REFLEXIVE MODULES OVER QF-3' RINGS*

By

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Abstract. We characterize reflexive modules over QF-3' rings using a linear compactness condition relative to the Lambek torsion theory, and we also give a necessary and sufficient condition for a left QF-3' maximal quotient ring to be right QF-3'.

1. Introduction.

The problem of finding the reflexive modules over generalizations of QF rings (and, in particular, over QF-3 rings) has a long tradition. One of the first contributions is due to Morita [10], who determined the finitely generated reflexive modules over a right artinian QF-3 ring and, some years later, Masaike [8] extended this result by giving a characterization of reflexive modules over QF-3 rings with ACC (or DCC) on left annihilators. On the other hand, Müller [11] proved that if $rU_s$ is a bimodule that induces a Morita duality, then the $U$-reflexive modules are precisely the linearly compact modules and this applies, in particular, to the case in which $R=U$ is a PF ring. Recently, Masaike [9], extended this to QF-3 rings without chain conditions by showing that the reflexive modules over these rings are the modules of $R$-dominant dimension $\geq 2$ that satisfy a suitable linear compactness condition.

Recall that a ring is left QF-3 when it has a minimal faithful left module and left QF-3' when the injective envelope $E(R)$ is torsionless. When $R$ is left and right QF-3', we will simply say that it is a QF-3' ring (and a similar convention will be used for other classes of rings). QF-3' rings have been studied by a number of authors and their relation with Morita duality and the properties of the double dual functors has been analyzed by Colby and Fuller in a series of papers (see, e.g., [1] and its references). One of the aims of this paper is to show that a characterization of reflexive modules similar to Masaike's one may be given for the much larger class of QF-3' rings. In fact,

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we obtain a more general module-theoretic result that embraces also the theorem of Müller mentioned above. As a further application of the techniques developed here, we study the interplay between $R$ being right $QF$-3' and linear compactness conditions on the left, that leads to a necessary and sufficient condition for a left $QF$-3' ring to be right $QF$-3', and to a new one-sided characterization of $QF$-3 maximal quotient rings.

Throughout this paper, $R$ denotes an associative ring with identity and $R$-Mod (resp. Mod-$R$) the category of left (resp. right) $R$-modules. If $X$ and $M$ are left $R$-modules, $X$ is said to be finitely $M$-generated when it is a quotient of a finite direct sum of copies of $M$ and $X$ has $M$-dominant dimension \( \geq 2 \) (M-dom. dim $X \geq 2$) when there exists an exact sequence \( 0 \to X \to Y \to Z \), whith $Y$ and $Z$ isomorphic to direct products of copies of $X$.

We will call $\mathcal{E}_M$ to the localizing subcategory of $R$-Mod cogenerated by the injective envelope $E(M)$ of $M$. The corresponding quotient category of $R$-Mod will be denoted by $R$-Mod/$\mathcal{E}_M$ and its objects are precisely the modules of $E(M)$-dom. dim $\geq 2$. The most important case of this construction arises for $M=\Omega R$, and then $\mathcal{E}_M=\mathcal{L}$ is just the Lambek (or dense) localizing subcategory of $R$-Mod (see [15]).

2. Reflexive modules

We will fix a module $M \in R$-Mod and call $S=\text{End}_R(M)$. The $M$-dual functors $\text{Hom}_R(\_, M)$ and $\text{Hom}_S(\_, M)$ will be denoted by $(\_)^*$, and their composition in either order by $(\_)^{**}$. For each $X \in R$-Mod there is a canonical (evaluation) morphism $\sigma_X : X \to X^{**}$; $\sigma_X$ is a monomorphism precisely when $X$ is $M$-cogenerated and when $\sigma_X$ is an isomorphism, $X$ is said to be $M$-reflexive (or just reflexive if we take $M=\Omega R$).

We are interested in characterizing reflexive modules and, not surprisingly, a certain form of linear compactness plays a key role in this characterization. Recall from [3] that an object of a Grothendieck category $\mathcal{A}$ is said to be linearly compact when, for each inverse system \( \{ p_i : X_i \to X \} \) in $\mathcal{A}$ such that the $p_i$ are epimorphisms, the induced morphism $\varinjlim p_i : X \to \varinjlim X_i$ is also an epimorphism (this just gives ordinary linear compactness when $\mathcal{A}=R$-Mod). We will also use the following related concept (introduced by Hoshino and Takashima in [5]): An $R$-module $X$ will be called $\mathcal{E}_M$-linearly compact when, for each inverse system \( \{ p_i : X_i \to X \} \) in $R$-Mod such that the $X_i$ are $M$-cogenerated and $\text{Coker} p_i \in \mathcal{E}_M$, $\text{Coker} (\varinjlim p_i) \in \mathcal{E}_M$. It is not difficult to show that when every finitely $M$-generated submodule of $E(M)$ is $M$-cogenerated and $M$ is an object of $R$-
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Mod/\mathcal{T}_M (M rationally complete), then M is \mathcal{T}_M-linearly compact if and only if it is linearly compact in the category R-Mod/\mathcal{T}_M. When a module is \mathcal{L}-linearly compact, we will also say that it is Lambek linearly compact.

\mathcal{T}_M-linearly compact modules have the following useful property:

PROPOSITION 2.1. Let M be a left R-module such that each finitely M-generated submodule of E(M) is M-cogenerated. Then, for each \mathcal{T}_M-linearly compact R-module X, Coker \sigma_X \in \mathcal{T}_M.

PROOF. The proof is essentially the same of [5, Corollary 2.2], where this is shown in the case M=R. \Box

LEMMA 2.2. Let X \in R-Mod, Y an M-reflexive module, and I a set. If f : X \rightarrow Y^I is a homomorphism, then there exists a homomorphism g : X^{**} \rightarrow Y^I such that g \circ \sigma_X = f.

PROOF. Let, for each i \in I, p_i : Y^I \rightarrow Y be the canonical projection and consider the homomorphism g_i := \sigma_Y^{-1}(p_i \circ f)^{**} : X^{**} \rightarrow Y. Since \sigma_Y^{-1} \circ p_i \circ f = (p_i \circ f)^{**} \circ \sigma_X we see that p_i \circ f = \sigma_Y^{-1}(p_i \circ f)^{**} \circ \sigma_X = g_i \circ \sigma_X for each i \in I and so, calling g : X^{**} \rightarrow Y^I to the unique homomorphism such that p_i \circ g = g_i \forall i \in I, we see that p_i \circ f = p_i \circ g \circ \sigma_X \forall i \in I and hence that f = g \circ \sigma_X. \Box

PROPOSITION 2.3. Let M \in R-Mod be such that every finitely M-generated submodule of E(M) is M-cogenerated and let X \in R-Mod a \mathcal{T}_M-linearly compact module. Then X is M-reflexive if and only if M-dom. dim X \geq 2.

PROOF. The necessity is clear, for if X is M-reflexive and S(c) \rightarrow S(t) \rightarrow X^* \rightarrow 0 is a free presentation of X* in Mod-S, then applying ( )^* we get an exact sequence in R-Mod: 0 \rightarrow X \oplus X^{**} \rightarrow M^t \rightarrow M' and so M-dom. dim X \geq 2.

To prove the sufficiency, assume that X is \mathcal{T}_M-linearly compact and that there exists an exact sequence in R-Mod: 0 \rightarrow X \rightarrow X^{**} \rightarrow M^t \rightarrow M'. By Proposition 2.1, Coker \sigma_X \in \mathcal{T}_M and, as X^{**} is \mathcal{T}_M-torsionfree, it is clear that \sigma_X is an essential monomorphism. On the other hand, by Lemma 2.2 we see that there exists a homomorphism g : X^{**} \rightarrow M such that u = g \circ \sigma_X and, as \sigma_X is essential, g is a monomorphism. Therefore, Coker \sigma_X is a \mathcal{T}_M-torsion module which is isomorphic to a submodule of the M-cogenerated module Coker u and so Coker \sigma_X = 0. Thus \sigma_X is an isomorphism and X is M-reflexive. \Box

In the case M=R, the preceding result has been observed by Hoshino and
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Takashima in [5, Remark, p. 9]. In the following proposition we denote by \( \mathcal{D}_M \) the localizing subcategory of Mod-\( S \) cogenerated by \( E(M_S) \).

**Proposition 2.4.** Let \( M \subseteq R\text{-Mod} \). Then \( E(RM) \) is \( M \)-cogenerated if and only if, for every monomorphism \( g \) of \( R\text{-Mod} \), \( \text{Coker } g^* \subseteq \mathcal{D}_M \).

**Proof.** The proof can be easily adapted from that of [4, Theorem 1.1], where a similar result is proved in the case \( M = R \). \( \square \)

We can now give our main result characterizing \( M \)-reflexive modules. Recall that a bimodule \( \_R M_S \) is called faithfully balanced when \( R = \text{End}(M_S) \) and \( S = \text{End}(\_R M) \).

**Theorem 2.5.** Let \( \_R M_S \) be a faithfully balanced bimodule such that both \( E(RM) \) and \( E(M_S) \) are \( M \)-cogenerated, and let \( X \subseteq R\text{-Mod} \). Then \( X \) is \( M \)-reflexive if and only if it is \( \mathcal{D}_M \)-linearly compact and \( M \)-dom. \( \dim X \geq 2 \).

**Proof.** Applying Proposition 2.3, the only thing that remains to be proved is that any \( M \)-reflexive left \( R \)-module is \( \mathcal{D}_M \)-linearly compact. Assume then that \( X \) is \( M \)-reflexive and let \( \{ p_i : X_i \rightarrow X \}_I \) be an inverse system with \( X_i \) \( M \)-cogenerated and \( \text{Coker } p_i \subseteq \mathcal{D}_M \), for each \( i \in I \). Since \( \sigma_X \) is an isomorphism, we can identify the inverse system \( \{ p_i^* \}_I \) with the inverse system \( \{ \sigma_{X_i}^* p_i \}_I \) and we have:

\[
\lim \sigma_{X_i} \lim p_i = \lim p_i^* = (\lim p_i^*)^*.
\]

Since \( \text{Coker } p_i \subseteq \mathcal{D}_M \), the \( p_i^* \) are monomorphisms and so is \( \lim p_i^* \). Now, since \( E(M_S) \) is \( M \)-cogenerated and \( R = \text{End}(M_S) \), it follows from Proposition 2.4 that \( \text{Coker } (\lim p_i^*) \subseteq \mathcal{D}_M \). But, on the other hand, as \( \lim \) is a left exact functor, we have that \( \lim \sigma_{X_i} \) is a monomorphism and so \( \text{Coker } (\lim p_i) \subseteq \text{Coker } (\lim p_i^*) \). Thus \( \text{Coker } (\lim p_i) \subseteq \mathcal{D}_M \) and so \( X \) is \( \mathcal{D}_M \)-linearly compact. \( \square \)

Specializing Theorem 2.5 to the case \( M = R \), we obtain the promised characterization of reflexive modules over \( QF-3' \) rings.

**Corollary 2.6.** Let \( R \) be a \( QF-3' \) ring and \( X \subseteq R\text{-Mod} \). Then \( X \) is reflexive if and only if it is Lambek linearly compact and \( R \)-dom. \( \dim X \geq 2 \).

As we have remarked after Proposition 2.3, the "if" part of Corollary 2.6 has been proved by Hoshino and Takashima in [5], assuming only that every finitely generated submodule of \( E(R_R) \) is torsionless. The "only if" part, however, does not hold even in the case that \( R \) has this property on both sides.
An easy example is the following. Let $R = \mathbb{Z}$ be the ring of rational integers and $X$ a countable direct sum of copies of $\pi R$. Then it is clear that $X$ is not Lambek linearly compact, but $X$ is reflexive by a theorem of E. Specker [14].

3. Right QF-3' rings.

It is easy to infer from the proof of Theorem 2.5 that a right QF-3' ring is Lambek linearly compact on the left, and now we want to go in the opposite direction and, similarly to what is done in [9, Theorem 5] (see also [4, Theorem 2.2]) to give conditions on the left for a left QF-3' ring to be QF-3' (on both sides). Since the property of being QF-3' does not pass well from the maximal quotient ring of $R$ to $R$, we will assume that $R$ is, furthermore, a left maximal quotient ring. We will also need a stronger linear compactness condition that appeared in [3]. Assuming that $R \in R$-Mod/$\mathcal{L}$, let $\sigma_L^\infty[R]$ be the full subcategory of $R$-Mod/$\mathcal{L}$ consisting of the subobjects of quotients of finite direct sums of copies of $R$ in this category (this is just the smallest finitely closed, i.e., closed under subobjects, quotient objects, and finite direct sums-subcategory of $R$-Mod/$\mathcal{L}$ containing $R$). We will say that $\sigma_L^\infty[R]$ is a linearly compact subcategory of $R$-Mod/$\mathcal{L}$ if, for each inverse system $\{p_t : X_t \to Y_t\}_t$ in $R$-Mod/$\mathcal{L}$ with the $p_t$ epimorphisms and $X_t \in \sigma_L^\infty[R]$, the morphism $\lim p_t$ is also an epimorphism of $R$-Mod/$\mathcal{L}$.

**Theorem 3.1.** Let $R$ be a left maximal quotient ring. Then the following statements hold:

i) If $\sigma_L^\infty[R]$ is a linearly compact subcategory of $R$-Mod/$\mathcal{L}$, then $R$ is right QF-3' if and only if every finitely generated submodule of $E(RR)$ is torsionless.

ii) If every finitely generated submodule of $E(RR)$ is torsionless, then $R$ is right QF-3' if and only if $\sigma_L^\infty[R]$ is a linearly compact subcategory of $R$-Mod/$\mathcal{L}$.

**Proof.** i) Assume that each finitely generated submodule of $E(RR)$ is torsionless. Then, using Proposition 2.4 and [4, Theorem 1.1], it is enough to prove that if $j : X \to Y$ is a monomorphism in Mod-$R$, then Coker $j^* \in \mathcal{L}$, assuming that the analogous property holds for monomorphisms in Mod-$R$ that have finitely generated codomain. Thus, let $j : X \to Y$ be a monomorphism of Mod-$R$ and write $Y = \lim Y_i$, where $\{Y_i\}_i$ is the direct system of all the finitely generated submodules of $Y$. For each $i \in I$, set $X_i := X_i \cap Y_i$, with inclusions $j_i : X_i \to Y_i$. Using A5 we see that $j = \lim j_i$ and, taking $R$-duals, that $j^* = (\lim j_i)^* = \lim j_i^*$. Since the $Y_i$ are finitely generated right $R$-modules, we have that Coker $j_i^* \in \mathcal{L}$ for each $i \in I$ and, since $R$ is a maximal quotient ring, the

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$X^\dagger$ and $Y^\dagger$ are objects of $R$-$\text{Mod}/\mathcal{L}$, so that we have an inverse system of epimorphisms $j^\dagger: Y^\dagger \to X^\dagger$ in $R$-$\text{Mod}/\mathcal{L}$, with $Y^\dagger \in \sigma_{\mathcal{L}}^r[R]$. Now, as $\sigma_{\mathcal{L}}^r[R]$ is a linearly compact subcategory of $R$-$\text{Mod}/\mathcal{L}$, we see that $j^\dagger = \lim j^\dagger$ is an epimorphism of $R$-$\text{Mod}/\mathcal{L}$ and so $\text{Coker } j^\dagger \in \mathcal{L}$, completing the proof of i).

ii) Assume first that every finitely generated submodule of $E_\lambda(R)$ is torsionless and $R$ is right QF-3'. Since $R$ is, furthermore, a left maximal quotient ring, it follows from [4, Theorem 1.5] that every object of $\sigma_{\mathcal{L}}^r[R]$ is reflexive. Thus if we have an inverse system of epimorphisms $\{p_i: X_1 \to X_i\}$ in $R$-$\text{Mod}/\mathcal{L}$ with $X_i \in \sigma_{\mathcal{L}}^r[R]$, we may identify each $p_i$ with $p_i^\ast\ast$ and we have $\lim p_i = (\lim p_i)^\ast\ast$. Since $\text{Coker } p_i \in \mathcal{L}$, each $p_i^\ast\ast$ is a monomorphism in $\text{Mod}-R$, and hence so is $\lim p_i$. Now, as $R$ is right QF-3', we have by Proposition 2.4 $\text{Coker } (\lim p_i) \in \mathcal{L}$ and so $\sigma_{\mathcal{L}}^r[R]$ is linearly compact. Finally, assume that every finitely generated submodule of $E_\lambda(R)$ is torsionless and $\sigma_{\mathcal{L}}^r[R]$ is linearly compact. Then $R$ is a linearly compact object of $R$-$\text{Mod}/\mathcal{L}$ and by [4, Theorem 2.2], we have that every finitely generated submodule of $E_\lambda(R)$ is torsionless, so that, applying i) we see that $R$ is right QF-3'. □

Recall that a right $R$-module $P_R$ is called dominant if it is a finitely generated faithful projective module such that if $T = \text{End}(P_R)$, then $\tau P$ cogenerates all the simple left $T$-modules [7]. Then, assuming again that $R$ is a left maximal quotient ring, the existence of a dominant right module is equivalent to $R$-$\text{Mod}/\mathcal{L}$ being a module category by [7]. As it is well known, the left minimal faithful module over a left QF-3 ring is dominant [13] and so we may use the preceding theorem to characterize QF-3 maximal quotient rings. This is an important class of rings for, according to the Ringel-Tachikawa theorem [12], they correspond to Morita dualities. We next show that QF-3 maximal quotient rings can be characterized by conditions on the left that are similar to, but weaker than, those given by Masaike [9, Theorem 5] for QF-3 rings that are not necessarily maximal quotient rings.

**Corollary 3.2.** Let $R$ be a left maximal quotient ring. Then $R$ is QF-3 if and only if the following conditions hold:

i) $R$ is left QF-3'

ii) $R$ is left Lambek linearly compact

iii) $R$-$\text{Mod}/\mathcal{L}$ is a module category (equivalently, $R$ has a dominant right module).

**Proof.** It is clear from what we have already said that if $R$ is QF-3, then all three conditions above hold. Conversely, if conditions ii) and iii) hold, then
it follows from [6, Theorem 7.1] that \( \sigma L_2(R) \) is a linearly compact subcategory of \( R\text{-Mod}/\mathcal{L} \) and then, if i) also holds, we see from Theorem 3.1 that \( R \) is a \( QF-3' \) ring. Now, using [2, Corollary 6], we see that \( R \) is a \( QF-3 \) ring. □

**Remarks.**

i) The hypothesis that \( R \) is a left maximal quotient ring cannot be dropped from Theorem 3.1 and Corollary 3.2. Indeed, the ring \( R = \begin{pmatrix} Z & Q \\ 0 & Q \end{pmatrix} \)
satisfies i), ii) and iii) of Corollary 3.2 but is neither \( QF-3 \) nor right \( QF-3' \).

ii) Assume that \( R \) is a left maximal quotient ring which is linearly compact as an object of \( R\text{-Mod}/\mathcal{L} \). Then, a sufficient condition for \( \sigma L_2(R) \) to be a linearly compact subcategory of \( R\text{-Mod}/\mathcal{L} \) is that \( R\text{-Mod}/\mathcal{L} \) has a projective generator, as can be seen in the proof of [3, Corollary 7]. Thus an argument similar to the one used in the proof of Corollary 3.2 gives that if \( R \) is a left maximal quotient ring such that every finitely generated submodule of \( E(pR) \) is torsionless, \( R\text{-Mod}/\mathcal{L} \) has a projective generator, and \( R \) is Lambek linearly compact, then \( R \) is right \( QF-3' \).

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