KILLING VECTOR FIELDS AND THE HOLONOMY ALGEBRA IN SEMIRIEMANNIAN MANIFOLDS

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Abstract In this paper we generalize some results of Kostant [2] to semiriemannian manifolds of signature s. We also prove that any Killing vector field on a semiriemannian homogeneous compact flat manifold is parallel.

0. Introduction.

Let \((M^n, g)\) be a semiriemannian manifold of dimension \(n\) and signature \(s\). Let \(X\) be a Killing vector field on \(M\). The \(A_X\)-operator provides a skew symmetric endomorphism of \(TM\). It is well known that

\[\nabla_Y A_X = R_{XY}.\]

This fact and the Ambrossse-Singer theorem (Wo) show that the \(A_X\)-operator lies infinitesimally in the holonomy algebra \(h\) of \(M\).

We ask ourselves whether or not \(A_X\) lies in \(h\).

In the riemannian case the question has an affirmative answer on compact manifolds [2]. We obtain here a similar result in the semiriemannian case.

Finally we study the holonomicity of a Killing vector field on semiriemannian manifolds of constant curvature. If the curvature is non zero, the holonomy algebra can be represented as \(\text{so}(n, s)\), that is the skew symmetric endomorphisms of \(TM\). In this case each Killing vector field is holonomic.

There are flat manifolds and Killing vector fields on them such that the \(A_X\)-operator does not lie in the holonomy algebra, that is \(A_X \notin h\). Take, for instance, \(R^n\). In the usual coordinates on \(R^n\), \(X\) is a Killing vector field if

\[X = \sum_{i,j} \varepsilon_i K_i^j x_i \frac{\partial}{\partial x_j}\]

where \(K_i^j = -K_j^i\) are constants, \(\varepsilon_i = g(\partial/\partial x_i, \partial/\partial x_i) = \pm 1\) and \(x_n = 1\). There are nonholonomic Killing vector fields on \(R^n\): nonparallel vector fields are nonholo-
nomic because flatness implies $h=0$.

However, the assumption of compactness and homogeneity of $M$ allows us to state that any Killing vector field on a compact homogeneous semiriemannian flat manifold is parallel.

1. Main theorem.

Let $(M^n, g)$ be a semiriemannian manifold of dimension $n$ and signature $s$. If $p \in M$ and $A, B \in \text{End}(T_pM)$ we denote by $\phi$ the trace form

$$\phi(A, B) = -\text{trace}(A \cdot B).$$

Note that,

i) $\phi$ is nondegenerate on $\text{po}(n, s)$,

ii) $\phi$ is parallel.

From now on for any $\Omega \subset \text{po}(n, s)$, $\Omega^\perp$ will denote its orthogonal complementary with respect to $\phi$.

**Theorem 1.** Let $(M^n, g)$ be a semiriemannian manifold compact orientable manifold and $X$ a Killing vector field on $M$. If $\phi$ is nondegenerate on the holonomy algebra $h$, then the $A_X$-operator decomposes as

$$A_X = h + B_X$$

where $h \in h$, $B_X \in h^\perp$ and $\phi(B_X, B_X) = 0$.

**Proof.** The nondegenerate character of $\phi$ on $h$ allows us to decompose

$$\text{po}(n, s) = h + h^\perp$$

and $A_X = h + B_X$ in a unique way.

For any field $Y$ on $M$.

$$R_{XY} = \nabla_Y A_X = \nabla_Y h + \nabla_Y B_X.$$

$R_{XY}$ and $\nabla_Y h$ lie in $h$ and $\nabla_Y B_X$ lies in $h^\perp$. Thus $\nabla_Y B_X = 0$ and $B_X$ is parallel.

And accordingly

$$\text{div} B_X X = \text{trace}(B_X \cdot B_X) = \phi(B_X, B_X).$$

But $\phi(B_X, B_X)$ is constant because

$$Y \phi(B_X, B_X) = 2\phi(\nabla_Y B_X, B_X) = 0.$$

Finally, the integral of $\text{div} B_X X$ on $M$ gives
Killing Vector Fields and Holonomy Algebra

\[ 0 = \int_M \text{div} \langle B_X X \rangle = k \text{vol} (M). \]

That is

\[ 0 = k = \phi(B_X, B_X). \]

(Q. E. D.)

Remark. This theorem still holds without the assumption of orientability because the covering of the orientations is \( (2:1) \) and is also a local isometry.

We gave in [1] some examples of compact semiriemannian manifolds with nonholonomic Killing vector fields.

2. The flat case.

Let \( (M^*_1, g) \) be a compact flat manifold. If \( X \) is a Killing vector field on \( M \), by Theorem 1

\[ \phi(A_X, A_X) = 0. \]

On the assumption of homogeneity, we will see in this section that \( A_X = 0 \).

We recall

Lemma 2 [2]. Let \( (M^*_1, g) \) be a semiriemannian manifold; if \( X \) is a Killing vector field on \( M \), assume that

\[ 2f = g(X, X). \]

Then,

i) \[ \text{grad} f = A_X X. \]

ii) \[ H'(V, W) = g(\nabla_Y X, \nabla_W X) + g(R_{XY} X, W) = g(\nabla_V (A_X X), W) \]

iii) \[ \Delta f = -\phi(A_X, A_X) - \text{Ric} (X, X). \]

Proposition 3 (Marsden) [2]. A homogeneous compact semiriemannian manifold is complete.

Corollary 4. A homogeneous compact flat semiriemannian manifold is geodesically convex. (i.e. given any two points there is a geodesic joining them).

Proof. By Proposition 3 the universal covering of \( M \) is a flat complete simply connected manifold; thus it is \( \mathbb{R}^n \). In order to obtain a geodesic \( \sigma \) joining \( p \in M \) and \( q \in M \), take \( \tilde{p} \), in the fiber of \( p \) and \( \tilde{q} \) in the fiber of \( q \) and project on \( M \) the straight line \( \tilde{pq} \).

(Q. E. D.)

Proposition 5. Let \( X \) be a Killing vector field on a homogeneous compact flat semiriemannian manifold \( (M^*_1, g) \). The product \( g(X, X) \) is constant on \( M \).
PROOF. $A_X X$ is a Jacobi field.
Let $\gamma$ be a geodesic; then

$$\nabla_\gamma(\nabla_\gamma(A_X X)) = \nabla_\gamma((\nabla_\gamma A_X)X + A_X(\nabla_\gamma X)) = \nabla_\gamma(R_{X\gamma} X - A_X A_X \gamma')$$

$$= -\nabla_\gamma(A_X(A_X \gamma')) = -(\nabla_\gamma A_X)(A_X \gamma') - A_X(\nabla_\gamma(A_X \gamma'))$$

$$= R_{X\gamma}(A_X \gamma') - A_X((\nabla_\gamma A_X)(\gamma') - A_X A_X(\nabla_\gamma \gamma')) = 0.$$

Assume that $2f = g(X, X)$. Since $M$ is compact, $f$ reaches at least a maximum and a minimum at $p$ and $q$ respectively. By Corollary 4 there is a geodesic joining $p$ and $q$. Call it $\sigma$. $A_X X$ is a Jacobi field on $\sigma$ which cancels at $p$ and $q$ (Lemma 2). Because of the flatness of $M$, $A_X X = 0$.

Then $f(p) = f(q)$ and $f$ must be constant on $M$. (Q.E.D.)

THEOREM 6. Let $X$ be a Killing vector field on a semi-riemannian homogeneous compact flat manifold $M^n$. Then $X$ is parallel.

PROOF. 1st step. The universal covering of $M^n$ is $\mathbb{R}^n$. Then $M \cong \mathbb{R}^n / \Gamma$ where $\Gamma$ is a properly discontinuous subgroup of the motions of $\mathbb{R}^n$. Let $\tilde{X}$ be the lift of $X$ on $\mathbb{R}^n$; $\tilde{X}$ is a Killing vector field on $\mathbb{R}^n$ and it is $\Gamma$-invariant.

2nd step. Take $f = (1/2)g(X, X) = (1/2)g(\tilde{X}, \tilde{X})$.

From Lemma 2 and because of the flatness of $M$,

$$H'(V, W) = g(\nabla_V X, \nabla_W X) = g(A_X V, A_X W) = -g(A_X A_X V, W).$$

On the other hand, by Proposition 5

$$H'(V, W) = 0 \quad \forall V, W.$$ 

Thus $A_X \cdot A_X = 0$ and $A_\tilde{X} \cdot A_\tilde{X} = 0$.

3rd step. Let $p$ be a point of $\mathbb{R}^n$. We can choose a basis of $T_p M \ v_1, w_1, \ldots, v_r, w_r, u_1, \ldots, u_t$ in which the $A_\tilde{X}$ matrix has the form

$$\begin{bmatrix}
0 & v_1 & w_1 & v_r & w_r & u_1 & u_t \\
1 & 0 & \cdot & \cdot & \cdot & 0 \\
& & & & & 1 & 0 \\
& & & & & & 0 \\
& & & & & & \cdot \\
& & & & & & 0
\end{bmatrix}$$

(*)
Using parallel transport and because of the flatness we can assume that we have a coordinate system \( x_1, y_1, \ldots, x_r, y_r, z_1, \ldots, z_t \) on \( R^n_\mathbb{R} \) such that the matrix of the \( A_{x_i} \)-operator in the associated frame is (*) . In this coordinate system the nonparallel part of \( \tilde{X} \) is

\[
\tilde{X} = x_1 \frac{\partial}{\partial y_1} + \ldots + x_r \frac{\partial}{\partial y_r}.
\]

There is no lost of generality in assuming that

\[
\tilde{X} = x_1 \frac{\partial}{\partial y_1} + \ldots + x_r \frac{\partial}{\partial y_r}.
\]

4th Step. Let us now consider the new system

\[
(x_1, \ldots, x_r, y_1, \ldots, y_r, z_1, \ldots, z_t).
\]

Let \( \mu \) be an element of \( \Gamma \); because of the homogeneity of \( M^\mathbb{R}_r \), \( \Gamma \) is a group of pure translations (see [4] pg. 135). In our new coordinate system

\[
\mu = \begin{pmatrix} x & y & z \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} = \begin{pmatrix} M \\ N \end{pmatrix} + \begin{pmatrix} I \end{pmatrix} \begin{pmatrix} U \end{pmatrix}
\]

where \( I \) is the identity matrix.

The \( \Gamma \)-invariance of \( \tilde{X} \) is reflected on the \( \mu \)-matrix by the fact that \( M=0 \), so that the dimension of the subspace spanned by the translation components of the elements of \( \Gamma \) is not greater than \( n-r \). If \( r \neq 0 \) the translation components of the elements of \( \Gamma \) do not generate \( R^n_\mathbb{R} \). But this is impossible because \( M \) is compact.

Consequently, \( r=0 \) and \( \tilde{X} \) and \( X \) are parallel. (Q. E. D.)

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Bibliography


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