ON COMPACTA WHICH ARE l-EQUIVALENT TO $I^n$

By

Akira Koyama and Toshinao Okada

1. Introduction.

All spaces considered in this paper are assumed to be metrizable. A compactum is a compact space. A continuum is a connected compactum, and a mapping is a continuous function. For a space $X$ we denote by $C(X)$ the space of all real-valued mappings on $X$ with the topology of uniform convergence. Then by Milutin's interesting work [8], we have known that for each pair of uncountable compacta $X$ and $Y$, $C(X)$ is linearly isomorphic to $C(Y)$ (see [12] for the details and generalizations). On the other hand, for space $X$ we denote by $C_p(X)$ the space of all real-valued mappings on $X$ with the topology of pointwise convergence. Spaces $X$ and $Y$ are said to be $l$-equivalent [1] provided that $C_p(X)$ is linearly isomorphic to $C_p(Y)$, written $C_p(X)\cong C_p(Y)$. Recently, Pavlovskii [11] showed the following.

1.1. Theorem. (1) If locally compact spaces $X$ and $Y$ are $l$-equivalent, then for each non-empty open or closed set $\tilde{X}$ of $X$, there exists a non-empty open set in $\tilde{X}$ which can be embedded in $Y$. Therefore, dim $X=\dim Y$ (see also [4] and [13]).

(2) Non-zero-dimensional compact polyhedra $P$ and $Q$ are $l$-equivalent if and only if $\dim P=\dim Q$.

(3) Let $B$ be the Pontryagin's 2-dimensional continuum with the property $\dim (B\times B)=3$. Then $B$ is not $l$-equivalent to $I^n$, where $I$ is the unit interval [0, 1].

Being motivated by Theorem 1.1 (2), readers may consider that for $n\geq 1$, all $n$-dimensional compact ANR's are $l$-equivalent to $I^n$. However, by Theorem 1.1 (1) and [3, Theorem VI. (6.1)], we can easily see that for each $n\geq 1$, there exists a collection of $2^{2^n}$ $n$-dimensional compact AR's in $R^{n+1}$ which are not $l$-equivalent to each other. On the other hand, let $X$ be a compactification of the half-open interval $[0, 1)$ whose remainder is $I^n$. Then $X$ is $l$-equivalent to $I^n$, although $X$ is not even locally connected. Therefore it seems to be difficult to

Received April 10, 1986.
control \( n \)-dimensional compacta which are \( l \)-equivalent to \( I^n \).

In this paper we will show a criterion of an \( n \)-dimensional locally compact space which is \( l \)-equivalent to an \( n \)-manifold. Concerning \( 1 \)-dimensional compacta, Lelek [7] introduced the class of finitely Suslinian compacta, which contains all hereditarily locally connected continua, and therefore all \( 1 \)-dimensional compact \( \text{ANR's} \). We will also show a simple criterion of a curve (=\( 1 \)-dimensional continuum) which is \( l \)-equivalent to a finitely Suslinian compactum. Hence we can easily see that neither the Cantor fan nor the Knaster indecomposable curve are \( l \)-equivalent to any finitely Suslinian compacta. Moreover, we will investigate a class of curves which are \( l \)-equivalent to \( I \). So we have a desired class of special compact \( \text{ANR's} \) which contains all graphs, and show that every continuum which is a one-to-one continuous image of \( [0, \infty) \) is \( l \)-equivalent to \( I \).

Most of our results can be applied to the theory of free topological groups in the sense of Graev [5]. So we may have interesting examples concerning free topological groups in the sense of Graev.

We denote by \( \dim X \) the \emph{covering dimension} of a space \( X \). Let \( A \) be a \( \text{subset of a space} \) \( X \). We denote its \emph{interior} and \emph{closure} in \( X \) by \( \text{int} \ A \) and \( \text{cl} \ A \), respectively. The symbol \( \text{ANR} \) is used to specify an \emph{absolute neighborhood retract} for the class of all metric spaces. Undefined terms and notations in continuum theory may be found in [6] and [14].

The authors would like to express their thanks to Professor A. Okuyama for his valuable and kind suggestions.

2. Criterions for being \( l \)-equivalent to special spaces.

First, we will discuss a compactum which is \( l \)-equivalent to \( I^n \). A space \( X \) is \emph{locally contractible at a point} \( x \) of \( X \) if for every open neighborhood \( U \) of \( x \) in \( X \), there exists an open neighborhood \( V \) of \( x \) in \( X \) such that \( V \subseteq U \) and \( V \) is contractible in \( U \). We denote the set of all points of \( X \) at which \( X \) are locally contractible by \( L_c(X) \). Now we have

2.1. Theorem. Let \( X \) be an \( n \)-dimensional locally compact space and \( \bar{X} \) be the closure of the set of all points of \( X \) whose local dimensions are exactly \( n \). If \( X \) is \( l \)-equivalent to an \( n \)-manifold, then \( L_c(\bar{X}) \) is dense in \( \bar{X} \).

Proof. Note that \( \dim A=n \) for any non-empty open subset \( A \) of \( \bar{X} \). Suppose that \( X \) is \( l \)-equivalent to an \( n \)-manifold \( M \). First, we show that for an arbitrary open subset \( U \) of \( \bar{X} \), there is an open subset of \( U \) which is contractible in \( U \). By Theorem 1.1 (1), there exists a non-empty open subset \( V \) of
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$U$ and there exist maps $f : V \to M$ and $g : f(V) \to V$ such that $gf = 1_V$. Since $f(V)$ is the $n$-dimensional subset of $M$, $\text{int} f(V) \neq \emptyset$. Hence there is a point $x_0$ of $V$ and there is an open subset $W$ of $M$ such that $f(x_0) \in W \subset \text{cl} W \subset \text{int} f(V)$ and $\text{cl} W$ is homeomorphic to $I^n$. Particularly, $W$ is contractible in $f(V)$, and therefore there is a homotopy $G : W \times I \to f(V)$ such that $g(y, 0) = y$ and $G(y, 1) = f(x_0)$ for all $y \in W$. Take an open subset $V_0$ in $V$ such that $x_0 \in V_0$ and $f(V_0) \subset W$ and define a homotopy $H : V_0 \times I \to U$ by $H(x, t) = gG(f(x), t)$ for $(x, t) \in V_0 \times I$. Then $H(x, 0) = x$ and $H(x, 1) = x_0$ for all $(x, t) \in V_0 \times I$. Hence $V_0$ is contractible in $U$.

Next, we show that $L_c(\tilde{X})$ is dense in $\tilde{X}$. Let $U$ an arbitrary non-empty open subset of $\tilde{X}$. By the first part of the proof, we have a sequence $\{U_n\}_{n \geq 0}$ of non-empty open subsets of $\tilde{X}$ such that for every $n = 0, 1, 2, \ldots$,

(1) $\text{cl} U_{n+1} \subset U_n$, where $U_0 = U$

(2) $\text{diam} [U_n] < \frac{1}{n}$, and

(3) $U_{n+1}$ is contractible in $U_n$.

Then by (1) and (2), we have a point $x^* \in \bigcap_{n \geq 0} U_n \subset U$, and by (2) and (3), we can see that $x^* \in L_c(\tilde{X})$. Therefore $L_c(\tilde{X})$ is dense in $\tilde{X}$.

2.2. COROLLARY. Let $X$ be an $n$-dimensional compactum and $\tilde{X}$ be the closure of the set of all points of $X$ whose local dimensions are exactly $n$. Then if $X$ is $l$-equivalent to $I^n$, $L_c(\tilde{X})$ is dense in $\tilde{X}$.

Next, we will consider the case of curves. A compactum $X$ is finitely Suslinian [7] if for every $\varepsilon > 0$, each collection of pairwise disjoint subcontinua of $X$ having diameters greater than $\varepsilon$ is finite. We note that every finitely Suslinian continuum is at most 1-dimensional, and that every hereditarily locally connected continuum is finitely Suslinian. Hence every 1-dimensional compact ANR is finitely Suslinian, and there exist finitely Suslinian compacta which are not ANR's. In order to show a criterion of a curve which is $l$-equivalent to $I$, we introduce a notation as follows. A space $X$ is locally connected at a point $x$ of $X$ if for every open neighborhood $U$ of $x$ in $X$, there exists a connected open neighborhood $V$ of $x$ in $U$. By $L(X)$, we denote the set of all points of $X$ at which $X$ is locally connected. Clearly a space $X$ is locally connected if and only if $L(X) = X$. Then we have

2.3. THEOREM. If a curve $X$ is $l$-equivalent to a finitely Suslinian compactum, then the following conditions are satisfied:
(i) $L(X)$ is dense in $X$, and
(ii) $L(X)$ has non-empty interior in $X$.

**Proof.** Suppose that $X$ is $l$-equivalent to a finitely Suslinian compactum $Y$ but $L(X)$ is not dense in $X$. Then there is a non-empty open subset $U$ of $X$ such that $U \cap L(X) = \emptyset$. By Theorem 1.1 (1), there is a non-empty open subset $V$ of $U$ such that $clV \subset U$ and there exists an embedding $f : clV \to Y$. Since $V \cap L(X) = \emptyset$, by [14, Theorem I.12.1], there exist a positive number $\varepsilon > 0$ and a sequence $K_0, K_1, K_2, \ldots$ of pairwise disjoint subcontinua of $clV$ such that

$$\text{diam} [K_i] > \varepsilon \quad \text{for all} \quad i \geq 0, \quad \text{and} \quad K_0 = \lim_i K_i.$$

Then the sequence $f(K_0), f(K_1), f(K_2), \ldots$ consists of pairwise disjoint subcontinua in $Y$ and satisfies the following properties:

$$f(K_0) = \lim_i f(K_i), \quad \text{and} \quad \text{diam} [f(K_i)] > 0.$$

But this contradicts to the assumption that $Y$ is finitely Suslinian, because $\text{diam} [f(K_i)] \geq 1/2 \text{diam} [f(K_0)]$ for almost all $i \geq 1$. Namely, the curve $X$ satisfies the condition (i).

If $\text{int} L(X) = \emptyset$, then $X - L(X)$ is dense in $X$. Hence we can similarly prove that the condition (ii) is satisfied.

2.4. **Corollary.** Neither the Cantor fan nor the Knaster indecomposable curve (see [6, Example 1, p. 204]) are $l$-equivalent to any finitely Suslinian compactum.

A space $X$ has a decomposable local system if every non-empty open subset of $X$ contains a non-degenerate decomposable continuum. For example, $n$-manifolds, polyhedra, hereditarily decomposable continua, the Knaster indecomposable curve, the dyadic solenoid have decomposable local system. By Theorem 1.1 (1), we can easily show the following.

2.5. **Lemma.** No compactum which has a decomposable local system is $l$-equivalent to any hereditarily indecomposable continuum.

Considering the arc, the Knaster indecomposable curve and the pseudo-arc [2], by Corollary 2.4 and Lemma 2.5, we have.

2.6. **Corollary.** There exist three arc-like continua which are not $l$-equivalent to each other.

Finally, we will construct a finitely Suslinian continuum which is not locally
contractible at any point. Namely, for a curve $X$, the density of $L(X)$ is a criterion for being $l$-equivalent to a finitely Suslinian compactum but is not one for being $l$-equivalent to $I$.

2.7. Example. Let $S_0$ be the unit circle in the plane $R^2$. Let $\{a_i\}_{i \geq 1}$ be a countable dense subset of $S_0$. Then we can take a sequence $\{S_{i,t}\}_{t \geq 1}$ of pairwise disjoint circles inside of $S_0$ satisfying the conditions:

(1) $S_{i} \cap S_{i,t} = \{a_i\}$ for every $i \geq 1$, and

(2) $\text{diam} [S_{i,t}] \leq \frac{1}{2^i}$ for every $i \geq 1$.

Define

$$X_i = S_0 \cup (\bigcup_{t \geq 1} S_{i,t}).$$

For $n \geq 1$, assume that we have constructed a sequence $\{S_{n,i}\}_{t \geq 1}$ of pairwise disjoint circles and a continuum $X_n$ of the form $X_{n-1} \cup (\bigcup_{t \geq 1} S_{n,i})$, where $X_0 = S_0$, such that for every $i \geq 1$,

(3) $X_{n-1} \cap S_{n,i} = \{a_{n,i}\}$, $X_{n-2} \cap S_{n,i} = \emptyset$,

(4) $\text{diam} [S_{n,i}] \leq \frac{1}{n \cdot 2^i}$,

(5) $\{a_{n,i}\}_{i \geq 1}$ is dense in $X_{n-1}$.

Then for every $i \geq 1$, take a countable subset $\{b_{i,j}\}_{j \geq 1}$ of $S_{n,i} - X_{n-1}$ which is dense in $S_{n,i}$. Further let us take a sequence $\{S_{n,i,j}\}_{j \geq 1}$ of pairwise disjoint circles inside of $S_{n,i}$ such that for every $i \geq 1$,

(6) $X_n \cap S_{n,i,j} = \{b_{i,j}\}$, and

(7) $\text{diam} [S_{n,i,j}] \leq \frac{1}{(n+1) \cdot 2^{i+1}}$.

Then define

$$X_{n+1} = X_n \cup [\bigcup_{i \geq 1} (\bigcup_{j \geq 1} S_{n,i,j})].$$

It is easily seen that $X_{n+1}$ can be represented in the form which satisfies the inductive assumptions (3)-(5) in replacement of $X_n$ by $X_{n+1}$. So we define a curve

$$X = \bigcup_{n \geq 1} X_n.$$

Now we can rewrite $X$ as follows;

$$Y_i = S_{i,t} \cup (\bigcup_{j \geq 1} S_{i,j}) \cup (\bigcup_{k \geq 1} S_{i,t,k}) \cup \ldots \quad \text{for } i \geq 1, \quad \text{and } X = \bigcup_{i \geq 1} Y_i.$$

By the construction, every subcontinuum of $X$ having diameter greater than $1/2^i$, which intersects $Y_i$, must contain $a_i$. Hence it is easily seen that $X$ is
3. Curves which are $l$-equivalent to $I$.

In this section we will show that certain curves are $l$-equivalent to $I$. We need the following lemma as elementary and key tools for calculations.

3.1. Lemma (Pavlovskii [8]). (1) For a closed subset $S$ of $I$, $C_p(I) = C_p(S) \times C_p(I;S)$, where for a subset $A$ of a space $X$, we define $C_p(X;A) = \{ f \in C_p(X) | f(A) = 0 \}$, and if $A = \{a\}$, we write $C_p(X;A) = C_p(X;\{a\})$.

(2) Let $A$ be a closed subset of a space $X$, which is a neighborhood retract of $X$. Then $C_p(X) = C_p(A) \times C_p(X;A)$.

(3) Let $X_1$ and $X_2$ be closed subsets of a space $X$ such that $X = X_1 \cup X_2$, $X_a = X_1 \cap X_2$ is a neighborhood retract of $X$ and $C_p(X) = C_p(X_1) \times C_p(X_2)$. Then $C_p(X) = C_p(X_1) \times C_p(X_2)$.

(4) $C_p(I) \times C_p(I) = C_p(I)$. 

3.2. Theorem. Every dendrite (=1-dimensional compact AR) with finite ramification points is $l$-equivalent to $I$.

Proof. By Theorem 1.1 (2), we consider only a dendrite which is not a tree. Let $X$ be a dendrite with ramification points $x_1, x_2, \ldots, x_n$. Let $A$ be a tree in $X$ which contains all $x_i$. Then by Lemma 3.1 (2) and (4),

\[ C_p(X) = C_p(A) \times C_p(X;A) \cong C_p(I) \times C_p(X/A;\{A\}) \]

\[ \cong C_p(I) \times R \times C_p(X/A;\{A\}) \]

\[ \cong C_p(I) \times C_p(X/A), \]

where $\{A\}$ is the identification point of $A$ in $X/A$. Since $X/A$ is a dendrite with exactly one ramification point, by Lemma 3.1 (4), it suffices to show the case of dendrites with exactly one ramification point.

Let $p$ be the pole (i.e., the origin) in the polar coordinate system in the plane $R^2$. Define in the polar coordinate $(r, \theta)$,

\[ p_n = \left( \frac{1}{n}, \frac{1}{n} \right) \quad \text{for every } n \geq 1, \]

and let

\[ Y = \bigcup_{n \geq 1} \overline{xy}, \]

where $\overline{xy}$ stands for the straight line segment joining $x$ and $y$. Now it is easily
seen that every dendrite, which is not a tree and has exactly one ramification point, is homeomorphic to $Y$. Hence it suffices to prove that

\[ (*) \quad C_p(Y) \cong C_p(I). \]

Let $S = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$. Then by Lemma 3.1 (2),

\[ C_p(I) \cong C_p(S) \times C_p(S; 0) \cong R \times C_p(S; 0) \times C_p(I; S). \]

We note that we can identify each $\alpha \in C_p(S; 0)$ with the sequence $\{a_n\}_{n \geq 1}$ defined by $a_n = \alpha(1/n)$, which converges to 0. So for each $(\alpha, f) \in C_p(S; 0) \times C_p(I; S)$, we define $\varphi(\alpha, f) \in C_p(Y; p)$ by the formula;

\[ \varphi(\alpha, f)(r, \frac{1}{n}) = f(r + \frac{1}{n}, 1) + n r a_n \quad \text{for each } r, 0 \leq r \leq \frac{1}{n}, n \geq 1. \]

Namely, we have the continuous linear function $\varphi : C_p(S; 0) \times C_p(I; S) \to C_p(Y; p)$. On the other hand, for each $g \in C_p(Y; p)$, $\varphi_1(g) \in C_p(S; 0)$ and $\varphi_2(g) \in C_p(I; S)$ are defined as follows;

\[ \varphi_1(g)(t) = \begin{cases} g(p_n) & \text{if } t = \frac{1}{n} \text{ for some } n \geq 1, \\ 0 & \text{if } t = 0, \end{cases} \]

\[ \varphi_2(g)(t) = \begin{cases} g((n + 1)t - 1, \frac{1}{n}) & \text{if } t \in \left[ \frac{1}{n + 1}, \frac{1}{n} \right] \text{ for some } n \geq 1, \\ 0 & \text{if } t = 0. \end{cases} \]

Hence we have the continuous linear function $\psi : C_p(Y; p) \to C_p(S; 0) \times C_p(I; S)$ given by $\psi(g) = (\varphi_1(g), \varphi_2(g))$. Then we can see that $\varphi \psi = 1_{C_p(S; 0) \times C_p(I; S)}$ and $\psi \varphi = 1_{C_p(Y; p)}$. Hence $C_p(S; 0) \times C_p(I; S) \cong C_p(Y; p)$. Therefore we have

\[ (*) \quad C_p(I) \cong R \times C_p(S; 0) \times C_p(I; S) \cong R \times C_p(Y; p) \cong C_p(Y). \]

3.3. COROLLARY. Every 1-dimensional compact ANR with finite ramification points is $l$-equivalent to $I$.

PROOF. Let $X$ be a 1-dimensional compact ANR with finite ramification points. By Lemma 3.1 (4) and (3), we may assume that $X$ is connected. We will prove by the induction on the number of simple closed curves in $X$. If there is no simple closed curve in $X$, then $X$ is a dendrite. Hence by Theorem 3.2, the assertion holds.

Assume that the assertion holds for ANR's which has at most $n - 1$ simple
closed curves, where \( n \geq 1 \). Let \( X \) be 1-dimensional compact ANR which has \( n \) simple closed curves. Take a simple closed curve \( L \) in \( X \). Then \( X/L \) is a 1-dimensional compact ANR and has at most \( n-1 \) simple closed curves, because a 1-dimensional locally connected continuum with the finite Betti number is an ANR. Hence by the assumption, Theorem 1.1 (2) and Lemma 3.1,

\[
C_p(X) \cong C_p(L) \times C_p(X/L) \cong C_p(I) \times C_p(X/L) \cong C_p(I) \times C_p(I) 
\]

Therefore \( X \) is also \( l \)-equivalent to \( I \). The induction is completed.

3.4. Corollary. Let \( X \) be a dendrite. If there exists an increasing finite sequence \( X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{n+1} = X, n \geq 0, \) of subcontinua of \( X \) such that

1. \( X \) has at most finite ramification points, and
2. for \( i = 0, 1, \ldots, n \), the continuum \( X_{i+1}/X_i \) has at most finite ramification points,

then \( X \) is \( l \)-equivalent to \( I \).

Next, we will give other curves which are \( l \)-equivalent to \( I \).

3.5. Theorem. Every continuum which is a one-to-one continuous image of \([0, \infty)\) is \( l \)-equivalent to \( I \).

Proof. Let \( X \) be a continuum which admits a bijective map \( f : [0, \infty) \to X \). Then by [9, Structure Theorem and its Remark], \( X \) can be written in the form \( X = \alpha \cup C \cup L \), where \( \alpha \) is an arc or a point, \( C \) is an arc-like continuum with at most two arc-components, \( L \) is an arc, \( L \cap C \) is exactly the two non-cutpoints of \( L \) which are also opposite endpoints of \( C \), and \( \alpha \cap (C \cup L) \) is a single point of \( C \) which is a non-cutpoint of \( \alpha \) and which, if \( C \) is not an arc (i.e., \( C \cup L \) is not a simple closed curve), is the non-cutpoint not in \( L \cap C \) of the arc-component of \( C \) which is an arc. In fact, by the proof, there are real numbers \( 0 \leq a \leq b < c \) such that \( \alpha = f([0, a]) \), \( C = f([a, b]) \cup f([c, \infty)) \) and \( L = f([b, c]) \).

If \( a = b \), namely, \( C \cup L \) is a simple closed curve, by Theorem 1.1 (2), \( X \) is \( l \)-equivalent to \( I \). So we may assume that \( a < b \). Let define

\[
X_1 = \alpha \cup C, \\
X_2 = f([0, d]) \text{, where } d \text{ is an arbitrary real number with } d > c.
\]

Then by Lemma 3.1 (2) and (4),
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$C_p(X) \cong C_p(f([0, b])); C_p(X, f([0, b])); [f([0, b])]$

$\equiv C_p(I) \times C_p(I; 0)$

$\equiv C_p(I)$

Note that $X = X_1 \cup X_2$ and $X_0 = X_1 \cap X_2$ is a disjoint union of two arcs. Hence by Lemma 3.1 (3) and (4),

$$C_p(X) \equiv C_p(X_1) \times C_p(X_2) \equiv C_p(I) \times C_p(I) \equiv C_p(I)$$

Therefore such a curve $X$ is $l$-equivalent to $I$.

3.6. COROLLARY. Every continuum which is a one-to-one continuous image of the real line $R$ is $l$-equivalent to $I$.

Curves described in Theorem 3.5 and Corollary 3.6 are called half-real curves and real curves, respectively [10]. By Theorem 3.5 and Corollary 3.6, we see that the property of being $l$-equivalent to $I$ does not imply even local connectivity. Hence Theorem 2.1 and Theorem 2.3 may be interesting properties. As mentioned in Introduction, for each $n \geq 1$, there exist uncountable many $n$-dimensional compact AR's which are not $l$-equivalent to each other. Hence characterizations of continua or compact AR's which are $l$-equivalent to $I^*$ are important. In the case of curves we pose the following problem related to our result:

**Problem.** Characterize dendrites which are $l$-equivalent to $I$. Particularly, is the converse of Corollary 3.4 valid?

References


[9] Nadler, Jr., S.B., Continua which are a continuous one-to-one image of $[0, \infty)$, Fund. Math. 75 (1972), 123-133.
Akira Koyama and Toshinao Okada


A. KOYAMA
Department of Mathematics
Osaka Kyoiku University
Ikeda-city, Osaka, 563
Japan

T. OKADA
Department of Mathematics
Faculty of Education
Kagawa University
Takamatsu-city, Kagawa, 760
Japan