a real algebraic number field $k$ and $p \geq 5$. Then $\exp(D(\mathcal{R}C_{p^e})) = p^{e-1}$.

In fact, let $\tau$ denote the endomorphism of $\mathcal{R}C_{p^e}$ induced by $\sigma \mapsto \sigma^{-1}$, where $C_{p^e} = \langle \sigma \rangle$. Then $D(\mathcal{R}C_{p^e})$ can be regarded as a $\langle \tau \rangle$-module. For every $\langle \tau \rangle$-module $M$, we put $M^e = \{ m \in M \mid m^e = m \}$.

Let $V$ be the kernel of the natural surjection $D(\mathcal{R}C_{p^{e+1}}) \rightarrow D(\mathcal{R}C_{p^e})$. Then, along the almost same line as in ([4]), we can show that $V^{\cong} = \bigoplus_{a=1}^{e} (Z/p^a Z)^{v_a}$, where $v_a = (1/2)[k : Q](p-1)^2 p^{e-a-1}$ for $a \leq e$ and $v_e = (1/2)[k : Q](p-1)-g$, $g$ is the number of prime ideals in $\mathcal{R}$ over $p$.

**Proposition 1.5.** Suppose that $p$ is unramified in $\mathcal{R}$. Then
Class Groups of Group Rings Whose Coefficients

\[ D(RC_{p^i})^{(p^i)} \cong D(RC_p)^e \quad \text{(direct sum)}. \]

**Proof.** Let \( \mathcal{O}_i = R[\zeta_{p^i}], 1 \leq i \leq e \). Then \( \mathcal{O}_i \) is a Dedekind domain and \( \bigoplus_{i=1}^e \mathcal{O}_i \) is a maximal \( R \)-order in \( kC_{p^i}/(\Sigma_{p^i}) \) containing \( RC_{p^i}/(\Sigma_{p^i}) \), and the product \( p_i \) of all prime ideals over \( p \) in \( \mathcal{O}_i \) equals \( (1 - \zeta_{p^i}) \), \( 1 \leq i \leq e \). Hence we get

\[
D(RC_{p^i}) \cong \prod_{i=1}^e U(\mathcal{O}_i, p) / \prod_{i=1}^e U(\mathcal{O}_i) u(R, C_{p^i})
\]

\[
\cong \left[ \prod_{i=1}^e \frac{U(R/pR)}{\varphi_i(U(\mathcal{O}_i))} \right] \times \left[ \prod_{i=1}^e \frac{1 + p_i \mathcal{O}_i, p}{\prod_{i=1}^e U(\mathcal{O}_i) u(R, C_{p^i})} \right],
\]

where \( \varphi_i \) is induced by the natural surjection \( \mathcal{O}_i - \mathcal{O}_i / p_i \cong R/pR \) and \( \mathcal{U}(\mathcal{O}_i) = \text{Ker} \varphi_i = U(\mathcal{O}_i) \cap (1 + p_i \mathcal{O}_i, p) \). Then it is easily seen that the former factor is isomorphic to \( D(RC_{p^i})^{(p^i)} \). On the other hand, \( |D(RC_{p^i})^{(p^i)}| = d(RC_p)^e \) by (1.1) and (1.2), and so we have

\[
U(R/pR) / \varphi_i(U(\mathcal{O}_i)) \cong D(RC_p), \quad 1 \leq i \leq e.
\]

Thus we complete the proof.

§ 2.

Hereafter, let \( k \) denote \( \mathbb{Q}(\sqrt{m}) \), a quadratic field, where \( m \) is a square-free integer, and \( R \) be the ring of integers of \( k \). We write \( w_m = \sqrt{m} \) (resp. \( \sqrt{m} + 1/2 \)) if \( m \equiv 1 \pmod{4} \) (resp. \( m \equiv 1 \pmod{4} \)).

Let \( \mathcal{O}_i \) be the maximal \( R \)-order in \( k[\zeta_{p^i}] \) and \( p_i \) be the product of all the prime ideals over \( p \) in \( \mathcal{O}_i \), \( 1 \leq i \leq e \). Then

\[
D(RC_{p^i}) \cong \prod_{i=1}^e U(\mathcal{O}_i, p) / \prod_{i=1}^e U(\mathcal{O}_i) u(R, C_{p^i})
\]

\[
\cong \left[ \prod_{i=1}^e \frac{U(\mathcal{O}_i / p_i)}{\varphi_i(U(\mathcal{O}_i))} \right] \times \left[ \prod_{i=1}^e \frac{1 + p_i \mathcal{O}_i, p}{\prod_{i=1}^e U(\mathcal{O}_i) u(R, C_{p^i})} \right],
\]

where \( \varphi_i : U(\mathcal{O}_i) ightarrow U(\mathcal{O}_i / p_i) \) is the natural map and \( \mathcal{U}(\mathcal{O}_i) = \text{Ker} \varphi_i, 1 \leq i \leq e \). It is easily seen that the latter factor is isomorphic to \( D(RC_{p^i})^{(p^i)} \).

**Proposition 2.1.** Let \( p \) be unramified in \( R \), i.e. \( p \nmid m \) if \( p \equiv 2 \) and \( m \equiv 1 \pmod{4} \) if \( p = 2 \). Then

\[
\exp(D(RC_{p^i})^{(p^i)}) \mid p^{e-1} \quad \text{and} \quad D(RC_{p^i})^{(p^i)} \cong D(RC_p)^e.
\]

**Proof.** This is a special case of (1.4) and (1.5).
We write \( p^* = (-1)^{p-1/p} p \).

**Proposition 2.2.** Let \( p|m \) and \( m \equiv p^* \) if \( p \neq 2 \), and let \( m \equiv 1 \pmod{4} \) and \( m \equiv -1, \pm 2 \) if \( p = 2 \). Then

i) The exponent of \( D(RC_p)^{(p^*)} \) divides

\[
\begin{cases}
2^{e+1} & \text{if } p=2, m \equiv 2 \pmod{4} \text{ and } e > 1, \text{ or } \\
p^* & \text{otherwise}.
\end{cases}
\]

ii) For the case \( p \neq 2 \) and \( m = np^* \),

\[
D(RC_p)^{(p^*)} \equiv D(R'p)^e \quad \text{where } R' = \mathbb{Z}[w_n].
\]

iii) For the case \( p = 2 \) and \( m = -n \) where \( n \equiv 1 \pmod{4} \),

\[
D(RC_2)^{(p^*)} \equiv D(R'C_2)^{e-1} \quad \text{where } R' = \mathbb{Z}[w_n].
\]

iv) For the case \( p = 2 \) and \( m = 2n \) or \(-2n \) where \( n \equiv 1 \pmod{4} \),

\[
D(RC_2)^{(p^*)} \equiv \begin{cases}
0 & \text{if } e = 1, 2 \\
D(R'C_2)^{e-2} & \text{if } e \geq 3,
\end{cases}
\]

where \( R' = \mathbb{Z}[w_n] \).

**Proof.** If \( p \neq 2 \) and \( m = np^* \), then we see that \( O_i = \mathbb{Z}[w_n, \zeta_{p^i}] \), \( p_i = (1 - \zeta_{p^i}) \) and \( p_i R \subseteq R[\zeta_{p^i}], 1 \leq i \leq e \). Hence we get that \( \exp(D(RC_p)^{(p^*)}) | p^e \) and \( D(RC_p)^{(p^*)} \equiv D(R'C_p)^{p^e} \equiv D(R'C_p)^{p^e} \), where \( R' = \mathbb{Z}[w_n] \).

If \( p = 2 \), \( m = -n \) and \( n \equiv 1 \pmod{4} \), then we see that \( O_i = R \) and \( O_i = \mathbb{Z}[w_n, \zeta_{2^i}] \) for \( i \geq 2 \). Then, it is easy to see that \( \exp(D(RC_2)^{(p^*)}) | 2^e \) and \( D(RC_2) = 0 \). For \( e \geq 2 \), we have

\[
D(RC_2)^{(p^*)} \equiv D(R'C_2)^{p^e},
\]

where \( R' = \mathbb{Z}[w_n] \), and so, by (2.1),

\[
D(RC_2)^{(p^*)} \equiv D(R'C_2)^{p^e-1}.
\]

If \( p = 2 \), \( m = 2n \) or \(-2n \) and \( n \equiv 1 \pmod{4} \), then \( O_i = R \), \( O_i = \mathbb{Z}[\sqrt{m}, \sqrt{-1}, \sqrt{m+\sqrt{-1}/2}] \) and \( O_i = \mathbb{Z}[w_n, \zeta_{2^i}] \) for \( i \geq 3 \). The assertion can be shown similarly for this case.

**Proposition 2.3.** Let \( m = p^* \) if \( p \neq 2 \) and let \( m = -1, \pm 2 \) if \( p = 2 \). Then the exponent of \( D(RC_p) \) divides \( p^e \). Especially, it divides \( p^{e-1} \) if \( p = 3, 5 \) or \( p = 2 \) and \( m = -1 \).

**Proof.** Put \( O_i = \mathbb{Z}[\zeta_{p^i}], 1 \leq i \leq e \). If \( p \neq 2 \), then \( R \bigoplus \bigoplus_{i=1}^e (O_i \oplus O_i) \) is a maximal \( R \)-order in \( kC_{p^e} \) containing \( RC_{p^e} \), and so we have
Let $R$ be the ring of integers of $k=\mathbb{Q}(\sqrt{m})$. In the case $m>0$, we denote a fundamental unit of $R$ by $\varepsilon_m$. $\varepsilon_m$ can be written as $a+b\sqrt{m}$, $a, b \in \mathbb{Z}$ or $(a+b\sqrt{m})/2$, $a, b \in \mathbb{Z}$, $2 \nmid ab$.

Here we investigate $D(RC_p)$ more precisely.

There is an exact sequence

$$0 \longrightarrow D(RC_p) \longrightarrow C(RC_p) \longrightarrow C(R) \oplus C(R) \longrightarrow 0$$

and $T(RC_p)=D(RC_p)$. Further we have easily

**Proposition 3.1.**

<table>
<thead>
<tr>
<th>$D(RC_p)$</th>
<th>$m&lt;0$</th>
<th>$m&gt;0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}/3\mathbb{Z}$</td>
<td>$m \equiv 5 \pmod{8}$ and $m&lt;-3$</td>
<td>$m \equiv 5 \pmod{8}$ and $\varepsilon_m \in \mathbb{Z}[\sqrt{m}]$</td>
</tr>
<tr>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
<td>$m \equiv 2$ or $3 \pmod{4}$ and $m&lt;-1$</td>
<td>$m \equiv 2$ or $3 \pmod{4}$ and $2 \not</td>
</tr>
<tr>
<td>$0$</td>
<td>$m \equiv 1 \pmod{8}$, $m=-1$ or $-3$</td>
<td>$m \equiv 1 \pmod{8}$, $m=5$ (mod 8) and $\varepsilon_m \notin \mathbb{Z}[\sqrt{m}]$ or $m=2$ or $3$ (mod 4) and $2 \not</td>
</tr>
</tbody>
</table>

From now on, $p$ is assumed to be an odd prime. From the pullback diagram
we have exact sequences

\[
\begin{align*}
\psi : U(R[\sigma]) &\longrightarrow U(F_p[\sqrt{m}]) \\
\xi &\longrightarrow T(RC_p) \\
0 &\longrightarrow D(RC_p) \\
&\longrightarrow D(R[\sigma]) \\
&\longrightarrow 0.
\end{align*}
\]

Here \(\psi\) is the restriction of the canonical surjection \(\tilde{\psi} : R[\sigma] \to R[\sigma]/(\sigma - 1)\),
\(\xi(\tilde{\psi}(x)) = \text{the class of the ideal } (x, \Sigma_p)\) and

\[
F_p[\sqrt{m}] = \begin{cases} 
F_p \oplus F_p & \text{if } \left(\frac{m}{p}\right) = 1 \\
F_p^2 & \text{if } \left(\frac{m}{p}\right) = -1,
\end{cases}
\]

where \(\left(\frac{m}{p}\right)\) is the quadratic residue symbol.

Let \(p \nmid m\) and let \(r\) be an element of \(R[\sigma] = R[\zeta_p]\) such that

i) if \(\left(\frac{m}{p}\right) = 1\), then \(\phi(r) = (a, 1) \in U(F_p) \oplus U(F_p)\),
where \(a\) is a generator of \(U(F_p)\),

ii) if \(\left(\frac{m}{p}\right) = -1\), then \(\phi(r)\) is a generator of \(U(F_p)\).

Noticing that \(\phi(U(Z[\zeta_p])) = U(F_p)\), we have

**Lemma 3.2.** In the case \(p \nmid m\), \(D(RC_p) = T(RC_p)\) is a cyclic group generated by the class of \((r, \Sigma_p)\), where \(r\) is given as above. Its order divides \(p - 1\) (resp. \(p + 1\)) if \(\left(\frac{m}{p}\right) = 1\) (resp. \(\left(\frac{m}{p}\right) = -1\)).

For an imaginary abelian field \(K\), let \(K_0\) be the maximal real subfield of \(K\). Denote by \(U\) (resp. \(U_0\)) the group of units in the ring of integers of \(K\) (resp. \(K_0\)) and denote by \(W\) the group of roots of unity contained in \(K\). Then the unit index \(Q_K\) of \(K\) is defined by the index \([U : WU_0]\). It is known that \(Q_K = 1\) or 2. (cf. [3, §20-26])

Assume that \(p \nmid m\) and \(m < 0\). Let \(K = Q(\zeta_p, \sqrt{m})\) and \(K_1 = Q(\zeta_p + \zeta_p^{-1})\). Let \(\text{Gal}(K/Q) = \langle \sigma, \tau | \sigma^{p-1} = \tau^2 - 1, \sigma \tau = \tau \sigma \rangle\), \(\text{Gal}(K/K_0) = \langle \sigma^{p-1/2}, \tau \rangle\) and \(\text{Gal}(K/K_1) = \langle \sigma^{p-1/2}, \tau \rangle\). The characters of \(K\) are given as follows:

i) the characters of \(K/K_0\);
ii) the characters of $K_0$

\[
\begin{aligned}
&\sigma \mapsto \zeta_p^{i-1}, \\
&\tau \mapsto 1, \quad 1 \leq i \leq p-1 \text{ and } 2 \not| i.
\end{aligned}
\]

\[
\begin{aligned}
&\sigma \mapsto \zeta_p^{i-1/2}, \\
&\tau \mapsto -1, \quad 1 \leq j \leq \frac{p-1}{2}.
\end{aligned}
\]

Then we see that $K/K_0$ is unramified at $p$. Since we can compute the absolute discriminants of $K_1$ and $K_0$, we see that the discriminant $d_{K/K_1} = (\pi^m)$, where $\pi = \zeta_p - \zeta_p^{-1}$ and $m = \begin{cases} m & \text{if } m \equiv 1 \pmod{4} \\
\frac{4m}{m} & \text{otherwise} \end{cases}$.

Thus, $(p)$ is totally ramified in $K_0/Q$, and so there is a unique prime ideal $\mathfrak{P}$ over $(p)$ in $K_0$. It is easy to see that $\mathfrak{P} = (\pi^2, \sqrt{m})$.

**Proposition 3.3.** Assume that $p \not| m$ and $m < 0$. Let $K = \mathbb{Q}(\zeta_p, \sqrt{m})$ and let $\mathcal{O}$ be the ring of integers of $K$. Then the following conditions are equivalent.

i) $Q_K = 2$.

ii) $\mathfrak{P}$ is a principal ideal in $K_0$.

iii) There exists a unit of $\mathcal{O}$ of the form $(\pi x + \sqrt{my})/2$, where $x, y \in \mathbb{Z} [\zeta_p + \zeta_p^{-1}]$.

**Proof.** It is easy to see that (i) is equivalent to

i') $K = K_0(\sqrt{\varepsilon})$ for some unit $\varepsilon$ of the ring $\mathcal{O}_0$ of integers of $K_0$.

On the other hand, we have

\[
K = K_0(\zeta_p) = K_0(\sqrt{\zeta_p - \zeta_p^{-1}}) \quad \text{and} \quad \langle (\zeta_p - \zeta_p^{-1})^2 \rangle = \mathfrak{P}^2 \quad \text{as ideals in } \mathcal{O}_0.
\]

Thus we get the equivalence between i) and ii). Let $K_1 = \mathbb{Q}(\zeta_p + \zeta_p^{-1})$ and $\mathcal{O}_1$ be the ring of integers of $K_1$. Then $K_0 = K_1(\pi \sqrt{m})$, and so $\mathcal{O}_1$ has an integral basis with respect to $\mathcal{O}_0$, since $d_{K_0/K_1} = (\pi^2)$ ([6]). As the discriminant of $\pi \sqrt{m}$ with respect to $K_0/K_1$ is equal to $4\pi^2 m$, we see that $[\mathcal{O}_0 : \mathcal{O}_1[\pi \sqrt{m}]] = 1$ or 2. Thus every element of $\mathcal{O}_0$ can be written uniquely with the form...
Assume the condition ii). Let $\alpha = (x' + \pi \sqrt{m}y)/2$, $x'$, $y \in \mathcal{O}$, be a generator of $\mathcal{O}$. Since $\mathcal{O} = \pi \mathcal{O}$, $\pi$ must divide $x'$, and so we have $\pi | x'$. Thus $\alpha$ can be written as $(\pi x + \sqrt{m}y)/2$. Then it is clear that $(\pi x + \sqrt{m}y)/2$ is a unit of $\mathcal{O}$, hence we have iii). Conversely, if there exists such a unit $\varepsilon$ in $\mathcal{O}$, then $\pi \varepsilon$ is an element of $K_0$ and generates $\mathcal{P}$. This completes the proof.

**Theorem 3.4.** Assume that $p \not \equiv m$ and $m < 0$. Let $K = \mathbb{Q}(\sqrt{m})$. Then

i) The case where $m \neq -1$, $-3$:

$$d(RC_p) = \begin{cases} 
\frac{p - 1}{2} & \text{if } \left(\frac{m}{p}\right) = 1 \text{ and } Q_K = 1 \\
\frac{p + 1}{2} & \text{if } \left(\frac{m}{p}\right) = -1 \text{ and } Q_K = 2 \\
\frac{p + 1}{2} & \text{if } \left(\frac{m}{p}\right) = -1 \text{ and } Q_K = 2 .
\end{cases}$$

ii) The case where $m = -1$:

$$d(RC_p) = \begin{cases} 
\frac{p - 1}{2} & \text{if } p \equiv 1 \pmod{4} \\
\frac{p + 1}{4} & \text{if } p \equiv 3 \pmod{4}.
\end{cases}$$

iii) The case where $m = -3$:

$$d(RC_p) = \begin{cases} 
\frac{p - 1}{3} & \text{if } p \equiv 1 \pmod{3} \\
\frac{p + 1}{6} & \text{if } p \equiv -1 \pmod{3}.
\end{cases}$$

**Proof.** There is an exact sequence

$$U(\mathcal{O}) \xrightarrow{\phi} U(\mathbb{F}_p[\sqrt{m}]) \rightarrow D(RC_p) \rightarrow 0 .$$

Since $\langle p \rangle$ is totally ramified in $K_0/\mathbb{Q}$, we see that $\phi(U(\mathcal{O})) \subseteq U(\mathbb{F}_p)$. On the other hand, we have $\phi(U(\mathcal{O})) = U(\mathbb{F}_p)$. Let $m \neq -1$, $-3$. If $Q_K = 1$, then $U(\mathcal{O}) = \langle \zeta_p \rangle U(\mathcal{O})$. Thus we have

$$d(RC_p) = \begin{cases} 
p - 1 & \text{if } \left(\frac{m}{p}\right) = 1 \\
p + 1 & \text{if } \left(\frac{m}{p}\right) = -1.
\end{cases}$$
If $Q_K=2$, then $U(\mathcal{O})=\langle \zeta_p, \varepsilon=(\pi x+my)/2 \rangle U(\mathcal{O}_0)$, for some $\varepsilon \in U(\mathcal{O})$ by (3.3). $\phi(\varepsilon) \in U(F_p)$ and $\varepsilon^4 \in \langle \zeta_p \rangle U(\mathcal{O}_0)$, hence we see that

$$d(RC_p) = \begin{cases} \frac{p-1}{2} & \text{if } (\frac{m}{p})=1 \\ \frac{p+1}{2} & \text{if } (\frac{m}{p})=-1. \end{cases}$$

Let $m=-1$. Then $K=\mathbb{Q}(\zeta_p, \zeta_3)$ and $U(\mathcal{O})=\langle \zeta_p, \sqrt{-1}, \varepsilon=1-\sqrt{-1} \zeta_p \rangle U(\mathcal{O}_0)$. $\phi(\varepsilon)$ is of order 4 in $U(F_p[\sqrt{-1}])/U(F_p)$. Thus we see that

$$d(RC_p) = \begin{cases} \frac{p-1}{4} & \text{if } p \equiv 1 \pmod{4} \\ \frac{p+1}{4} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Let $m=-3$. Then $K=\mathbb{Q}(\zeta_p, \zeta_3)$ and $U(\mathcal{O})=\langle \zeta_p, \zeta_3, \varepsilon=1-\zeta_3 \zeta_p \rangle U(\mathcal{O}_0)$. $\phi(\varepsilon)$ is of order 6 in $U(F_p[\sqrt{-3}])/U(F_p)$. Since $\left(\frac{-3}{p}\right)=\left(\frac{p}{3}\right)$, we see that

$$d(RC_p) = \begin{cases} \frac{p-1}{6} & \text{if } p \equiv 1 \pmod{3} \\ \frac{p+1}{6} & \text{if } p \equiv -1 \pmod{3}. \end{cases}$$

**Remark 3.5.** 1) Assume that $p \equiv 3 \pmod{4}$ and $m \neq -1, -3$. Let $M=\mathbb{Q}(\sqrt{-p}, \sqrt{m})$ and let $\varepsilon>0$ be a fundamental unit of $M_0=\mathbb{Q}(\sqrt{m})$. Then the following conditions are equivalent.

i) $Q_K=2$  ii) $Q_m=2$  iii) $\sqrt{-\varepsilon} \in M$.

2) Assume that $p \equiv 1 \pmod{4}$ and $m=-q$, where $q$ is a prime and $q \equiv 3 \pmod{4}$. Then $Q_K=2$.

**Proof.** 1) The equivalence between ii) and iii) is clear. By [3, Satz 29], ii) implies i). Let $p$ be the unique prime ideal over $(p)$ in $M_0$. Then it is easy to see that ii) is equivalent to the condition that $p$ is principal. If $Q_K=2$, then we can take a generator $\alpha$ of $\mathcal{O}$. Since $N_{K/\mathbb{Q}}(\alpha) = p$, we see that $N_{K/\mathbb{Q}}(\alpha)$ generates $p$. This establishes 1). 2) We may assume that $q \neq 3$. Let $b$ be a primitive root modulo $q$, and let $\varepsilon=\prod_{t=0}^{q-1} (1-\zeta_q^{xt} \zeta_p)$. Then $\varepsilon \in U(\mathcal{O})$ and $\phi(\varepsilon) = \tilde{\phi}(\prod_{t=0}^{q-1} (1-\zeta_q^{xt})) = \tilde{\phi}(\pm \sqrt{-q}) \in U(F_p)$. Hence $\varepsilon \in \langle \zeta_p \rangle U(\mathcal{O}_0)$, and so $Q_K=2$.

The next result is a special case of [3, Satz 22]. We give a direct proof based on the idea of T. Miyata ([7, (2.6)]).
Lemma 3.6. Suppose that \( m > 0 \) and \( w_m \in \mathbb{Z}[\zeta_p] \). Then \( U(\mathbb{Z}[w_m, \zeta_p]) = \langle \zeta_p \rangle U(\mathbb{Z}[w_m, \zeta_p + \zeta_p^{-1}] \rangle \).

Proof. Let \( \tau \) be the complex conjugation. For every \( u \in U(\mathbb{Z}[w_m, \zeta_p]) \), \( (u^\tau/u)(u^\tau/u)^{-1} = 1 \). So we see that \( u^\tau/u \) is a root of unity in \( U(\mathbb{Z}[w_m, \zeta_p]) \). Since \( u^\tau/u \) is mapped to 1 by the map \( \phi: U(\mathbb{Z}[w_m, \zeta_p]) \to U(F_p[w_m]) \), \( u^\tau/u = \zeta_p^i \) for some \( i \). Then there is an integer \( j \) such that \( (\zeta_p^i u)^\tau = \zeta_p^j u \). Hence we see that \( U(\mathbb{Z}[w_m, \zeta_p]) = \langle \zeta_p^i \rangle U(\mathbb{Z}[w_m, \zeta_p + \zeta_p^{-1}] \rangle \rangle \).

Let \( p \mid m, m > 0 \), \( N_{q(x)/q}(\epsilon_m) = -1 \) and \( p \equiv 1 \mod 4 \). Then a system of fundamental units of \( \mathbb{Z}[w_p, w_m] \) is given as one of the following three types ([5, Satz 11]):

(a) \( \epsilon_p, \epsilon_m \) and \( \epsilon_{pm} \),
(b) \( \epsilon_p, \epsilon_m \) and \( \sqrt{\epsilon_{pm}} \) (in this case, \( N_{q(x)/q}(\epsilon_{pm}) = 1 \)), or
(c) \( \epsilon_p, \epsilon_m \) and \( \sqrt{\epsilon_p \epsilon_m \epsilon_{pm}} \) (in this case, \( N_{q(x)/q}(\epsilon_m) = -1 \)).

Theorem 3.7. Suppose that \( p \mid m \) and \( m > 0 \). Then

i) If \( N_{q(x)/q}(\epsilon_m) = 1 \), then \( D(RC_p)^{(2)} = 0 \).

ii) If \( p \equiv 3 \mod 4 \) and \( N_{q(x)/q}(\epsilon_m) = -1 \), then \( D(RC_p)^{(2)} = 0 \).

iii) If \( p \equiv 1 \mod 4 \) and \( N_{q(x)/q}(\epsilon_m) = -1 \), then \( D(RC_p)^{(2)} \neq 0 \)

when the type of fundamental units of \( \mathbb{Z}[w_p, w_m] \) is (a) or (b), and \( D(RC_p)^{(2)} = 0 \)

when the type of fundamental units of \( \mathbb{Z}[w_p, w_m] \) is (c) and \( p \equiv 5 \mod 8 \).

Proof. Let \( \varphi: U(\mathbb{Z}[w_m, \zeta_p + \zeta_p^{-1}]) \to U(F_p[\sqrt{m}]) \) be the restriction of \( \Phi \) to \( U(\mathbb{Z}[w_m, \zeta_p + \zeta_p^{-1}]) \). Then, by force of (3.5), \( D(RC_p) \cong \ker \varphi \). There is a commutative diagram with surjective vertical maps

\[
\begin{array}{ccc}
U(\mathbb{Z}[w_m, \zeta_p + \zeta_p^{-1}]) & \xrightarrow{\varphi} & U(F_p[\sqrt{m}]) \\
\downarrow N_1 & & \downarrow N_1' \\
U(\mathbb{Z}[w_m]) & \xrightarrow{\varphi'} & U(F_p[\sqrt{m}])^{p-1/2} \\
\downarrow N_2 & & \downarrow N_2' \\
U(\mathbb{Z}) & \xrightarrow{\varphi''} & U(F_p)^{p-1/2} \\
\end{array}
\]

where \( N_1 = N_{q(x)/(\epsilon_{pm})}(\epsilon_m), N_2 = N_{p(x)/(\epsilon_{pm})}, N_2'(x) = x^{p-1/2}, N_2'' = N_{p(x)/(\epsilon_{pm})} \), \( \varphi' \) and \( \varphi'' \) are the restrictions of \( \varphi \) to \( \ker N_1 \) and \( \ker N_2 \) respectively. If \( N_2(\epsilon_m) = 1 \), then \( \ker N_2 = \ker N_1 = 1 \), and so \( 2 | \ker \varphi \). For the case where \( N_2(\epsilon_m) = -1 \), \( p \equiv 3 \mod 4 \) and \( \left( \frac{m}{p} \right) = 1 \), \( \varphi'' N_2(\epsilon_m) = (1, -1) \) or \( (-1, 1) \) in \( U(F_p[\sqrt{m}]) \equiv U(F_p) \times U(F_p) \). Since \( |U(F_p[\sqrt{m}])^{(2)}| = 4 \), this shows that \( (\ker \varphi)^{(2)} = 0 \). For the case where \( N_2(\epsilon_m) = -1 \), \( p \equiv 3 \mod 4 \) and \( \left( \frac{m}{p} \right) = -1 \), \( U(F_p[\sqrt{m}])^{(2)} = \langle \varphi(\epsilon_m) \rangle^{(2)} \).
because $\varepsilon_n^{p+1} = -1 \pmod{p}$, and therefore we see that $(\text{Coker } \varphi)^{(3)} = 0$.

To prove iii), we form the following commutative diagram with surjective vertical maps:

\[
\begin{array}{ccc}
U(Z[w_m, \zeta_p^{1} + \zeta_p^{-1}]) & \xrightarrow{\varphi} & U(F_p[\sqrt{m}]) \\
N_1 \downarrow & & \downarrow \\
U(Z[w_m, w_p]) \cong \text{Im } N_1 & \xrightarrow{\text{Im } N_1} & U(F_p[\sqrt{m}])^{p-1/4} \\
N_2 \downarrow & & \downarrow \\
U(Z[w_{pm}]) \cong \text{Im } N_2 & \xrightarrow{\text{Im } N_2} & U(F_p)^{p-1/4} \\
U(Z) \cong \text{Im } N_3 & \xrightarrow{\varphi'} & U(F_p)^{p-1/2},
\end{array}
\]

where $N_i$, $i=1, 2, 3$, are the norm maps and the other maps are natural. For the case of type (a), $\text{Im } N_1 \leq \langle -1, \varepsilon_p \rangle$, and for the case of type (b), $\text{Im } N_2 \leq \langle -1, \varepsilon_{pm} \rangle$ and $N_3(\varepsilon_{pm})=1$. Hence, for either case, $\text{Im } \varphi' * N_2 * N_3 * N_1 = \{1\}$, and so $2 | \text{Coker } \varphi$. If $p \equiv 5 \pmod{8}$, $U(F_p[\sqrt{m}])^{(3)} = (U(F_{\sqrt{m}}))^{p-1/4}(3)$. Now consider the case of type (c). If $\left(\frac{m}{p}\right) = 1$, $U(F_p[\sqrt{m}])^{p-1/4} = Z/4Z \oplus Z/4Z$. Since

\[
\varphi' * N_2 * N_3(\sqrt{\varepsilon_p \varepsilon_{pm} \varepsilon_{pm}}) = \varphi' * N_2 * N_3(\varepsilon_{pm} \varepsilon_{pm}^{p-1/4}) = \varphi'(-1) = -1, \quad \varphi(\sqrt{\varepsilon_p \varepsilon_{pm} \varepsilon_{pm}^{p-1/4}}) \text{ is of type } (\pm 1, c) \text{ or } (c, \pm 1) \text{ in } U(F_p) \times U(F_p), \quad \text{where } c \text{ is of order } 4 \text{ in } U(F_p).
\]

Hence (Coker $\varphi)^{(3)} = 0$, because $\text{Im } \varphi \supseteq \{(a, a) | a \in U(F_p)\}$. If $\left(\frac{m}{p}\right) = -1$, $U(F_p[\sqrt{m}])^{p-1/4} = Z/(p+1)Z$. We see that the order of $(\sqrt{\varepsilon_p \varepsilon_{pm} \varepsilon_{pm}^{p-1/2}})$ is 8, because $\varepsilon_p^{n+1} = -1 \pmod{p}$. This shows that (Coker $\varphi)^{(3)} = 0$, and thus the proof is completed.

**Remark 3.8.** For the case where the type of fundamental units of $Z[w_p, w_m]$ is (c) and $p \equiv 1 \pmod{8}$, we do not know whether $D(RC_p)^{(3)} = 0$ or not.

**Proposition 3.9.** Suppose that $p | m$ and write $m = np$. Then

i) $D(RC_p) \cong T(RC_p) \oplus D(RC_{p/(\Sigma_p)})$.

ii) $T(RC_p) \cong \left\{ \begin{array}{ll} Z/pZ & \text{if } m < -3 \text{ or } m > 0 \text{ and } p | b \\
0 & \text{otherwise.} \end{array} \right.$

iii) $D(RC_p/(\Sigma_p))^{(p)}$ is an elementary $p$-group of rank $\leq (p-3)/2$. Especially, if $n = 1$, then $D(RC_p/(\Sigma_p))$ is an elementary $p$-group of rank $\leq \max(0, (p-7)/2)$.

**Proof.** ii) There is a commutative diagram
\[ U(R[\bar{a}]) \xrightarrow{\phi} U(F_p[\sqrt{m}]) \cong \mathbb{Z}/p(p-1)\mathbb{Z} \xrightarrow{T(RC_p)} 0 \]

where \( N(f(\bar{a})) = \prod_{i=1}^{p-1} f(\bar{a}^i) \) for every \( f(\bar{a}) \in U(R[\bar{a}]) \), \( N'(x) = x^{p-1} \) for every \( x \in U(F_p[\sqrt{m}]) \) and \( \phi' \) is the restriction of \( \phi \) to \( \text{Im} \, N \). Then Coker \( \phi \cong \mathbb{Z}/p\mathbb{Z} \) or 0, and Coker \( \phi \cong \mathbb{Z}/p\mathbb{Z} \) if and only if Coker \( \phi' \cong \mathbb{Z}/p\mathbb{Z} \). If \( m > 0 \), then \( \phi' \circ N(e_m) = \phi'(e_{p^{-1}}) = \mathbb{Z}/p\mathbb{Z} \) if and only if \( p \mid b \). If \( m < -3 \), then \( U(R) = \{ \pm 1 \} \), and hence Coker \( \phi' \cong \mathbb{Z}/p\mathbb{Z} \). For \( m = -3 \), we can compute directly that \( \phi \) is surjective.

i) The conclusion follows from ii) and (2.2 i).

iii) Let \( n \neq 1 \). Then we can write as

\[ D(RC_p/(\Sigma_p)^{(p)}) = \frac{1+pS_p}{U(\bar{S})(1+pS_p)} \]

where \( S = \mathbb{Z}[w_{\bar{a}}, z_p] \), \( \bar{S} = \mathbb{Z}[w_a, z_p] \), \( p \) is the unique prime ideal over \( \bar{p} \) in \( S \) and \( \bar{p} = pS \). Then the conclusion follows from (2.2 i) and the fact that \( |1+pS_p| = p^{p-\xi_p} \). Next assume that \( n = 1 \). By force of (2.3), we may assume that \( p \geq 7 \). Then

\[ D(RC_p/(\Sigma_p)) = \frac{(1+\pi_{C_p}) \times (1+\pi_{C_p})}{U(\bar{S})(1+pS_p)} \subseteq \{ \frac{(1+\pi_{C_p}) \times (1+\pi_{C_p})}{U(\bar{S})(1+pS_p)} \} \]

where \( \pi = z_p - 1 \) and \( \mathcal{O} = \mathbb{Z}[z_p] \). The map \( U(RC_p/(\Sigma_p)) \subseteq \{ \frac{(1+\pi_{C_p}) \times (1+\pi_{C_p})}{U(\bar{S})(1+pS_p)} \} \) is surjective where \( \varphi(x, y) = x \). Since \( U(\bar{S}) \) contains \( 1+\pi \) and \( 1+\pi^2 - \pi z_p^{-1} \), each element of \( D(RC_p/(\Sigma_p)) \) has a representative of the form \( \frac{(1+\pi_{C_p}) \times (1+\pi_{C_p})}{U(\bar{S})(1+pS_p)} \), \( x \in \mathcal{O}_p \). The conclusion follows from this, because \( U(\bar{S}) \subseteq \{ 1 \} \times \{ 1+\pi^{p-1} \} \).

Remark 3.10. If \( p = 5 \) and \( n > 1 \), then \( D(RC_5/(\Sigma_5)^{(5)}) \cong \mathbb{Z}/5\mathbb{Z} \). In fact, since \( U(\bar{S}) = \mathbb{Z}[w_{\bar{a}}, w_\epsilon] \), it is easy to see that \( U(\bar{S}) = U(\mathbb{S}) \cong 1+pS_p \). On the other hand, there are examples of \( n \) for which \( D(RC_5/(\Sigma_5)^{(5)}) = 0 \), e.g. \( n = -1, -3, -7 \) or \(-11 \).

§ 4.

In this section, we shall determine completely the structure of \( D(RC_5) \).

Lemma 4.1. Let \( m > 0 \) and \( 3 \nmid m \). Put \( e_m = (a+b \sqrt{m})/2 \). Then
Class Groups of Group Rings Whose Coefficients

i) \(3 \nmid a \) or \(3 \nmid b\).

ii) If \(m \equiv 1 \pmod{3}\), then \(3 \mid ab\).

iii) \(N_{K/Q}(\varepsilon_m) = 1\) if and only if \(m \equiv 1 \pmod{3}\) and \(3 \nmid a\) or \(m \equiv -1 \pmod{3}\) and \(3 \mid ab\).

**Proof.** The results follow from the facts that \(N_{K/Q}(\varepsilon_m) \equiv a^2 - b^2 \pmod{3}\) if \(m \equiv 1 \pmod{3}\) and that \(N_{K/Q}(\varepsilon_m) \equiv a^2 + b^2 \pmod{3}\) if \(m \equiv -1 \pmod{3}\).

We can refine (3.4) and (3.7) as follows.

**Theorem 4.2.** Suppose that \(3 \mid m\). Then \((\sqrt{m}, \Sigma_2)\) (resp. \((-1 + \sqrt{m}, \Sigma_2)\)) is a Representative of a generator of \(D(RC_3)\) if \((m/p) = 1\) (resp. \((m/p) = -1\)), and

<table>
<thead>
<tr>
<th>(D(RC_3))</th>
<th>(m &lt; 0)</th>
<th>(m &gt; 0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(m \equiv 1 \pmod{3}) and (A), or (m \equiv -1 \pmod{3}) if (N_{K/Q}(\varepsilon_m) = -1)</td>
<td>(m \equiv 1 \pmod{3}) and (3 \mid b), or (m \equiv -1 \pmod{3}), (3 \mid a)</td>
</tr>
<tr>
<td>(\mathbb{Z}/2\mathbb{Z})</td>
<td>(m \equiv 1 \pmod{3}) and not (A), or (m \equiv -1 \pmod{3}), (m \equiv -1 \pmod{3}) and (A)</td>
<td>(m \equiv 1 \pmod{3}) and (3 \mid b), or (m \equiv -1 \pmod{3}), (3 \mid a)</td>
</tr>
<tr>
<td>(\mathbb{Z}/4\mathbb{Z})</td>
<td>(m \equiv -1 \pmod{3}) and not (A)</td>
<td>(m \equiv -1 \pmod{3}) and (3 \mid b)</td>
</tr>
</tbody>
</table>

where, for \(m < 0\), (A) means the property that \(\sqrt{-m} \in U(\mathbb{Z}[w_m, \zeta_3])\).

**Proof.** For the case \(m < 0\), the result follows from (3.4), since the condition (A) is equivalent to the condition \(Q_K = 2\), where \(K = Q(\zeta_3, \sqrt{m})\). For the case \(m > 0\), we see that \(D(RC_3) = U(F_3[\sqrt{m}])/\varphi(U(\mathbb{Z}[w_m]))\) by (3.6), and so \(D(RC_3)\) is a 2-group. Therefore the result follows from (3.7 ii) and (4.1).

For the case \(m = -3n\), we have, by (3.9) and (2.2),

\[
D(RC_3) \equiv T(RC_3) \oplus D(RC_3/(\Sigma_3))
\]

and

\[
D(RC_3/(\Sigma_3)) \equiv \begin{cases} 0 & \text{if } n = 1 \\ D(R'C_3) & \text{if } n \neq 1, \text{ where } R' = \mathbb{Z}[w_m]. \end{cases}
\]

Hence we have

**Theorem 4.3.** Suppose that \(3 \mid m\) and write \(m = -3n\). Then
<table>
<thead>
<tr>
<th>$D(RC_3)$</th>
<th>$n &lt; 0$</th>
<th>$n &gt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$n \equiv 1 \pmod{3}$, (A) and $3 \nmid d$, or $n = -1$</td>
<td>$n = 1$</td>
</tr>
<tr>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
<td>$n \equiv 1 \pmod{3}$, not (A) and $3 \nmid d$, or $n \equiv -1 \pmod{3}$, $n = -1$, (A) and $3 \nmid d$</td>
<td></td>
</tr>
<tr>
<td>$\mathbb{Z}/3\mathbb{Z}$</td>
<td>$n \equiv 1 \pmod{3}$, (A) and $3 \nmid d$</td>
<td>$n \neq 1$ and $N(\varepsilon_n) = -1$</td>
</tr>
<tr>
<td>$\mathbb{Z}/4\mathbb{Z}$</td>
<td>$n \equiv -1 \pmod{3}$, (A) and $3 \nmid d$</td>
<td></td>
</tr>
<tr>
<td>$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$</td>
<td>$n \equiv 1 \pmod{3}$, not (A) and $3 \nmid d$, or $n \equiv -1 \pmod{3}$, $n = -1$, not (A) and $3 \nmid d$</td>
<td>$n \equiv 1 \pmod{3}$, $n \neq 1$ and $3 \nmid b$, or $n \equiv -1 \pmod{3}$, and $3 \nmid a$</td>
</tr>
<tr>
<td>$\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$</td>
<td>$n \equiv -1 \pmod{3}$, not (A) and $3 \nmid d$</td>
<td>$n \equiv -1 \pmod{3}$, $3 \nmid b$</td>
</tr>
</tbody>
</table>

where $\varepsilon_n = (a + b\sqrt{n})/2$ if $n > 1$, $\varepsilon_n = (c + d\sqrt{-3})/2$ if $m > 0$, (A) means the property that $\sqrt{-\varepsilon_n} \in U(\mathbb{Z}[w_n, \zeta_n])$ if $m > 0$, and $N = N_{Q(\sqrt{-3})}$.

Next we want to know representatives of generators of $D(RC_3)$ in the case of $3|n$. Since $T(RC_3)$ is generated by the class of $(1 + \sqrt{n}, \Sigma_3)$, we have only to consider $D(RC_3 \setminus \Sigma_3)$. Write $m = -3n$ assume that $n \neq \pm 1$. Let $R = \mathbb{Z}[w_n, \zeta_n]$ and $\tilde{S} = \mathbb{Z}[w_n, \zeta_n]$. Then we see that $\tilde{S}$ is the integral closure of $S$. Put $p = (\sqrt{-3}, \sqrt{-3n})$ (resp. $(\sqrt{-3}, 1 + (1 + \sqrt{-3n})/2)$) if $n \equiv 1 \pmod{4}$ (resp. $n \equiv 1 \pmod{4}$). Then we see that $p$ is a unique prime ideal of $S$ which contains $p^2 = (\sqrt{-3})p$ and $p\tilde{S} = (\sqrt{-3})$. First we note

**Lemma 4.4.** An invertible ideal $C$ of $S$ such that $C\tilde{S}$ is principal in $\tilde{S}$ is isomorphic to some $p$-primary invertible ideal of $S$ not contained in $p^2$.

**Proof.** Let $C'$ be an invertible ideal of $S$ such that $C' \cong C^{-1}$. Since $p$ is a unique non-invertible prime ideal of $S$, we have $S[1/3] = \tilde{S}[1/3]$. Hence, there is $c' \in C'$ such that $c'S[1/3] = (c')$ in $S[1/3]$, and so there is a $p$-primary invertible ideal $g$ of $S$ such that $(c') = C'g$ in $S$. Since $p^2 = (\sqrt{-3})p$, $g$ is isomorphic to a $p$-primary invertible not contained in $p^2$. Since $C \cong C'^{-1} \cong g$, this completes the proof.

Put $a = (3, \sqrt{-3n})$ (resp. $(3, 1 + (1 + \sqrt{-3n})/2)$) if $n \equiv 1 \pmod{4}$ (resp. $n \equiv 1 \pmod{4}$). Then $a\tilde{S} = (\sqrt{-3})$ and $a^2 = (3)$.
**Lemma 4.5.** The following statements are equivalent:

i) $a$ is non-principal in $S$.

ii) In the case where $n < 0$, $U(Z[w_n, \zeta_n]) = \langle -1, \zeta_3, \varepsilon_{-3n} \rangle$, and in the case where $n > 0$, $\varepsilon_n = (a + b\sqrt{n})/2$, $3 \nmid a, 3 \nmid b$.

**Proof.** Let $n \equiv 1 \pmod{4}$. If $a$ is principal, then we can write as $a = (3x + y\sqrt{-3n})$ for some $x, y \in Z[\zeta_n]$. Further we see that $\sqrt{-3} \parallel y$ and $(x, y) = (1)$. Since $3 \nmid a$,

$$(x' + z\sqrt{-3n})(3x + y\sqrt{-3n}) = 3$$

for some $x', z \in Z[\zeta_n]$.

Hence we have that $3xz + yv' = 0$, and so $v' = 3v$ for some $v \in Z[\zeta_n]$. Then the equality $xz + yz = 0$ implies that $v = ux$ and $z = uy$ for some $u \in Z[\zeta_n]$. Therefore we can write $a = (3x + y\sqrt{-3n})$ and $u = (x, y) = (1)$.

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Combining (4.3), (4.5) and (4.6), we have

**Theorem 4.7.** Suppose that $3|m$ and $m \not\equiv \pm 3$. Let

$$a = \begin{cases} (3, \sqrt{m}) & \text{if } m \equiv 1 \pmod{4} \\ (3, 1 + \frac{1 + \sqrt{m}}{2}) & \text{if } m \equiv 1 \pmod{4} \end{cases},$$

and

$$g = \begin{cases} (3, \sqrt{-3} + \sqrt{m}) & \text{if } m \equiv 1 \pmod{4} \\ (3, \sqrt{-3} + \frac{3 + \sqrt{m}}{2}) & \text{if } m \equiv 1 \pmod{4}. \end{cases}$$

Then $D(RC_5/(\Sigma_5))$ is generated by the class of $a$ (resp. $g$) if $m/3 \equiv -1 \pmod{3}$ (resp. $m/3 \equiv 1 \pmod{3}$).

**Remark 4.8.** We can also determine the structure of $D(RC_5)$ for the case that $k$ is a real quadratic field. Let $R = \mathbb{Z}[w_m]$, $S = \mathbb{Z}[w_m, w_5]$, where $5 \nmid m > 0$, and $\varepsilon_m = (a + b \sqrt{m}/2)$. Then a system of fundamental units of $S$ is given as one of the following:

(a) $\varepsilon_5, \varepsilon_m, \varepsilon_{5m}$,
(b) $\varepsilon_5, \varepsilon_m, \sqrt{\varepsilon_{5m}}$ (in this case $N(\varepsilon_{5m}) = 1$ and $\left(\frac{m}{5}\right) = 1$),
(c) $\varepsilon_5, \varepsilon_m, \sqrt{\varepsilon_{5}\varepsilon_{5m}}$ (in this case $N(\varepsilon_m) = N(\varepsilon_{5m}) = -1$, or $N(\varepsilon_m) = 1, \left(\frac{m}{5}\right) = -1$ and $5 \nmid b$), or
(d) $\varepsilon_5, \varepsilon_m, \sqrt{\varepsilon_{5m}\varepsilon_{5m}}$ (in this case $N(\varepsilon_m) = N(\varepsilon_{5m}) = 1, \left(\frac{m}{5}\right) = -1$ and $5 \nmid b$),

where, for a square-free positive integer $d$, $N(\varepsilon_d) = N_{\mathbb{Q}(\sqrt{d})/\mathbb{Q}}(\varepsilon_d)$.

We have a following table:

<table>
<thead>
<tr>
<th>$D(RC_5)$</th>
<th>$\left(\frac{m}{5}\right) = 1$</th>
<th>$\left(\frac{m}{5}\right) = -1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$N(\varepsilon_m) = -1$ and (c)</td>
<td>$N(\varepsilon_m) = -1$, (c) and $5 \nmid b$</td>
</tr>
<tr>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
<td>(a) and $5 \nmid b$, or (b)</td>
<td>(a) and $5 \nmid b$, or $N(\varepsilon_m) = 1$ and (c) or (d)</td>
</tr>
<tr>
<td>$\mathbb{Z}/3\mathbb{Z}$</td>
<td></td>
<td>$N(\varepsilon_m) = -1$, (c) and $5</td>
</tr>
<tr>
<td>$\mathbb{Z}/4\mathbb{Z}$</td>
<td>(a) and $5</td>
<td>b$</td>
</tr>
<tr>
<td>$\mathbb{Z}/6\mathbb{Z}$</td>
<td></td>
<td>(a) and $5</td>
</tr>
</tbody>
</table>

where (a) means that the type of fundamental units of $S$ is (a).

Further, let $R' = \mathbb{Z}[w_{5m}]$ and $\varepsilon_{5m} = (c + d \sqrt{5m})/2$. Then
Class Groups of Group Rings Whose Coefficients

\[ D(R'C_6) \cong D(RC_6) \oplus \mathbb{Z}/5\mathbb{Z} \oplus T(R'C_6) \]

and

\[ T(R'C_6) \cong \begin{cases} 
0 & \text{if } 5 \nmid d \\
\mathbb{Z}/5\mathbb{Z} & \text{if } 5 \mid d.
\end{cases} \]

References


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