UNIVERSAL TRANSITIVITY OF CERTAIN CLASSES OF REDUCTIVE PREHOMOGENEOUS VECTOR SPACES

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Introduction.

Let \( k \) be a field of characteristic zero. Let \( \tilde{G} \) be a connected \( k \)-split linear algebraic group, \( \rho: \tilde{G} \to GL(X) \) with \( X = \text{Aff}^n \) a \( k \)-homomorphism. If there exists a Zariski-dense \( \rho(\tilde{G}) \)-orbit \( Y \), we say that \( \langle \tilde{G}, \rho, X \rangle \) is a prehomogeneous vector space (abbrev. P. V.). When each irreducible component is castling equivalent to a non-trivial reduced irreducible prehomogeneous vector space or each irreducible component is a regular prehomogeneous vector space, we have completed a classification of reductive prehomogeneous vector spaces over a complex number field \( \mathbb{C} \) (see [4], [5]).

Put \( G = \rho(\tilde{G}) \). Let \( l \) be the number of \( G(k) \)-orbits in \( Y(k) \), i.e., \( l = |G(k) \backslash Y(k)| \). We say that \( Y \) is a universally transitive open orbit if \( l = |G(k) \backslash Y(k)| = 1 \) for all \( k \) satisfying \( H^1(k, \text{Aut}(SL_2)) \neq 0 \), i.e., there exists a nonsplit quaternion \( k \)-algebra. This condition is satisfied by every local field \( k \) other than \( \mathbb{C} \). Actually our classification depends on the transitivity of \( G(k) \) on \( Y(k) \) for just one \( k \) satisfying \( H^1(k, \text{Aut}(SL_2)) \neq 0 \) (see Remarks 2.13 and 3.5). In [1] and [2], all irreducible regular prehomogeneous vector spaces with universally transitive open orbits are classified. In [10], we have classified simple or 2-simple prehomogeneous vector spaces with universally transitive open orbits.

In this paper, we shall classify reductive prehomogeneous vector spaces with universally transitive open orbits when each irreducible component is castling equivalent to a non-trivial reduced irreducible prehomogeneous vector space or each irreducible component is a regular prehomogeneous vector space.

This paper consists of the following three sections.

§1. Preliminaries.

§2. Reductive P. V.'s with universally transitive open orbits: the case I.

§3. Reductive P. V.'s with universally transitive open orbits: the case II.

The results are given in Theorems 2.11, 3.4 and Corollary 2.12.

§ 1. Preliminaries.

**Proposition 1.1.** We have $l=1$ for $(\tilde{G}, \rho_1 \oplus \rho_3, X_1 \oplus X_3)$ if and only if (1) $l=1$ for $(\tilde{G}, \rho_1, X_1)$ and (2) $l=1$ for $(H, \rho_2|_H, X_3)$ where $H$ is a generic isotropy subgroup of $(\tilde{G}, \rho_1, X_1)$.

**Proof.** See Proposition 1.5 in [10]. Q.E.D.

**Corollary 1.2.** Assume that $l=1$ for $(\tilde{G}, \rho_1, X_1)$ and $(H^0, \rho_1|_{H^0}, X_2)$ where $H^0$ is the connected component of a generic isotropy subgroup $H$ of $(\tilde{G}, \rho_1, X_1)$. Then we have $l=1$ for $(\tilde{G}, \rho_1 \oplus \rho_3, X_1 \oplus X_3)$.

**Remark 1.3.** Assume that $l=1$ for $(G, \rho, X)$. Then $l=1$ for $(\tilde{G}, \tilde{\rho}, X)$ with $\tilde{\rho}(\tilde{G}) \supset \rho(G)$.

**Proposition 1.4.** The number $l=|G(k) \backslash Y(k)|$ is invariant under a castling transformation.

**Proof.** See [2]. Q.E.D.

**Theorem 1.5.** ([1], [2]) A regular irreducible P.V. has a universally transitive open orbit (i.e. $l=1$) if and only if it is castling equivalent to one of the following P.V.'s.

(1.1) $(H \times GL(n), \rho \otimes A_i)$ where $\rho$ is an $n$-dimensional irreducible representation of $H$.

(1.2) $(GL(2m), A_2)$ with $m \geq 2$.

(1.3) $(Sp(n) \times GL(2m), A_4 \otimes A_3)$ with $n \geq 2m$.

(1.4) $(GL(1) \times SO(2n), A_4 \otimes A_3)$ with $n \geq 2$.

(1.5) $(GL(1) \times Spin(7), A_4 \otimes Spin$ rep.).

(1.6) $(GL(1) \times Spin(9), A_4 \otimes Spin$ rep.).

(1.7) $(Spin(10) \times GL(2), \text{half-spin rep.} \otimes A_4)$.

(1.8) $(GL(1) \times E_6, A_4 \otimes A_3)$.

**Theorem 1.6.** Any non-regular irreducible P.V., which is not castling equivalent to $(Sp(n) \times GL(2), A_4 \otimes 2A_3)$, has the universally transitive open orbit.

**Proof.** See Corollary 3.22 in [10]. Q.E.D.
§ 2. Reductive P.V.'s with universally transitive open orbits: the case I.

In this section, we shall consider the case when each irreducible component is castling equivalent to a non-trivial reduced irreducible P.V.

**Theorem 2.1. ([4])** Let \((G, \rho, V)\) be an indecomposable reductive P.V. Assume that each irreducible component of \((G, \rho, V)\) is castling equivalent to a non-trivial reduced irreducible P.V. If \((G, \rho, V)\) does not contain an irreducible P.V. with \(l \geq 2\), then it is castling equivalent to one of the following P.V.'s.

1. \((GL(1)^{\delta} \times SL(2m+1), A_1 + A_2)\) with \(m \geq 2\).
2. \((GL(1)^{\delta} \times Spin(8), \text{half-spin rep. + vector rep.})\).
3. \((GL(1)^{\delta} \times Spin(10), \text{half-spin rep. + vector rep.})\).
4. \((GL(1)^{\delta} \times Spin(10), \text{even half-spin rep. + even half-spin rep.})\).
5. \((GL(1)^{\delta} \times Sp(n) \times SL(m), A_i \otimes 1 + A_i \otimes A_i)\) with \(n \geq m \geq 1\).
6. \((GL(1)^{\delta} \times Sp(n) \times SL(2m+1), A \otimes A_1 \otimes 1 + A \otimes A_i)\) with \(2n > 2m + 1 \geq 5\).
7. \((GL(1)^{\delta} \times Sp(n) \times SL(2m+1), (A_1 + A_i) \otimes 1 + A_i \otimes A_i)\) with \(n \geq 2m + 1 \geq 1\).

(ii)

8. \((GL(1)^{\delta} \times Sp(n) \times SL(2m+1) \times Sp(n'), A_i \otimes A_i \otimes 1 + 1 \otimes A_i \otimes A_i)\) with \(2n, 2n' > 2m + 1 \geq 5\).
9. \((GL(1)^{\delta} \times Sp(n) \times SL(3) \times Sp(n'), A_i \otimes A_i \otimes 1 + 1 \otimes A_i^{(\delta)} \otimes A_i)\).
10. \((GL(1)^{\delta} \times Sp(n) \times SL(3) \times Sp(n'), A_i \otimes 1 \otimes 1 + A_i \otimes A_i \otimes 1 + 1 \otimes A_i^{(\delta)} \otimes A_i)\).
11. \((GL(1)^{\delta} \times Sp(n) \times Sp(n') \times SL(3) \times Sp(m), (A_i \otimes 1 + 1 \otimes A_i) \otimes A_i \otimes 1 + 1 \otimes 1 \otimes A_i^{(\delta)} \otimes A_i)\).
12. \((GL(1)^{\delta} \times Sp(n) \times Sp(n') \times SL(3) \times Sp(m), (A_i \otimes 1 + 1 \otimes A_i) \otimes A_i \otimes 1 + 1 \otimes 1 \otimes (A_i^{(\delta)} \otimes A_i + 1 \otimes A_i))\).
13. \((GL(1)^{\delta} \times Sp(n) \times Sp(n') \times Sp(n') \times SL(3) \times Sp(m), (A_i \otimes 1 \otimes 1 + 1 \otimes A_i \otimes 1 + 1 \otimes 1 \otimes A_i) \otimes A_i \otimes 1 + 1 \otimes 1 \otimes 1 \otimes A_i^{(\delta)} \otimes A_i)\).

(iii)

14. \((GL(1)^{\delta} \times SL(2m) \times Sp(n) \times SL(2m' + 1), A_i \otimes A_i \otimes 1 + 1 \otimes A_i \otimes A_i)\) with \(n \geq 2m, 2m' + 1 \geq 2\).
(2.15) \((GL(1)^3 \times SL(2m) \times Sp(n) \times SL(3) \times Sp(n'))\),
\(A_i \otimes A_i \otimes 1 \otimes 1 \otimes A_i \otimes 1 \otimes 1 \otimes A_i \otimes A_i\) with \(n \geq 2m \geq 2\).

(2.16) \((GL(1)^4 \times SL(2m+1) \times Sp(n) \times SL(2) \times Sp(n') \times SL(2m'+1))\),
\(A_i \otimes A_i \otimes 1 \otimes 1 \otimes A_i \otimes A_i \otimes 1 \otimes 1 \otimes A_i \otimes 1 \otimes 1 \otimes A_i \otimes A_i\) with \(n \geq 2m+1 \geq 1\) and \(n' \geq 2m'+1 \geq 1\).

(2.17) \((GL(1)^3 \times SL(2m+1) \times Sp(n) \times SL(3) \times Sp(n'))\),
\(A_i \otimes A_i \otimes 1 \otimes 1 \otimes A_i \otimes A_i \otimes 1 \otimes 1 \otimes A_i \otimes 1 \otimes 1 \otimes A_i \otimes A_i\) with \(n \geq 2m+1 \geq 1\).

(2.18) \((GL(1)^4 \times SL(2m+1) \times Sp(n) \times SL(2) \times Sp(n') \times SL(2m'+1))\),
\(SL(2m'+1)\),
\(A_i \otimes A_i \otimes 1 \otimes 1 \otimes A_i \otimes A_i \otimes 1 \otimes 1 \otimes A_i \otimes 1 \otimes 1 \otimes A_i \otimes A_i\) with \(n \geq 2m+1 \geq 1\) and \(n' \geq 2m'+1 \geq 1\).

(2.19) \((GL(1)^i \times SL(2) \times H_i \times \cdots \times H_i)\),
\(A_i \otimes (\tau_1 \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes \tau_1 \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes \tau_i)\)
where \((H_j, \tau_j)\) is one of \((SL(2n+1), A_2)\) \((n \geq 2)\), \((Sp(n), A_1)\) \((n \geq 2)\) and \((Spin(10), \text{half-spin rep.})\).

(2.20) \((GL(1)^{i+k} \times (G' \times SL(2)) \times (H_i \times \cdots \times H_i))\),
\((\rho_1 + \cdots + \rho_n) \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes A_i \otimes (\tau_1 \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes \tau_i)\)
where \((GL(1)^k \times G' \times SL(2), \rho_1 + \cdots + \rho_n)\) is one of \((2.14)\) with \(m = 1\), \((2.15)\), \((2.16)\), \((2.17)\) and \((2.18)\), and \((H_j, \tau_j)\) is one of \((SL(2n+1), A_2)\) \((n \geq 2)\), \((Sp(n), A_1)\) \((n \geq 2)\) and \((Spin(10), \text{half-spin rep.})\).

**Proposition 2.2.** We have \(l=1\) for P.V.'s in (i), i.e., \((2.1)\sim(2.7)\) in Theorem 2.1.

**Proof.** By [10], we obtain our assertion. Q.E.D.

**Proposition 2.3.** We have \(l=1\) for P.V.'s in (ii), i.e., \((2.8)\sim(2.13)\) in Theorem 2.1.

**Proof.** By Corollary 3.8 in [10], we have \(l=1\) for \((2.8)\) (resp. \((2.9), (2.10), (2.11), (2.12), (2.13)\)) if and only if \(l=1\) for \((GL(1)^3 \times SL(2m+1), A_3 + A_2)\) (resp. \((GL(1)^3 \times SL(3), A_1 + A_1^{(*)})\), \((GL(1)^3 \times SL(3), A_1 + A_1^{(*)})\), \((GL(1)^3 \times SL(3), A_1 + A_1^{(*)})\), \((GL(1)^3 \times SL(3), A_1 + A_1^{(*)})\), \((GL(1)^3 \times SL(3), A_1 + A_1^{(*)})\).
Universal Transitivity of Reductive P. V.'s


PROPOSITION 2.4. We have \( l = 1 \) for a P. V. (2.14) in Theorem 2.1.

Proof. By Corollary 3.8 in [10], we have \( l = 1 \) for a P. V. (2.14) \( \cong \) (Sp(n) × (GL(2m) × GL(2m' + 1)), \( A \otimes (A \otimes 1 + A \otimes A) \) if and only if \( l = 1 \) for (GL(2m) × GL(2m' + 1), \( A \otimes 1 + A \otimes A_1 + 1 \otimes A_2 \)). Since an irreducible P. V. (GL(2m), \( A_2 \)) has a universally transitive open orbit and a generic isotropy subgroup of (GL(2m), \( A_2 \)) is isomorphic to Sp(m), it is enough to show that (Sp(n) × (GL(2m) × GL(2m' + 1), \( A \otimes 1 + A \otimes A_1 + 1 \otimes A_2 \)) has a universally transitive open orbit. By Proposition 4.4 in [10], we may assume that \( 2m > 2m' + 1 \). Then, by Corollary 3.8 in [10], it has a universally transitive open orbit if and only if (GL(2m' + 1), \( A_1 + A_2 \)) has a universally transitive open orbit. By the proof of Proposition 2.15 in [10], we have \( l = 1 \) for (GL(2m' + 1), \( A_1 + A_2 \)). Q. E. D.

PROPOSITION 2.5. We have \( l = 1 \) for a P. V. (2.15) in Theorem 2.1.

Proof. By Corollary 3.8 in [10], we have \( l = 1 \) for a P. V. (2.15) if and only if \( l = 1 \) for (GL(1) × SL(2m) × Sp(n) × SL(3), \( A \otimes A \otimes 1 + 1 \otimes A \otimes A_1 + 1 \otimes A_2 \)). By the same argument as in the proof of Proposition 2.4, it is equivalent to say that \( l = 1 \) for a P. V. (GL(3) × GL(1), \( A_1 + A_2 \otimes 1 + A_3 \otimes A_3 \) \( \cong \) (GL(3) × GL(1), \( A \otimes (1 + A_1) \)) if \( m \geq 2 \), (SL(2) × GL(3) × GL(1), \( A \otimes A_1 + 1 \otimes A_2 \otimes 1 + 1 \otimes A \otimes A_3 \)) if \( m = 1 \) and \( n \geq 3 \), (GL(1) × SL(2) × Sp(2) × GL(3) × GL(1), \( A \otimes A_1 \otimes A_2 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes A \otimes A_3 \)) if \( m = 1 \) and \( n = 2 \). By Lemma 2.2 in [10], we have \( l = 1 \) for (GL(3) × GL(1), \( A \otimes (1 + A_1) \)). By the proof of Proposition 2.4, (SL(2) × GL(3), \( A \otimes A_1 + 1 \otimes A_2 \)) has a universally transitive open orbit, and the (GL(3))-part of its generic isotropy subgroup is locally isomorphic to \( \left\{ \begin{array}{ccc} -\alpha & \beta & \gamma \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{array} \right\} \). Thus we have \( l = 1 \) for (SL(2) × GL(3) × GL(1), \( A \otimes A_1 + 1 \otimes A_2 \otimes 1 + 1 \otimes A \otimes A_3 \)). Also, by Proposition 2.4, we have \( l = 1 \) for (GL(1) × SL(2) × Sp(2) × GL(3), \( A \otimes A \otimes A_1 \otimes 1 + 1 \otimes 1 \otimes A_2 \otimes A_3 \)), and the (GL(3))-part of its generic isotropy subgroup is locally isomorphic to \( \left\{ \begin{array}{ccc} -\alpha & \beta & \gamma \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{array} \right\} \). Thus we have \( l = 1 \) for (GL(1) × SL(2) × Sp(2) × GL(3) × GL(1), \( A \otimes A \otimes A \otimes A_1 \otimes 1 + 1 \otimes 1 \otimes A_2 \otimes A_3 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes A \otimes A_3 \)). Q. E. D.

LEMMA 2.6. We have \( l = 1 \) for a P. V. (GL(1) × H × SL(2), \( \rho \otimes \tau + 1 \otimes A_3 \)) if and only if \( l = 1 \) for a P. V. (GL(1) × H × SL(2) × Sp(n) × SL(2m + 1), \( \rho \otimes \tau \otimes 1 + 1 \otimes A_1 \otimes A_1 + 1 \otimes 1 \otimes A_1 \otimes A_1 \otimes A_4 \)) \( (n > m \geq 0) \).
Proof. By calculations similar to the proof of Lemma 2.7 in [10], we obtain that a generic isotropy subgroup of \((\text{Sp}(n) \times \text{GL}(2m), A_1 \otimes A_1)\) is isomorphic to \(\{(\begin{pmatrix} A_1 & 0 \\ 0 & B \end{pmatrix}, A_1^-) \mid A_1 \in \text{Sp}(m), B \in \text{Sp}(n-m)\} \cong \text{Sp}(m) \times \text{Sp}(n-m)\) (cf. p. 101 in [13]). Thus, by Proposition 1.1, we have \(l=1\) for a P.V. \((\text{GL}(1)^3 \times \text{H} \times \text{SL}(2) \times \text{Sp}(n) \times \text{SL}(2m+1), A_1 \otimes A_1 \otimes A_1 \otimes 1 + 1 \otimes A_1 \otimes A_1 \otimes 1 + 1 \otimes A_1 \otimes A_1 \otimes A_1)\) if and only if \(l=1\) for a P.V. \((\text{GL}(1)^3 \times \text{H} \times \text{SL}(2) \times \text{Sp}(n-1) \times \text{GL}(2m+1), A_1 \otimes \rho \otimes \tau \otimes 1 \otimes 1 \otimes 1 \otimes (A_1 \otimes A_1 \otimes A_1 \otimes 1 + 1 \otimes A_1 \otimes A_1 \otimes A_1)\). By Corollary 3.8 and Lemma 4.3 in [10], a P.V. \((\text{GL}(1)^3 \times \text{H} \times \text{SL}(2) \times \text{GL}(2m+1) \times \text{Sp}(n-1), A_1 \otimes \rho \otimes \tau \otimes 1 \otimes 1 \otimes (A_1 \otimes A_1 \otimes A_1 \otimes 1 + 1 \otimes A_1 \otimes A_1 \otimes A_1)\) has a universally transitive open orbit if and only if a P.V. \((\text{GL}(1)^3 \times \text{H} \times \text{SL}(2) \times \text{GL}(2m+1), A_1 \otimes \rho \otimes \tau \otimes 1 + 1 \otimes (A_1 \otimes A_1 \otimes A_1 \otimes 1 + 1 \otimes (A_1 \otimes A_1 \otimes A_1)\) has a universally transitive open orbit. By Proposition 4.4 in [10], it is equivalent to say that a P.V. \((\text{GL}(1)^3 \times \text{H} \times \text{SL}(2), \rho \otimes \tau + 1 \otimes A_1)\) has a universally transitive open orbit. Hence we obtain our assertion.

Q.E.D.

Proposition 2.7. We have \(l=1\) for a P.V.'s (2.16) and (2.18) in Theorem 2.1.

Proof. By Theorem 2.19 in [10], a P.V. \((\text{GL}(1)^3 \times \text{SL}(2), A_1 + A_1)\) (resp. \((\text{GL}(1)^3 \times \text{SL}(2), A_1 + A_1 + A_1)\)) has a universally transitive open orbit, and hence, by Lemma 2.6, we have \(l=1\) for (2.16) (resp. (2.18)).

Q.E.D.

Proposition 2.8. We have \(l=1\) for a P.V. (2.17) in Theorem 2.1.

Proof. By Lemma 2.6 and Corollary 3.8 in [10], we have \(l=1\) for a P.V. (2.17) if and only if \(l=1\) for \((\text{GL}(1)^3 \times \text{SL}(2) \times \text{Sp}(n') \times \text{SL}(3), A_1 \otimes 1 \otimes 1 + 1 \otimes A_1 \otimes A_1 \otimes 1 + 1 \otimes A_1 \otimes A_1)\). By the proof of Lemma 2.6, it is enough to show that \(l=1\) for a P.V. \((\text{GL}(1)^3 \times \text{SL}(2) \times \text{GL}(3) \times \text{GL}(1), A_1 \otimes A_1 \otimes 1 \otimes 1 \otimes (A_1 \otimes A_1 + 1 \otimes A_1 \otimes 1 \otimes A_1)\) (if \(n \geq 3\)), \((\text{GL}(1)^3 \times \text{SL}(2) \times \text{Sp}(2) \times \text{GL}(3) \times \text{GL}(1), A_1 \otimes A_1 \otimes A_1 \otimes 1 \otimes 1 \otimes (A_1 \otimes A_1 \otimes A_1)\) (if \(n=2\)). By the proof of Proposition 2.5, \((\text{SL}(2) \times \text{GL}(3) \times \text{GL}(1), (A_1 \otimes A_1 \otimes 1 \otimes A_1 \otimes 1 \otimes A_1)\) and \((\text{GL}(1)^3 \times \text{SL}(2) \times \text{Sp}(2) \times \text{GL}(3) \times \text{GL}(1), A_1 \otimes A_1 \otimes 1 \otimes 1 \otimes A_1 \otimes A_1 \otimes 1 \otimes A_1 \otimes A_1)\) have universally transitive open orbits, and the \(\text{SL}(2)\)-parts of its generic isotropy subgroups are locally isomorphic to \(\text{SO}(2)\). Thus we have \(l=1\) for \((\text{GL}(1)^3 \times \text{SL}(2) \times \text{GL}(3) \times \text{GL}(1), A_1 \otimes A_1 \otimes 1 \otimes 1 \otimes (A_1 \otimes A_1 + 1 \otimes A_1) \otimes 1 \otimes 1 \otimes A_1 \otimes A_1)\) and \((\text{GL}(1)^3 \times \text{SL}(2) \times \text{Sp}(2) \times \text{GL}(3) \times \text{GL}(1), A_1 \otimes A_1 \otimes 1 \otimes 1 \otimes A_1 \otimes A_1 \otimes 1 \otimes 1 \otimes A_1 \otimes A_1)\).

Q.E.D.
Lemma 2.9. We have \( l=1 \) for a P. V. \((GL(1)^{k+1} \times (G' \times SL(2))) \times (H_1 \times \cdots \times H_i), (\rho_1 + \cdots + \rho_i) \otimes (1 \otimes \cdots \otimes 1) + (1 \otimes A_1) \otimes (1 \otimes \cdots \otimes 1 + 1 \otimes \tau_1 \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes \tau_i) \) if and only if \( l=1 \) for a P. V. \((GL(1)^{k} \times G' \times SL(2), \rho_1 + \cdots + \rho_i)\), where \((H_j, \tau_j) (1 \leq j \leq i)\) is one of \((SL(2n+1), A_{2}) (n \geq 2), (Sp(n), A_{1}) (n \geq 2)\) and \((Spin(10), \text{half-spin rep.})\).

Proof. We have \( l=1 \) for irreducible P. V.'s \((SL(2n+1) \times GL(2), A_{2} \otimes A_{1})\), \((Sp(n) \times GL(2), A_{1} \otimes A_{1})\) and \((Spin(10) \times GL(2), \text{half-spin rep.} \otimes A_{1})\), and each \(GL(2)\)-part of its generic isotropy subgroup contains \(SL(2)\) (see [13]). Hence, by Proposition 1.1 and Corollary 1.2, we obtain our assertion. Q.E.D.

Proposition 2.10. We have \( l=1 \) for P. V.'s (2.19) and (2.20)

Proof. By Propositions 2.4, 2.5, 2.7, 2.8 and Lemma 2.9, we obtain our assertion. Q.E.D.

Theorem 2.11. Let \((G, \rho, V)\) be an indecomposable reductive P. V. Assume that each irreducible component of \((G, \rho, V)\) is castling equivalent to a non-trivial reduced irreducible P. V. Then we have \( l=1 \) for \((G, \rho, V)\) if and only if \((G, \rho, V)\) does not contain any irreducible P. V. with \( \geq 2\), namely, it is castling equivalent to one of P. V.'s (2.1)~(2.20) in Theorem 2.1.

Proof. By Propositions 2.2, 2.3, 2.4, 2.5, 2.7, 2.8, 2.10 and Theorem 2.1, we obtain our assertion. Q.E.D.

Corollary 2.12. Let \((G, \rho, V)\) be a regular indecomposable reductive P. V. with a universally transitive open orbit. If each irreducible component of \((G, \rho, V)\) is castling equivalent to a non-trivial reduced irreducible P. V., then \((G, \rho, V)\) is castling equivalent to one of the following P. V.'s.

1. \((GL(1)^{k} \times Sp(n) \times SL(2m+1), A_{2} \otimes 1 + A_{1} \otimes A_{1})\) with \( n \geq 2m+1 \geq 1 \).
2. \((GL(1)^{k} \times Spin(8), \text{half-spin rep.} + \text{vector rep.})\).
3. \((GL(1)^{k} \times Spin(10), \text{half-spin rep.} + \text{vector rep.})\).
4. \((GL(1)^{k} \times Spin(10), \text{even half-spin rep.} + \text{even half-spin rep.})\).
5. \((GL(1)^{k} \times SL(2) \times H_1 \times \cdots \times H_i, A_{i} \otimes (1 \otimes \cdots \otimes 1 + 1 \otimes \tau_1 \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes \tau_i))\), where \((H_j, \tau_j) (1 \leq j \leq i)\) is one of \((Sp(n), A_{1}) (n \geq 2)\) and \((Spin(10), \text{half-spin rep.})\).

Remark 2.13. In [2], it is proved that, for \((GL(7), A_{1})\), \(Y(k)\) is \(G(k)\)-transitive for any local field \(k\) other than \(R\). And, \((GL(7), A_{1})\) is the only one irreducible P. V. which depends on the transitivity of \(G(k)\) on \(Y(k)\) for just one
§3. Reductive P.V.'s with universally transitive open orbits: the case II.

In this section, we shall consider the case when each irreducible component is a regular P.V.

**Proposition 3.1.** We have \( l = 1 \) for a P.V. \( (GL(1)^k \times G' \times SL(2), \rho_1 + \cdots + \rho_k) \) if and only if \( l = 1 \) for a P.V. \( (GL(1)^{k+1} \times (G' \times SL(2)) \times H, (\rho_1 + \cdots + \rho_k) \otimes 1 + (1 \otimes A_1) \otimes \tau) \), where \( (H, \tau) \) is one of \( (SL(2), A_1), (Sp(n), A_1) \) \((n \geq 2)\) and \( (Spin(10), \text{half-spin rep.})\).

**Proof.** By Theorem 1.5, irreducible P.V.'s \( (SL(2) \times GL(2), A_1 \otimes A_1), (Sp(n) \times GL(2), A_1 \otimes A_1) \) and \( (Spin(10) \times GL(2), \text{half-spin rep.}) \) have universally transitive open orbits, and each \( GL(2) \)-part of its generic isotropy subgroup contains \( SL(2) \) (see [13]). Hence, by Proposition 1.1 and Corollary 1.2, we obtain our assertion. Q.E.D.

**Proposition 3.2.** We have \( l = 1 \) for a P.V. \( (GL(1)^k \times H, \rho_1 + \cdots + \rho_k) \) if and only if \( l = 1 \) for a P.V. \( (GL(1)^{k+1} \times H \times SL(n), (\rho_1 + \cdots + \rho_k) \otimes 1 + \tau \otimes A_1) \), where \( \tau \) is an \( n \)-dimensional irreducible representation of a connected semi-simple algebraic group \( H \) satisfying the following condition: there is a simple normal algebraic subgroup \( K \) of \( H \) such that \( \tau|_K \neq 1 \) and \( \rho_i|_K \neq 1 \) for some \( i \) \((1 \leq i \leq k)\).

**Proof.** Since an irreducible P.V. \( (H \times GL(n), \tau \otimes A_1) \) \((\deg \tau = n)\) has a universally transitive open orbit and a generic isotropy subgroup of it is isomorphic to \( H \), we obtain our assertion by Proposition 1.1 and Corollary 1.2. Q.E.D.

**Proposition 3.3.** We have \( l = 1 \) for a P.V. \( (GL(1)^{k+1} \times H \times SL(n), (\rho_1 + \cdots + \rho_k) \otimes 1 + \tau \otimes A_1) \) if and only if \( l = 1 \) for a P.V. \( (GL(1)^{k+1} \times H \times SL(n), \sigma \otimes 1 + \tau \otimes A_1) \), where \( (GL(1)^k \times H, \sigma) \) is a direct sum of \( (GL(1)^k \times H', \rho_1 + \cdots + \rho_k) \) and a P.V. \( (GL(1)^k \times H', \rho_{k+1} + \cdots + \rho_s) \) \((t = k - s \geq 1)\) with \( l = 1 \) and \( \tau \) is an \( n \)-dimensional irreducible representation of \( H \) satisfying the following condition: there is a simple normal algebraic subgroup \( K' \) of \( H' \) such that \( \tau|_{K'} \neq 1 \) and \( \rho_i|_{K'} \neq 1 \) for some \( i \) \((s+1 \leq i \leq k)\).

**Proof.** By Proposition 3.2, we obtain our assertion. Q.E.D.
Universal Transitivity of Reductive P. V.'s

**Theorem 3.4.** Let \((G, \rho, V)\) be an indecomposable reductive P. V. with a universally transitive open orbit. If all irreducible components of \((G, \rho, V)\) are regular P. V.'s, then \((G, \rho, A)\) is obtained from the following P. V.'s (1)\(\sim\)(9) by a finite number of transformations in Propositions 3.1, 3.2, 3.3 and castling transformations (see Theorem 2.5 in [5]).

1. \((H \times GL(n), \rho \otimes A_1)\) where \(\rho\) is an \(n\)-dimensional irreducible representation of \(H\).
2. \((GL(2m), A_2)\) with \(m \geq 2\).
3. \((Sp(n) \times GL(2m), A_1 \otimes A_1)\) with \(n \geq 2m\).
4. \((GL(1) \times SO(2n), A_1 \otimes A_1)\) with \(n \geq 2\).
5. \((GL(1) \times Spin(7), A_1 \otimes spin rep.)\).
6. \((GL(1) \times Spin(9), A_1 \otimes spin rep.)\).
7. \((Spin(10) \times GL(2), \text{half-spin rep.} \otimes A_1)\).
8. \((GL(1) \times E_6, A_1 \otimes A_1)\).
9. \((GL(1)^3 \times Spin(8), \text{half-spin rep.} + \text{vector rep.})\).

**Proof.** By Theorem 2.5 in [5] and Theorem 1.5, Propositions 2.2, 3.1, 3.2, 3.3, we obtain our assertion. Q.E.D.

**Remark 3.5.** In our case II, there are P. V.'s which depend on the transitivity of \(G(k)\) on \(Y(k)\) for just one \(k\) satisfying \(H^\vee(k, Aut(SL_2)) \neq 0\). We can obtain a such P. V. from \((GL(7), A_1)\) and P. V.'s (1)\(\sim\)(9) in Theorem 3.4 by same procedures as in Theorem 3.4. Because \((GL(7), A_1)\) is the only one regular irreducible P. V. which depends on the transitivity of \(G(k)\) on \(Y(k)\) for just one \(k\) satisfying \(H^\vee(k, Aut(SL_2)) \neq 0\).

**References**


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