REALIZATIONS OF INVOLUTIVE AUTOMORPHISMS $\sigma$
AND $G^\sigma$ OF EXCEPTIONAL LINEAR LIE GROUPS $G$,
PART I, $G=G_2$, $F_4$ AND $E_6$

By
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M. Berger [1] classified involutive automorphisms $\sigma$ of simple Lie algebras $g$ and determined the type of the subalgebras $g^\sigma$ of fixed points. Now for connected exceptional universal linear Lie groups $G$, we shall find involutive automorphisms $\sigma$ and realize the subgroups $G^\sigma$ of fixed points explicitly. In this paper we consider the cases of type $G_2$, $F_4$ and $E_6$. Our results are as follows. (Results of $E_7$ will be soon appeared in this Journal).

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<td>$Spin(9)$</td>
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|\( \text{Spin}(8, 1) \) & \( \sigma \) |
|---|---|
|\( E_6^c \) | \( (S p(1, C) \times S L(6, C))/Z_2 \) & \( \gamma \) |
| | \( (C^* \times \text{Spin}(10, C))/Z_4 \) & \( \sigma \) |
| | \( F_4^c \) & \( \lambda \) |
| | \( S p(4, C)/Z_2 \) & \( \lambda \gamma \) |
|\( E_6^c \) | \( E_6 \) & \( \tau \lambda \) |
| \( (S p(1) \times S U(6))/Z_2 \) & \( \gamma \) |
| | \( (U(1) \times \text{Spin}(10))/Z_4 \) & \( \sigma \) |
| | \( F_4 \) & \( \lambda \) |
| | \( S p(4)/Z_2 \) & \( \lambda \gamma \) |
| | \( S p(4, R)/Z_2 \times 2 \) & \( \lambda \gamma \) |
| | \( S p(2, 2)/Z_2 \times 2 \) & \( \lambda \gamma \) |
|\( E_6^c \) | \( E_6^{(1)} \) & \( \tau \lambda \gamma \) |
| \( (S p(1) \times S U^*(6))/Z_3 \) & \( \gamma \) |
| | \( (S p(1, R) \times S L(6, R))/Z_4 \times 2 \) & \( \gamma \) |
| | \( (R^* \times \text{spin}(5, 5))/2 \) & \( \sigma \) |
| | \( F_{4(4)} \) & \( \lambda \) |
| | \( S p(4)/Z_2 \) & \( \lambda \gamma \) |
| | \( S p(4, R)/Z_2 \times 2 \) & \( \lambda \gamma \) |
| | \( S p(2, 2)/Z_2 \times 2 \) & \( \lambda \gamma \) |
|\( E_6^c \) | \( E_6^{(2)} \) & \( \tau \lambda \gamma \) |
| \( (S p(1) \times S U(6))/Z_2 \) & \( \gamma \) |
| | \( (S p(1, R) \times S U(3, 3))/Z_2 \times 2 \) & \( \gamma \) |
| | \( (S p(1) \times S U(2, 4))/Z_2 \) & \( \gamma \) |
| | \( (U(1) \times \text{spin}(6, 4))/Z_4 \) & \( \sigma \) |
| | \( U(1) \times \text{spin}^*(10))/Z_4 \) & \( \sigma \) |
| | \( F_{4(4)} \) & \( \lambda \) |
| | \( S p(4, R)/Z_2 \times 2 \) & \( \lambda \gamma \) |
| | \( S p(2, 2)/Z_2 \times 2 \) & \( \lambda \gamma \) |
|\( E_6^c \) | \( E_6^{(-1)} \) & \( \tau \lambda \sigma \) |
| \( (S p(1) \times S U(2, 4))/Z_2 \) & \( \gamma \) |
| | \( (S p(1, R) \times S U(5, 1))/Z_2 \) & \( \gamma \) |
| | \( (U(1) \times \text{spin}(10))/Z_4 \) & \( \sigma \) |
| | \( (U(1) \times \text{spin}(8, 2))/Z_4 \) & \( \sigma \) |
| | \( (U(1) \times \text{spin}^*(10))/Z_4 \) & \( \sigma \) |
| | \( F_{4(-10)} \) & \( \lambda \) |
| | \( S p(2, 2)/Z_2 \times 2 \) & \( \lambda \gamma \) |
|\( E_6^c \) | \( E_6^{(-10)} \) & \( \tau \) |
| \( (S p(1) \times S U^*(6))/Z_2 \) & \( \gamma \) |
| | \( R^* \times \text{Spin}(9, 1) \) & \( \sigma \) |
Realizations of involutive automorphisms

The proofs of some theorems about the complex Lie groups are somewhere obtained by the modifications of the preceding papers [4]~[7], but we give their proofs again. Notation ~ in Theorems, for example, \( \langle G_{3(3)} \rangle \sim (\langle T' \rangle)^{\gamma} \) in Theorem 1.3.5 means \( \langle G_{3(3)} \rangle_{\delta} \sim (\langle T' \rangle)^{\gamma} \) for some \( \delta \in G_3 \). Finally the author would like to thank Takeshi Miyasaka, Toshikazu Miyashita and Osamu Shikuzawa for their advices and encouragements.

0.1. Notations and preliminaries.

Let \( R, C = R \oplus Ri \) (\( i^2 = -1 \)) and \( H = C \oplus Cj \) (\( j^2 = -1 \)) be the fields of real, complex and quaternion numbers, respectively. We define \( R \)-algebras

\[
\begin{align*}
C' & = R \oplus Ri, \quad i^2 = 1, \\
H' & = C' \oplus Cj, \quad j^2 = -1, \quad H = C \oplus Cj', \quad i'^2 = 1,
\end{align*}
\]

called the algebras of split complex numbers and split quaternion numbers, respectively. \( H' \) and \( H' \) are isomorphic as algebras.

For a vector space \( V \) over \( R \), its complexification \( \{ u + iv \mid u, v \in V \} \) is denoted by \( V^c \). For an \( R \)-linear transformation \( f : V \rightarrow V \), its complexification \( f^c : V^c \rightarrow V^c \) is written by the same notation \( f \). The complex conjugation in \( V^c \) is denoted by \( \tau : \)

\[
\tau(u + iv) = u - iv, \quad u, v \in V.
\]

The complexification of \( R \) is briefly denoted by \( C : C = R^c \). The complexifications \( C^c, H^c \) of \( C, H \) have algebraic structures over \( C \). Note that these algebras have the natural conjugations \( \sim \), for example, \( \overline{a + bi} = a - bi, \ a + bi \in C \oplus Ci = C^c \).

We use the following notations.

- \( M(n, K) \) (resp. \( M(n, m, K) \)) : all of \( n \times n \) (resp. \( n \times m \)) matrices with entries in \( K, K = R, C, C', H, H', H, C, C^c, H^c \) etc..
- \( E \) : the \( n \times n \) unit matrix (\( n \) is arbitrary).

\[
\begin{align*}
J_{n} = \text{diag}(J, \ldots, J) & \in M(2n, R) \text{ where } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
I_{n} = \text{diag}(I, \ldots, I) & \in M(2n, R) \text{ where } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } J' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad \text{Hereafter the suffices } n \text{ of } J_{n}, I_{n} \text{ will be omitted (so } I_{n} \text{ will have no confusions with the following } I_{n}).
\end{align*}
\]

- \( I_{1} = \text{diag}(-1, 1, 1, \ldots), \ I_{2} = \text{diag}(-1, 1, 1, \ldots), \ \cdots \in M(n, R) \).
- \( \Gamma_{1} = \text{diag}(i, 1, 1, \cdots), \ \Gamma_{2} = \text{diag}(i, i, 1, 1, \cdots), \ \cdots \in M(n, C) \).
For a vector space $V$ over $K=\mathbb{R}$, $\mathbb{C}$, $\text{Iso}_K(V)$ denotes all of $K$-linear isomorphisms of $V$. For a $K$-linear transformation $f$ of $V$, $V_f$ denotes $\{v \in V | f(v) = v\}$. When $V$ has the non-degenerate inner product $(u, v)$, for a $K$-linear transformation of $f$ of $V$, $^t f$ denotes the transpose of $f : (^t f(u), v) = (u, f(v))$.

$\mathbb{Z}_r$ (resp. $\mathbb{Z}_r$): the cyclic group of order $r$.

Let $G$ be a group. For $a, b \in G$, $a \sim b$ means that $a$ and $b$ are conjugate in $G : da = bd$ for some $d \in G$.

For a topological group $G$, $G_s$ denotes the identity connected component and $G = G_s \times 2$ means that $G$ has two connected components. When $G$ is a transformation group of a space $X$, $G_s$ denotes the isotropy subgroup of $G$ at $x \in X$: $G_s = \{g \in G | gx = x\}$.

If two groups $G, G'$ (resp. algebras $A, A'$) are isomorphic: $G \cong G'$ (resp. $A \cong A'$), then $G, G'$ (resp. $A, A'$) are often identified: $G = G'$ (resp. $A = A'$).

We arrange here some of classical Lie groups used in this paper.

$SL(n, K) = \{ A \in M(n, K) | \det A = 1 \}$, $K=\mathbb{R}, \mathbb{C}$, $C_c$,

$SO(n, K) = \{ A \in M(n, K) | ^t AA = E, \det A = 1 \}$, $K=\mathbb{R}, \mathbb{C}$,

$O(m, n-m) = \{ A \in M(n, \mathbb{R}) | ^t A I_m A = I_m \}$,

$SO^*(2m) = \{ A \in M(2n, C) | ^t AA = E, JA = -(A^t) J, \det A = 1 \}$,

$SU(n, K) = \{ A \in M(n, K) | A^* A = E, \det A = 1 \}$, $K=\mathbb{C}, C'$, $C_c$,

$SU(m, n-m, K) = \{ A \in M(n, K) | A^* I_m A = I_m, \det A = 1 \}$, $K=\mathbb{C}, C'$, $C_c$,

$SU^*(2n, K) = \{ A \in M(2n, K) | JA = \bar{A} J, \det A = 1 \}$, $K=\mathbb{C}, C'$, $C_c$,

$Sp(n, K) = \{ A \in M(n, K) | A^* A = E \}$, $K=\mathbb{H}, \mathbb{H'}$, $\mathbb{H}^c$,

$Sp(m, n-m, K) = \{ A \in M(n, K) | A^* I_m A = I_m \}$, $K=\mathbb{H}, \mathbb{H'}$, $\mathbb{H}^c$,

$Sp(n, K) = \{ A \in M(2n, K) | ^t AJA = J \}$, $K=\mathbb{R}, \mathbb{C}$

where $^t A$ is the transposed matrix of $A$ and $A^* = ^t \bar{A}$. Usually the following notations are used.

$SO(n) = SO(n, \mathbb{R})$, $SU(n) = SU(n, \mathbb{C})$, $SU(m, n-m) = SU(m, n-m, \mathbb{C})$,

$SU^*(2n) = SU^*(2n, \mathbb{C})$, $Sp(n) = Sp(n, \mathbb{H})$, $Sp(m, n-m) = Sp(m, n-m, \mathbb{H})$.

The Lie algebra of a Lie group $G$ is denoted by the corresponding German small letter $g$. For example, $\mathfrak{su}(n)$ denotes the Lie algebra of $SU(n)$.

**Lemma 0.1.** $U(n, C') \cong U(m, n-m, C') \cong GL(n, \mathbb{R})$. 

Proof. $f: GL(n, \mathbb{R}) = \{ A \in M(n, \mathbb{R}) | \det A \neq 0 \} \to U(n, \mathbb{C'}) = \{ B \in M(n, \mathbb{C'}) | B^* B = E \}$,

$$f(A) = \varepsilon A + \varepsilon^t A^{-1}, \quad \varepsilon = \frac{1}{2} (1 + i')$$

is an isomorphism (note $\varepsilon^* = \varepsilon$, $\varepsilon^2 = \varepsilon$, $\varepsilon \bar{\varepsilon} = 0$, $\varepsilon + \bar{\varepsilon} = 1$). The inverse mapping $f^{-1}: U(n, \mathbb{C'}) \to GL(n, \mathbb{R})$ of $f$ is given by $f^{-1}(P + Q \varepsilon) = P + Q', \text{ } P, Q \in M(n, \mathbb{R})$.

Similarly, $f: GL(n, \mathbb{R}) \to U(m, n-m, \mathbb{C'}) = \{ B \in M(n, \mathbb{C'}) | B^* I_m B = I_n \}$, $f(A) = \varepsilon A + \varepsilon I_m A^{-1} A I_m$, is an isomorphism.

**Proposition 0.2.** (1) $SU(n, \mathbb{C'}) = SU(m, n-m, \mathbb{C'}) = SL(n, \mathbb{R})$, $SU^*(2n, \mathbb{C'}) = SL(2n, \mathbb{R})$.

(2) $SU(n, \mathbb{C}) = SU(m, n-m, \mathbb{C}) = SL(n, \mathbb{C})$, $SU^*(2n, \mathbb{C}) = SL(2n, \mathbb{C})$.

Proof. (1) The restriction $f: SL(n, \mathbb{R}) \to SU(n, \mathbb{C'})$ of $f$ in Lemma 0.1 is an isomorphism. In fact, the calculations of $\det(f(A)) = 1, A \in SL(n, \mathbb{R})$ and $\det(f^{-1}(B)) = 1, B \in SU(n, \mathbb{C'})$ follow from

**Lemma 0.3.** (1) For $A, B \in M(n, \mathbb{R})$, we have

$$\det(\varepsilon A + \varepsilon B) = \varepsilon \det A + \varepsilon \det B, \quad \varepsilon = \frac{1}{2} (1 + i').$$

(The above is also valid for $A, B \in M(n, \mathbb{C})$ and $\varepsilon = \frac{1}{2} (1 + i)$).

(2) Let $P(x_1, \ldots, x_m)$ be a polynomial with integral coefficients. If $P(p_1 + q_1 i', \ldots, p_m + q_m i') = 1$ for $p_1 + q_1 i' \in \mathbb{R} \oplus \mathbb{R} i' = \mathbb{C'}$ (resp. $p_1 + q_1 i \in \mathbb{C} \oplus \mathbb{C} i = \mathbb{C}$), then $P(p_1 + q_1, \ldots, p_m + q_m) = 1$.

Similarly, $f: SL(n, \mathbb{R}) \to SU(m, n-m, \mathbb{C'})$, $f(A) = \varepsilon A + \varepsilon I_m A^{-1} A I_m$ and $f: SL(2n, \mathbb{R}) \to SU^*(2n, \mathbb{C'})$, $f(A) = \varepsilon A - \varepsilon J A J$ where $\varepsilon = \frac{1}{2} (1 + i')$, are isomorphisms, respectively.

(2) These are corollaries of (1). In fact, for example, $f: SL(n, \mathbb{C}) \to SU(n, \mathbb{C})$, $f(A) = \varepsilon A + \varepsilon^t A^{-1}$ where $\varepsilon = \frac{1}{2} (1 + i)$, is an isomorphism.

**Proposition 0.4.** (1) $Sp(n, \mathbb{H'}) \equiv Sp(m, n-m, \mathbb{H'}) \equiv Sp(n, \mathbb{R})$.

(2) $Sp(n, \mathbb{H}) \equiv Sp(m, n-m, \mathbb{H}) \equiv Sp(n, \mathbb{C})$. In particular, $Sp(1, \mathbb{H'}) \equiv Sp(1, \mathbb{R}) = SL(2, \mathbb{R})$, $Sp(1, \mathbb{H}) \equiv Sp(1, \mathbb{C}) = SL(2, \mathbb{C})$.

Proof. (1) Let $k': M(n, \mathbb{H'}) \to \{ B \in M(2n, \mathbb{C'}) | JB = B J \}$ be the algebraic $\mathbb{R}$-isomorphism defined by
\[ k'(a+bj) = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \quad a, b \in \mathbb{C} \]

Then \( f^{-1}k' : \text{Sp}(n, \mathbb{H}') \to \text{Sp}(n, \mathbb{R}) \) is an isomorphism. In fact,

\[
\begin{align*}
\text{Sp}(n, \mathbb{H}') & = \{ D \in M(n, \mathbb{H}') \mid D^*D = E \} \\
\overset{k'}{\longrightarrow} & = \{ D \in \text{U}(2n, \mathbb{C}') \mid iBJB = J \} \\
\overset{f^{-1}}{\longrightarrow} & = \{ A \in \text{M}(2n, \mathbb{R}) \mid tAJA = J \} \quad (\text{Lemma 0.1}) = \text{Sp}(n, \mathbb{R}).
\end{align*}
\]

Similarly, \( \text{Sp}(m, \mathbf{n} - m, \mathbb{H}') = \{ D \in M(n, \mathbb{H}') \mid D*I_mD = I_m \} \to \{ D \in M(2n, \mathbb{C}') \mid B*I_m B = I_m, iBJI_mB = JI_m \} \) (since \( J \) and \( I_m \) are conjugate in \( O(2n) : J^n = JI_mJ_m' \) where \( J_m' = \text{diag}(J', \cdots, J, 1, \ldots, 1) \), by the correspondence \( B \to J_m'BJ_m' \equiv \{ B \in M(2n, \mathbb{C}') \mid B*I_m B = I_m, iBJB = J \} \) \( \overset{f^{-1}}{\longrightarrow} \) \( \{ A \in \text{M}(2n, \mathbb{R}) \mid tAJA = J \} = \text{Sp}(n, \mathbb{R}) \).

(2) These are corollaries of (1).

0.2. Automorphisms of a group.

Let \( G \) be a group and \( \sigma \) an automorphism of \( G \). \( G^\sigma \) denotes \( \{ g \in G \mid \sigma g = g \} \). For \( s \in G \), \( \bar{s} \) denotes the inner automorphism induced by \( s : \bar{s}(g) = sg^{-1}s^{-1} \), \( g \in G \), then \( G^\bar{s} = \{ g \in G \mid \forall g = gs \} \). Hereafter \( G^\sigma \) will be written by \( G^s \). Moreover when \( G \) is indicated, \( G^\sigma, G^\bar{s} \) will be written by \( \sigma, s \), respectively.

**Lemma 0.5.** Let \( \sigma_1, \sigma_2, \sigma_3 \) are involutive automorphisms of a group \( G \) satisfying \( \sigma_1\sigma_2 = \sigma_2\sigma_1 \), then

\[
(G^\sigma_1)^{\sigma_2} = (G^\sigma_2)^{\sigma_1}, \quad (G^{\sigma_1\sigma_2})^{\sigma_3} = (G^{\sigma_1})^{\sigma_2\sigma_3} = (G^\sigma_1)^{\sigma_2\sigma_3}, \quad (\sigma_1\sigma_2)^{\sigma_3\sigma_4} = (\sigma_1)^{\sigma_3\sigma_4} = (\sigma_2)^{\sigma_3\sigma_4},
\]

By the simple representation, these are written by \( (\sigma_1)^{\sigma_2} = (\sigma_2)^{\sigma_1}, (\sigma_1\sigma_2)^{\sigma_3} = (\sigma_1)^{\sigma_3\sigma_4}, (\sigma_2)^{\sigma_3\sigma_4} = (\sigma_2)^{\sigma_3\sigma_4}, \) respectively.

For a given group \( G \) and an involutive automorphism \( \sigma \) of \( G \), our aim is to determine the group structure of \( G^\sigma \). After this, for a homomorphism \( \phi : G' \to G^\sigma \) of groups, it needs often to prove that \( \phi \) is well-defined and onto. When \( G' \), \( G^\sigma \) are Lie groups, these properties can reduce to their Lie algebras, that is,

**Lemma 0.6.** Let \( \phi : G' \to G^\sigma \) be a homomorphism of Lie groups.

(1) When \( G' \) is connected, if \( d\phi : g' \to g^\sigma \) is well-defined, then \( \phi \) is so.

(2) When \( G^\sigma \) is connected, if \( d\phi : g' \to g^\sigma \) is onto, then \( \phi \) is so.
Realizations of involutive automorphisms

To use Lemma 0.6 (2), the following Lemma is useful.

**Lemma 0.7** (E. Cartan-P. K. Raševskii [3]). Let $G$ be a simply connected Lie group and $\sigma$ an involutive automorphism of $G$, then $G^\sigma$ is connected.

In the following we will somewhere try to give elementary proof not using Lemmas 0.6, 0.7. The author thinks that the elementary proof finds out occasionally essential properties of the group $G^\sigma$.

**Group $G_2$**

1.1. Cayley algebras and Lie groups of type $G_2$.

Let $\mathcal{G} = \mathbb{H} \oplus \mathbb{He}$ be the division Cayley algebra with the multiplication

$$(m + ae)(n + be) = (mn - ba) + (a\bar{n} + b\bar{m})e,$$

the conjugation $m + ae = \bar{m} - ae$ and the inner product $(m + ae, n + be) = (m, n) + (a, b) \left( = \frac{1}{2}((m\bar{n} + n\bar{m}) + (a\bar{b} + \bar{a}b)) \right)$. Another Cayley algebra $\mathcal{G}' = \mathbb{H} \oplus \mathbb{He}'$, called the split Cayley algebra, is defined as the algebra with the multiplication

$$(m + ae')(n + be') = (mn + \bar{b}a) + (a\bar{n} + b\bar{m})e',$$

the conjugation $m + ae' = \bar{m} - ae'$ and the inner product $(m + ae', n + be') = (m, n) - (a, b)$.

The connected linear Lie groups of type $G_2$ are obtained as the automorphism groups of the Cayley algebras, respectively.

$$G_2^\mathbb{C} = G_2(\mathbb{C}) = \{ \alpha \in \text{Iso}_c(\mathbb{C}) | \alpha(xy) = (\alpha x)(\alpha y) \},$$

$$G_2 = G_2(\mathbb{R}) = \{ \alpha \in \text{Iso}_c(\mathbb{R}) | \alpha(xy) = (\alpha x)(\alpha y) \},$$

$$G_2(\mathbb{C}) = G_2(\mathbb{C}) = \{ \alpha \in \text{Iso}_c(\mathbb{C}) | \alpha(xy) = (\alpha x)(\alpha y) \}.$$

(Similarly the group $G_2(\mathbb{H})$ is defined). $G_2^\mathbb{C}$, $G_2$ are simply connected (see Appendix).

1.2. Involutions of Lie groups of type $G_2$.

We define $\mathbb{R}$-linear transformations $\gamma$, $\gamma_c$, $\gamma_U$ of $\mathcal{G}$ by

$$\gamma(m + ae) = m - ae, \quad m + ae \in \mathbb{H} \oplus \mathbb{He} = \mathcal{G},$$

$$\gamma_c(m + ae) = \gamma cm + \gamma(\gamma a)e, \quad \gamma_U(m + ae) = \gamma_U m + (\gamma_U a)e,$$

where $\gamma_c, \gamma_U : \mathbb{H} \to \mathbb{H}$ are defined as $\gamma_c(x + yj) = \bar{x} + 5j, \gamma_U(x + yj) = x - yj, x + yj \in \mathbb{C} \oplus \mathbb{C}j = \mathbb{H}$, respectively. Then $\gamma, \gamma_c, \gamma_U \subseteq G_2 \subseteq G_2^\mathbb{C}$ and $\gamma^2 = \gamma_c^2 = \gamma_U^2 = 1$. 
LEMA 1.2.1. (1) \((H^C)_r = H, (H^C)_r T_c \cong H', (H^C)_r T_H \cong H'.\)

(2) \((G^C)_r = G, (G^C)_r \cong G'.\)

PROOF. For example, the correspondence

\[ (G^C)_r \ni m + iae \longrightarrow m + ae' \in G' \quad (m, a \in H) \]
gives an isomorphism as algebras.

The semi-linear transformations \(\tau, \tau'\) of \(G^C\) induce involutive automorphisms \(\tilde{\tau}, \tilde{\tau}'\) of \(G^C_G\):

\[ \tilde{\tau}(\alpha) = \tau \alpha \tau, \quad \tilde{\tau}'(\alpha) = \tau' \alpha \tau', \quad \alpha \in G^C_G. \]

THEOREM 1.2.2. \((G^C_G)^\tau = G_2, (G^C_G)^{\tilde{\tau}} = G_2(2).\)

PROOF. \((G^C_G)^\tau, (G^C_G)^{\tilde{\tau}}\) mean \((G^C_G)^\tau, (G^C_G)^{\tilde{\tau}}\), respectively. These are direct results of Lemma 1.2.1. (2).

PROPOSITION 1.2.3. \(\gamma, \gamma_c, \gamma_H, \gamma_T c, \gamma_T H\) are conjugate in \(G_2\) with one another (moreover \(\gamma\) is conjugate to the others under \(\delta = \delta^{-1} \in G_2\)).

PROOF. Define four \(R\)-linear isomorphisms \(\delta: \mathbb{C} \to \mathbb{C}\) satisfying \(\delta(1) = 1\) and

\[
\begin{align*}
i &\longrightarrow e, \quad i \longrightarrow i, \quad i \longrightarrow ie, \quad i \longrightarrow i, \\
j &\longrightarrow j, \quad j \longrightarrow e, \quad j \longrightarrow j, \quad j \longrightarrow je, \\
k &\longrightarrow -je, \quad k \longrightarrow ie, \quad k \longrightarrow -ke, \quad k \longrightarrow -ke, \\
e &\longrightarrow i, \quad e \longrightarrow j, \quad e \longrightarrow -e, \quad ie \longrightarrow -e, \\
ie &\longrightarrow -ie, \quad ie \longrightarrow k, \quad ie \longrightarrow i, \quad ie \longrightarrow -ie, \\
je &\longrightarrow -k, \quad je \longrightarrow -je, \quad je \longrightarrow -je, \quad je \longrightarrow j, \\
ke &\longrightarrow -ke, \quad ke \longrightarrow -ke, \quad ke \longrightarrow -k, \quad ke \longrightarrow -k
\end{align*}
\]

where \(k = ij\), respectively. Then \(\delta = \delta^{-1} \in G_2\) and \(\delta \gamma = \gamma c \delta, \delta \gamma = \gamma H \delta, \delta \gamma = \gamma_T c \delta, \delta \gamma = \gamma_T H \delta\), \(\delta \gamma = \gamma_T H \delta\), respectively.

1.3. Subgroups of type \(C_1 \oplus C_1\) of Lie groups of type \(G_2\).

PROPOSITION 1.3.1. \(G_2(H^C) \cong Sp(1, C)/Z_2.\)

PROOF. We define \(\phi: Sp(1, H^C) \to G_2(H^C)\) by
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\[ \phi(q)m = qmq, \quad m \in H^c. \]

It is clear that \( \phi \) is well-defined and a homomorphism. We shall show \( \phi \) is onto. Let \( \alpha \in G_2^c(H^c) \). Since \( H^c \) is a central simple \( C \)-algebra, by Noether-Skolem's theorem, there exists an invertible element \( q \in H^c \) such that \( am = qmq^{-1}, \quad m \in H^c \). We may assume \( q = 1 \), that is, \( q = Sp(1, H^c) \). Hence \( \phi \) is onto. \( \text{Ker} \phi = \{1, -1\} = Z_2 \). Thus we have \( G_2(H^c) = Sp(1, H^c)/Z_2 = Sp(1, C)/Z_2 \) (Proposition 0.4).

**Theorem 1.3.2.** \( (G_2^c)^r = (Sp(1, C) \times Sp(1, C))/Z_2, \quad Z_2 = \{1, 1\}, \quad (-1, -1) \).

**Proof** ([5]). We define \( \phi : Sp(1, H^c) \times Sp(1, H^c) \rightarrow (G_2^c)^r \) by

\[ \phi(p, q)(m + ae) = qmq + (paq)e, \quad m + ae \in H^c \oplus H^c, \quad e = \mathbb{C}. \]

It is easy to verify that \( \phi \) is well-defined and a homomorphism. We shall show \( \phi \) is onto. Let \( \alpha \in (G_2^c)^r \). Since \( (\mathbb{C})_r = H^c \) is invariant under \( \alpha \), \( \alpha \) induces an automorphism of \( H^c \). Hence there exists \( q = Sp(1, H^c) \) such that

\[ am = qmq, \quad m \in H^c \] (Proposition 1.3.1).

Put \( \beta = \phi(q, q)^{-1} \alpha \), then \( \beta \in (G_2^c)^r \) and \( \beta | H^c = 1 \). Since \( (\mathbb{C})_r = H^c \) is also invariant under \( \beta \), we can put

\[ \beta e = pe, \quad p \in H^c. \]

\( p \in Sp(1, H^c) \) because \( -1 = \beta(ee) = (\beta e)(\beta e) = (pe)(pe) = -ppe \), and \( \beta(m + ae) = m + a(\beta e) = m + a(pe) = m + (pa)e = \phi(p, 1)(m + ae) \), that is, \( \beta = \phi(p, 1) \). Hence \( \alpha = \phi(1, q) \beta = \phi(1, q) \phi(p, 1) = \phi(p, q) \). Therefore \( \phi \) is onto. \( \text{Ker} \phi = \{(1, 1), (-1, -1)\} = Z_2 \). Thus we have the required isomorphism. (Remark. \( (Sp(1, C) \times Sp(1, C))/Z_2 \cong SO(4, C) \)).

**Lemma 1.3.3.** \( \phi : Sp(1, H^c) \times Sp(1, H^c) \rightarrow G_2^c \) of Theorem 1.3.2 satisfies

1. \( \gamma = \phi(-1, 1), \quad \gamma_c = \phi(j, j), \quad \gamma_i = \phi(i, i). \)
2. \( \tau \phi(p, q) \tau = \phi(\tau p, \tau q), \quad \tau_c \phi(p, q) \tau_c = \phi(\tau_c p, \tau_c q). \)

**Theorem 1.3.4.** \( (G_2)^r = (Sp(1) \times Sp(1))/Z_2 \cong (G_2(2))^r \).

**Proof.** \( (G_2)^r = ((G_2)^r)^r \) (Theorem 1.2.2) = \( (Sp(1, H^c) \times Sp(1, H^c))^r \) (Lemma 0.5) = \( (Sp(1, H^c)) \times Sp(1, H^c))^r \) (Theorem 1.3.2). Hence for \( \alpha \in (G_2)^r \) there exist \( p, q = Sp(1, H^c) \) such that \( \alpha = \phi(p, q) \). From the condition \( \tau \alpha = \alpha \tau \), we have \( \phi(p, q) = \alpha = \tau \alpha = \tau \phi(p, q) \tau = \phi(\tau p, \tau q) \) (Lemma 1.3.3). Hence

\[ \tau p = p, \quad \tau q = q \quad \text{or} \quad \tau p = -p, \quad \tau q = -q. \]
The latter case is impossible. In fact, put \( p = ip', p' \in H \), then \( 1 = \beta p = (ip'(i\beta p') = -p'i \beta \leq 0 \), a contradiction. Therefore \( p, q \in Sp(1) \). Thus \((G_2)^c = (\phi(Sp(1, H^c) \times Sp(1, H^c))^c = \phi(Sp(1) \times Sp(1)) \cong (Sp(1) \times Sp(1))/Z_2 \), \((G_{2(1)})^c = ((G_2)^c)^c \) (Theorem 1.2.2) = \((G_2)^c^c \) (Lemma 0.5) \((Sp(1) \times Sp(1))/Z_2 \) (as above). (This fact is written as \((G_{2(1)})^c = (\tau \gamma)^c = (\tau)^c \). (REMARK. \((Sp(1) \times Sp(1))/Z_2 \cong SO(4)).

**THEOREM 1.3.5.** \((G_{2(1)})^c \cong (Sp(1, R) \times Sp(1, R))/Z_2 \times 2.

**Proof.** \( G_{2(1)} = (G_2)^c \cong (G_2)^c^c \).

In fact, since \( \gamma \) and \( \gamma_c \) are conjugate in \( G_2 : \gamma \gamma_c = \gamma \gamma_c \) (Proposition 1.2.3), the correspondence \((G_2)^c \gamma \cong \alpha \gamma \delta \gamma c = (G_2)^c \gamma c \) gives an isomorphism. Now let \( \alpha \in ((G_2)^c \gamma c) \), \( \alpha = \phi(p, q) \), \( p, q \in Sp(1, H^c) \) (Theorem 1.2.2). From the condition \( \tau \gamma c \alpha = \alpha \tau \gamma c \), we have \( \phi(\tau \gamma c, \tau \gamma c q) = \phi(p, q) \) (Lemma 1.3.3). Hence

\[
\tau \gamma c \alpha = p, \quad \tau \gamma c q = q \quad \text{or} \quad \tau \gamma c p = p, \quad \tau \gamma c q = -q.
\]

Therefore \( p, q \in Sp(1, H^c) \) or \( p, q \in iSp(1, H^c) \) (Lemma 1.2.1). Thus \((G_2)^c \gamma c \gamma^c = (Sp(1, H^c) \times Sp(1, H^c)) \cup iSp(1, H^c) / Z_2 \cong (Sp(1, R) \times Sp(1, R))/Z_2 \times 2 \) \((\phi, i) = \gamma(c) \). (REMARK. This group is isomorphic to the group \( SO(2, 2) = \{ A \in M(4, R) | ^tA I_A = A, \det A = 1 \} \).

**Group \( F_4 \)**

### 2.1. Jordan algebras and Lie groups of type \( F_4 \)

Let \( K \) be \( H, H^c, \mathbb{C}, \mathbb{C}' \) or \( \mathbb{C}^c \). \( \mathfrak{X}(K) \) denotes one of the Jordan algebras

\[
\mathfrak{X}(3, K) = \{ X \in M(3, K) | X^* = X \}, \\
\mathfrak{X}(1, 2, K) = \{ X \in M(3, K) | I^X X^* = X \}
\]

with the Jordan multiplication \( X \circ Y \), the inner product \( (X, Y) \) and the trilinear form \( \text{tr}(X, Y, Z) \):

\[
X \circ Y = \frac{1}{2}(XY + YX), \quad (X, Y) = \text{tr}(X \circ Y), \quad \text{tr}(X, Y, Z) = (X, Y \circ Z).
\]

In \( \mathfrak{X}(K) \), we define another multiplication \( X \odot Y \), called the Freudenthal multiplication, by

\[
X \odot Y = \frac{1}{2}(2X \circ Y - \text{tr}(X)Y - \text{tr}(Y)X + (\text{tr}(X) \text{tr}(Y) - (X, Y))E)
\]

and the trilinear form \( (X, Y, Z) \), the determinant \( \det X \) by

\[
(X, Y, Z) = (X, Y \times Z), \quad \det X = \frac{1}{3}(X, X, X).
\]
The algebra $\mathfrak{Z}(K)$ with the Freudenthal multiplication $X \times Y$ and the inner product $(X, Y)$ is called the Freudenthal algebra. In $\mathfrak{Z}(K)$, we have relations
\[ X \times (X \times X) = (\det X)E, \quad (X \times X) \times (X \times X) = (\det X)X. \]

An element $X \in \mathfrak{Z}(3, \mathfrak{C})$ has the form
\[
X = X(\xi, x) = \begin{pmatrix}
\xi_1 & x_3 & \xi_2 \\
\xi_2 & x_1 & \xi_3 \\
x_2 & \xi_3 & \xi_1
\end{pmatrix}, \quad \xi_i \in R, \ x_i \in \mathfrak{C}.
\]

We correspond such $X \in \mathfrak{Z}(3, \mathfrak{C})$ to an element $M + a \in \mathfrak{Z}(3, H) \oplus H^3$ such that
\[
\begin{pmatrix}
\xi_1 & m_3 & m_1 \\
m_3 & \xi_2 & m_1 \\
m_1 & m_1 & \xi_3
\end{pmatrix} + (a_1, a_2, a_3)
\]
where $x_i = m_i + a_i e \in H \oplus H e = \mathfrak{C}$. Then $\mathfrak{Z}(3, H) \oplus H^3$ has the multiplication and the inner product
\[
(M + a) \times (N + b) = \left( M \times N - \frac{1}{2} (a^* b + b^* a) \right) - \frac{1}{2} (a N + b M),
\]
where $(a, b) = \frac{1}{2} (ab^* + ba^*) = \frac{1}{2} \text{tr}(a^* b + b^* a)$, corresponding those of $\mathfrak{Z}(3, \mathfrak{C})$, that is, $\mathfrak{Z}(3, \mathfrak{C})$ is isomorphic to $\mathfrak{Z}(3, H) \oplus H^3$ as Freudenthal algebra. As for $\mathfrak{Z}(3, \mathfrak{C})$, the same arguments are valid as above: $\mathfrak{Z}(3, \mathfrak{C}^c) = \mathfrak{Z}(3, H^c) \oplus (H^c)^3$.

In $\mathfrak{Z}(3, K)$ we use the following notations.
\[
E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]
\[
F_1(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & x & 0 \end{pmatrix}, \quad F_4(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & x & 0 \\ x & 0 & 0 \end{pmatrix}, \quad F_2(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & x & 0 \\ x & 0 & 0 \end{pmatrix}.
\]

The tables of the Jordan and the Freudenthal multiplications among them are given as follows.
\[
\begin{align*}
E_i \cdot E_i &= E_i, & E_i \cdot E_j &= 0, & i \neq j, \\
E_i \cdot F_i(x) &= 0, & E_i \cdot F_j(x) &= \frac{1}{2} F_j(x), & i \neq j, \\
F_i(x) \cdot F_i(y) &= (x, y)(E_{i+1} + E_{i+2}), & F_i(x) \cdot F_{i+1}(y) &= \frac{1}{2} F_{i+3}(x y), \\
E_i \times E_i &= 0, & E_i \times E_{i+1} &= \frac{1}{2} E_{i+2}, \\
E_i \times F_i(x) &= -\frac{1}{2} F_i(x), & E_i \times F_j(x) &= 0, & i \neq j, \\
F_i(x) \times F_i(y) &= -(x, y) E_i, & F_i(x) \times F_{i+1}(y) &= \frac{1}{2} F_{i+3}(x y)
\end{align*}
\]

where the indexes are considered as mod 3.

The connected linear Lie groups of type \( F_4 \) are obtained as the automorphism groups of the Jordan algebras, respectively.

\[
\begin{align*}
F_4^c &= F_4(3(3, \mathbb{C}^c)) = \{ \alpha \in \text{Iso}_c(3(3, \mathbb{C}^c)) \mid \alpha(X \cdot Y) = \alpha X \cdot \alpha Y \}, \\
F_4 &= F_4(\mathbb{R}) = \{ \alpha \in \text{Iso}_\mathbb{R}(\mathbb{R}) \mid \alpha(X \cdot Y) = \alpha X \cdot \alpha Y \}, \\
F_{4(c)} &= F_4(3(3, \mathbb{C}')) = \{ \alpha \in \text{Iso}_\mathbb{R}(\mathbb{R}) \mid \alpha(X \cdot Y) = \alpha X \cdot \alpha Y \}, \\
F_{4(-20)} &= F_4(3(1, 2, \mathbb{C})) = \{ \alpha \in \text{Iso}_\mathbb{R}(\mathbb{R}) \mid \alpha(X \cdot Y) = \alpha X \cdot \alpha Y \}.
\end{align*}
\]

(Similarly the group \( F_4(3(3, \mathbb{H}^c)) \) is defined). \( F_4^c, F_4, F_{4(-20)} \) are simply connected (see Appendix). The group \( F_4^c \) naturally contains \( G_2^c \) as a subgroup, that is, for \( \alpha \in G_2^c \), define \( \tilde{\alpha} : 3 c \to 3 c \) by

\[
\tilde{\alpha} X(\xi, x) = X(\xi, \alpha x) \quad \text{where} \quad ax = a(x_1, x_2, x_3) = (ax_1, ax_2, ax_3),
\]

then \( G_2^c \equiv \{ \tilde{\alpha} \mid \alpha \in G_2^c \} \subset F_4^c \). Similarly \( G_2 \subset F_4, G_{2(12)} \subset F_{4(c)}, G_3 \subset F_{4(-20)}. \)

**Lemma 2.1.1.** For \( \alpha \in \text{Iso}_c(3c) \), the following three conditions are equivalent.

\[
\det \alpha X = \det X, \quad (a X, a Y, a Z) = (X, Y, Z), \quad a X \times a Y = a^{-1}(X \times Y),
\]

for \( X, Y, Z \in 3 c \).

**Lemma 2.1.2.** For \( \alpha \in F_4^c \), we have \( \alpha E = E \) and \( \text{tr}(\alpha X) = \text{tr}(X) \), \( X \in 3 c \).

**Proof ([4]).** \( \alpha E = E \) is trivial. Next we use the identity \( X(\alpha X)X = (\det X)E \), that is,

\[
X^*(X \times X) - \text{tr}(X)X + \frac{1}{2} (\text{tr}(X)^3 - \text{tr}(X^3))X = (\det X)E.
\]

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Apply (i) to $\alpha X$ and then operate $\alpha^{-1}$ on it, then

$$X^*(X\cdot X) - \text{tr}(\alpha X)X^2 + \frac{1}{2}(\text{tr}(\alpha X))^2 - \text{tr}((\alpha X)^2))X - (\det \alpha X)E. \quad (\text{ii})$$

By substruction (i)-(ii) we have

$$\text{tr}(\alpha X) - \text{tr}(X)X^2 + \frac{1}{2}(\text{tr}(X))^2 - \text{tr}(\alpha X)X^2 + \text{tr}((\alpha X)^2) - \text{tr}(X^2))X$$

$$= (\det X - \det(\alpha X))E.$$  

Note that as an additive generator of $\mathcal{3}^c$ we can choose $\mathcal{E} = \{E_i, F \in \mathcal{3}^c | \text{det}(F) = \text{det} F = 0, \text{diag } F = 0, F^2 = E + E_{i+1}, i = 1, 2, 3\}$. Now for $F \in \mathcal{E}$,

$$\text{tr}(\alpha F)(E_i + E_{i+1}) + \frac{1}{2}(\text{tr}(\alpha F)^2 + \text{tr}((\alpha F)^2) - 2)F = -(\det(\alpha F))E.$$  

Compare each term of both sides, then we have $\text{tr}(\alpha F) = 0$ (i.e. $\text{det}(F)$) and $\text{tr}((\alpha F)^2) = 2$. Hence $\text{tr}(\alpha E_i) = (\alpha(E - F_i(1)^2)) = \text{tr}(E) - \text{tr}(\alpha F_i(1)^2) = 3 - 2 = 1 = \text{tr}(E_i), i = 1, 2, 3$. Consequently $\text{tr}(\alpha X) = \text{tr}(X)$ for $X \in \mathcal{3}^c$.

**Proposition 2.1.3.**

$$F^c = \{\alpha \in \text{Iso}_c(\mathcal{3}^c) | \alpha(X \cdot Y) = \alpha X \cdot \alpha Y\} \quad (1)$$

$$= \{\alpha \in \text{Iso}_c(\mathcal{3}^c) | \text{tr}(\alpha X, \alpha Y, \alpha Z) = \text{tr}(X, Y, Z), (\alpha X, \alpha Y) = (X, Y)\} \quad (2)$$

$$= \{\alpha \in \text{Iso}_c(\mathcal{3}^c) | \text{det } \alpha X = \text{det } X, (\alpha X, \alpha Y) = (X, Y)\} \quad (3)$$

$$= \{\alpha \in \text{Iso}_c(\mathcal{3}^c) | \text{det } \alpha X = \text{det } X, \alpha E = E\} \quad (4)$$

$$= \{\alpha \in \text{Iso}_c(\mathcal{3}^c) | \alpha(X \cdot Y) = \alpha X \cdot \alpha Y\}. \quad (5)$$

**Proof.** (1) $\Rightarrow$ (2) $(\alpha X, \alpha Y) = \text{tr}(\alpha X \cdot \alpha Y) = \text{tr}(\alpha(X \cdot Y)) = \text{tr}(X \cdot Y)$ (Lemma 2.1.2)

$$= (X, Y). \quad \text{tr}(\alpha X, \alpha Y, \alpha Z) = (\alpha X, \alpha Y \cdot \alpha Z) = (\alpha X, \alpha(Y \cdot Z)) = (\alpha(X \cdot Y)) = \text{tr}(X, Y, Z).$$

(2) $\Rightarrow$ (1) $(\alpha X \cdot \alpha Y, \alpha Z) = \text{tr}(\alpha X, \alpha Y, \alpha Z) = (X \cdot Y, Z) = (\alpha(X \cdot Y), \alpha Z)$

holds for all $\alpha Z$, hence $\alpha X \cdot \alpha Y = \alpha(X \cdot Y)$.

(2) $\Rightarrow$ (3) Since we have already known (2) $\Rightarrow$ (1), we can use $\text{tr}(\alpha X) = \text{tr}(X)$ (Lemma 2.1.2). Now $3 \text{det } \alpha X = \text{tr}(\alpha X, \alpha X, \alpha X) - \frac{3}{2} \text{tr}(\alpha X)(\alpha X, \alpha X) + \frac{1}{2} \text{tr}(\alpha X)^2$

$$= \text{tr}(X, X, X) - \frac{3}{2} \text{tr}(X(X)X, X) + \frac{1}{2} \text{tr}(X)^2 = 3 \text{det } X.$$  

(3) $\Rightarrow$ (5) $(\alpha X \cdot Y, \alpha Z) = (X \cdot Y, Z) = (\alpha X, \alpha Y, \alpha Z)$ (Lemma 2.1.1)

$$= (\alpha X \cdot Y, \alpha Z) \quad \text{holds for all } \alpha Z, \text{ hence } \alpha X \cdot \alpha Y = \alpha(X \cdot Y).$$

(5) $\Rightarrow$ (4) $(\det \alpha X) \alpha X = (X \cdot X) \cdot (\alpha X \cdot \alpha X) = (\alpha(X \cdot X)(X \cdot X))$

$= (\det X) \alpha X$, hence $\det \alpha X = \det X$. Next, in $\alpha X \cdot \alpha E = \alpha(X \cdot E) = \frac{1}{2} \alpha(tr(X)E - X)$,

put $\alpha E = P = P(\rho, \rho)$, then
\[ aX \times P = \frac{1}{2} \text{tr}(X)P - \frac{1}{2} aX. \] \hfill (i)

Put \( X = \alpha^{-1} E_i \) in (i) and compare each term of both sides, then

\[ 0 = \mu \rho_i - 1, \quad \rho_3 = \mu \rho_2, \quad \rho_2 = \mu \rho_3, \quad -p_i = \mu p_1, \quad 0 = \mu p_2, \quad 0 = \mu p_3 \]

where \( \mu = \text{tr}(\alpha^{-1} E_i) \). Consequently we have \( p_2 = p_3 = 0 \). Similarly \( p_1 = 0 \). Again put \( X = \alpha^{-1} F_i \) in (i) and compare \( F_i \)-parts, then \( \rho_i = 1 \). Similarly \( \rho_2 = \rho_3 = 1 \). Thus \( \alpha E = E \).

(4) \( \rightarrow \) (2) \quad \text{tr}(\alpha X) = (\alpha X, E, E) = (\alpha X, \alpha E, \alpha E) = (X, E, E) = \text{tr}(X), \quad \frac{1}{2} (\text{tr}(X) \text{tr}(Y) - (X, Y)) = (X, Y, E) = (\alpha X, \alpha Y, E) = \frac{1}{2} (\text{tr}(\alpha X) \text{tr}(\alpha Y) - (\alpha X, \alpha Y)) = \frac{1}{2} (\text{tr}(X) \text{tr}(Y) - (X, \alpha Y)). \]

Hence \( (\alpha X, \alpha Y) = (X, Y) \). Finally using \( (X, Y, Z) = \text{tr}(X, Y, Z) - \frac{1}{2} \text{tr}(Y)(X, Z) - \frac{1}{2} \text{tr}(Z)(X, Y) - \frac{1}{2} \text{tr}(X) \text{tr}(Y) \text{tr}(Z) \), we have \( \text{tr}(\alpha X, \alpha Y, \alpha Z) = \text{tr}(X, Y, Z) \).

The Lie algebra \( \mathfrak{f}_i^c \) of the Lie group \( F_i^c \) has the following structure.

**Proposition 2.1.4** ([2]). \( \mathfrak{f}_i^c = h_i^c \oplus (\widetilde{m}^c)^- \)

where \( h_i^c = \{ \delta \in \mathfrak{f}_i^c | \delta E_i = 0, \ i = 1, 2, 3 \} \) is the complex Lie algebra of type \( D_4 \), \( (\widetilde{m}^c)^- = \{ A \in M(3, \mathbb{C}) | A^* = -A \} \) and for \( A \in (\widetilde{m}^c)^- \), \( \tilde{A} \) is the \( C \)-linear transformation of \( \mathfrak{z}^c \) defined by \( \tilde{A}X = AX -XA, \ X \in \mathfrak{z}^c \).

### 2.2. Involutions of Lie groups of type \( F_i \)

We define \( R \)-linear transformations \( \gamma, \sigma, \sigma' \) of \( \mathfrak{z}(3, \mathbb{C}) \) by

\[
\begin{align*}
\gamma X &= \gamma (\xi, x) = X(\xi, \gamma x), \quad X \in \mathfrak{z}(3, \mathbb{C}), \\
\sigma X &= \left( \begin{array}{c}
\xi_1 \\
-x_3 \\
-x_2 \\
-x_1 \\
\xi_3 \\
\xi_2 \\
\xi_1 \\
\xi_3 \\
\end{array} \right) = I_3 XI_1, \quad \sigma' X = \left( \begin{array}{c}
\xi_1 \\
x_3 \\
x_2 \\
x_1 \\
-x_3 \\
-x_2 \\
-x_1 \\
\xi_3 \\
\end{array} \right) = I_3 XI_2,
\end{align*}
\]

respectively. Then \( \gamma \in G_3 \subset F_i \subset F_i^c, \sigma, \sigma' \in F_i \subset F_i^c \) and \( \gamma^2 = \sigma^2 = \sigma'^2 = 1 \). Let \( \tau \) be the complex conjugation in \( \mathfrak{z}^c \) with respect to \( \mathfrak{z}(3, \mathbb{C}) \), then \( \tau, \tau \gamma, \tau \sigma \) induce involutive automorphisms \( \tilde{\tau}, \tilde{\tau} \gamma, \tilde{\tau} \sigma \) of \( F_i^c \):

\[
\tilde{\tau}(\alpha) = \tau \alpha \tau, \quad \tilde{\tau} \gamma(\alpha) = \tau \gamma \alpha \tau, \quad \tilde{\tau} \sigma(\alpha) = \tau \sigma \alpha \tau, \quad \alpha \in F_i^c.
\]

**Lemma 2.2.1.** \( (\mathfrak{z}^c)_\gamma = \mathfrak{z}(3, \mathbb{C}), (\mathfrak{z}^c)_\tau = \mathfrak{z}(3, \mathbb{C}'), (\mathfrak{z}^c)_\sigma = \mathfrak{z}(1, 2, \mathbb{C}). \)

**Proof.** The first two are trivial (Lemma 1.2.1. (2)). The correspondence
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$(3^C)_{rs} = \begin{pmatrix} \xi_1 & ix_3 & i\xi_2 \\ ix_3 & \xi_2 & x_1 \\ i\xi_2 & x_1 & \xi_3 \end{pmatrix} 
\rightarrow \begin{pmatrix} \xi_1 & x_3 & \xi_2 \\ -x_3 & \xi_2 & x_1 \\ -x_2 & \xi_1 & \xi_3 \end{pmatrix} \in \mathcal{S}(1, 2, \mathbb{C})$.

$\xi_i \in \mathbb{R}, \ x_i \in \mathbb{C}$, gives an isomorphism as Jordan algebras.

**Theorem 2.2.2.** $(F_4^C)^r = F_4^r, (F_4^C)^s = F_4^s = F_4^{s_0}$.

**Proof.** These are direct results of Lemma 2.2.1.

**Proposition 2.2.3.** (1) $\gamma$ and $\gamma\sigma$ are conjugate in $F_4$: $\delta \gamma = \gamma \sigma \delta$ (moreover under $\delta \in F_4$ such that $\delta \sigma = \sigma \delta$).

(2) $\sigma$ and $\sigma'$ are conjugate in $F_4$: $\delta \sigma = \sigma' \delta$ (moreover under $\delta = \delta^{-1} \in F_4$).

**Proof.** (1) Define $\delta : \mathfrak{X}(3, \mathbb{C}) \rightarrow \mathfrak{X}(3, \mathbb{C})$ by

$$
\delta X = \begin{pmatrix} x_3 e & -e\xi_2 e & \xi_3 e \\ -e\xi_3 e & x_2 e & -e\xi_1 e \\ \xi_3 e & -e\xi_1 e & x_2 e \end{pmatrix} = \bar{D} XD, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & e \\ 0 & e & 0 \end{pmatrix}.
$$

Then $\delta \in F_4$, $\delta \gamma = \gamma \sigma \delta$ and $\delta \sigma = \sigma \delta$.

(2) Define $\delta : \mathfrak{X}(3, \mathbb{C}) \rightarrow \mathfrak{X}(3, \mathbb{C})$ by

$$
\delta X = \begin{pmatrix} x_2 & \xi_2 & x_3 \\ x_1 & \xi_3 & \xi_2 \\ \xi_1 & x_2 & x_3 \end{pmatrix} = D XD, \quad D = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
$$

Then $\delta = \delta^{-1} \in F_4$ and $\delta \sigma = \sigma' \delta$.

### 2.3. Subgroups of type $C_4 \oplus C_3$ of Lie groups of type $F_4$.

**Lemma 2.3.1.** Any element $M \in \mathfrak{X}(3, H^C)$ such that $M^2 = M$, $\text{tr}(M) = 1$ can be transformed to any $E_i$ by a certain $A \in \text{Sp}(3, H^C)$: $AMA^* = E_i$ ($i = 1, 2, 3$).

**Proof.** Since $\text{Sp}(3, H^C)$ contains the subgroup $\text{Sp}(3)$, we may assume

$$
M = \begin{pmatrix} \mu_1 & im_3 & i\bar{m}_2 \\ im_3 & \mu_2 & im_1 \\ i\bar{m}_2 & im_1 & \mu_3 \end{pmatrix}, \quad \mu_i \in \mathbb{C}, \ m_i \in H,
$$

Then condition $M^2 = M$ is

$$
\mu_1 + \mu_2 + \mu_3 = 1.
$$
\[ \begin{pmatrix} \mu_1^2 - m_2 \bar{m}_2 - m_3 \bar{m}_3 & \bar{m}_3 \bar{m}_1 + i(\mu_1 + \mu_2)m_3 \\ * & \mu_2^2 - m_3 \bar{m}_3 - m_1 \bar{m}_1 \\ - \bar{m}_1 \bar{m}_1 + i(\mu_2 + \mu_1)m_2 & * \\ \mu_3^2 - m_1 \bar{m}_1 - m_3 \bar{m}_3 \end{pmatrix} = M. \]

Compare the diagonals, then each \( \mu_i \) is real. Hence we have

\[ m_1 m_2 = m_2 m_3 = m_3 m_1 = 0, \quad \mu_1 m_1 = \mu_2 m_2 = \mu_3 m_3 = 0. \]

If \( m_1 = m_2 = m_3 = 0 \) Lemma is clearly valid. Otherwise, for example, in the case \( m_3 \neq 0 \), we have \( m_1 = m_2 = 0, \mu_1 = \mu_2 = 1, \mu_3 = 0 \). Hence \( M \) has the form

\[ M = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} \mu & im \\ i\bar{m} & \nu \end{pmatrix}, \quad m\bar{m} = -\mu \nu, \mu + \nu = 1, \mu, \nu \in \mathbb{R}, m \in \mathbb{H}. \]

If \( \mu > 0, \nu < 0 \), this \( M \) can be transformed to \( E_1 \) by \( \begin{pmatrix} \bar{m}/\sqrt{-\nu} & i\sqrt{\nu}/\sqrt{\mu} \\ -i\bar{m}/\sqrt{\nu} & \sqrt{\mu} \end{pmatrix} \in \mathcal{S}(2, H^C) \). If \( \mu < 0, \nu > 0 \), then \( M \) can be transformed to \( E_2 \). Finally note that \( E_1, E_2, E_3 \) are transformed by \( \mathcal{S}(3, H^C) \) with one another. Thus Lemma is proved.

**Proposition 2.3.2.** \( F_4(\mathbb{S}(3, H^C)) \cong \mathcal{S}(3, C)/\mathbb{Z}_2. \)

**Proof ([6]).** We define \( \phi : \mathcal{S}(3, H^C) \to F_4(\mathbb{S}(3, H^C)) \) by

\[ \phi(A)M = AAM^*, \quad M \in \mathbb{S}(3, H^C). \]

It is clear that \( \phi \) is well-defined and a homomorphism. We shall show \( \phi \) is onto. Let \( \alpha \in F_4(\mathbb{S}(3, H^C)) \). Since \( \alpha E_i \in \mathbb{S}(3, H^C) \) satisfies \( (\alpha E_i)^2 = \alpha E_i, \text{tr}(\alpha E_i) = 1 \), there exists \( A_i \in \mathcal{S}(3, H^C) \) such that

\[ \alpha E_i = A_i E_i A_i^*, \quad i = 1, 2, 3 \quad (\text{Lemma 2.3.1}). \]

Let \( A_i \) be the \( i \)-th column vector of \( A_i \) and construct a matrix \( A = (a_1, a_2, a_3) \). Then we have \( \alpha E_i = A E_i A^* \), \( i = 1, 2, 3 \). Hence \( AA^* = A(E_1 + E_2 + E_3)A^* = (A E_1 + E_2 + E_3) = A E = E \), that is, \( A \in \mathcal{S}(3, H^C) \). Put \( \beta = 1\alpha^{-1} \), then \( \beta \in F_4(\mathbb{S}(3, H^C)) \) and satisfies \( \beta E_i = E_i, i = 1, 2, 3 \). \( \beta \) induces \( C \)-linear transformations \( \beta_i \) of \( H^C \) such that \( \beta F_i(m) = F_i(\beta_i m), m \in H^C \) from \( 2F_i, F_i(m) = F_i(m), j \neq i \), moreover \( \beta_i \) are orthogonal: \( \beta_i \in O(4, C) = O(H^C) \) from \( F_i(m) + F_i(n) = (m, n)(E_{i+1} + E_{i+2}) \). Furthermore \( \beta_1, \beta_2, \beta_3 \) satisfy

\[ (\beta_1 m)(\beta_2 n) = \beta_3(mn), \quad m, n \in H^C \]

from \( 2F_1(m) + F_2(n) = F_3(mn) \). Put \( p = \beta_1, q = \beta_2, 1 \), then \( p, q \in \mathcal{S}(1, H^C) \) and \( \beta_3(m) = \beta_3(m)q, \beta_3(m) = \beta_3(m)q, m \in H^C \). Again put \( \beta_i(m) = p \zeta(m), m \in H^C \), that is, \( \zeta \) satisfies \( \zeta(m)(\zeta(n) = \zeta(mn), m, n \in H^C \), that is, \( \zeta \) is an automorphism of \( H^C \). Hence there exists \( r \in \mathcal{S}(1, H^C) \) such that \( \zeta(m) = rm\bar{r}, m \in H^C \) (Proposition 1.3.1). Therefore

\[ \beta_1 m = prm\bar{r}, \quad \beta_2 m = r\bar{q}m\bar{r}, \quad \beta_3 m = qrm\bar{p} \]


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Construct a matrix $B = \text{diag}(\eta r, pr, r) \in S p(3, H^C)$, then $\beta M = B M B^*$, $M \in \mathfrak{S}(3, H^C)$, that is, $\beta = \phi(B)$. Hence $\alpha = \phi(A) \beta = \phi(A) \phi(B) = \phi(AB)$, $AB \in S p(3, H^C)$. Therefore $\phi$ is onto. $\text{Ker} \phi = \{E, -E\} = Z_2$. Thus $F_4(\mathfrak{S}(3, H^C)) \cong S p(3, H^C)/Z_2 \cong S p(3, C)/Z_2$.

**Theorem 2.3.3.** $(F_4)^r \cong (S p(1, C) \times S p(3, C))/Z_2$, $Z_2 = \{(1, E), (-1, -E)\}$.

**Proof.** ([6]). We define $\phi : S p(1, H^C) \times S p(3, H^C) \to (F_4)^r$ by

$$\phi(p, A)(M + a) = A M A^* + p A A^*, \quad M + a \in \mathfrak{S}(3, H^C) \oplus (H^C)^r = \mathfrak{S}^r.$$  

It is easy to verify that $\phi$ is well-defined (Proposition 2.1.3. (5)) and a homomorphism. We shall show $\phi$ is onto. Let $\alpha \in (F_4)^r$. Since $(\mathfrak{S}^r) = \mathfrak{S}(3, H^C)$ is invariant under $\alpha$, $\alpha$ induces an automorphism of $\mathfrak{S}(3, H^C)$. Hence there exists $A \in S p(3, H^C)$ such that

$$\alpha M = A M A^*, \quad M \in \mathfrak{S}(3, H^C) \quad \text{(Proposition 2.3.2).}$$

Put $\beta = \phi(1, A)^{-1} \alpha$, then $\beta \mathfrak{S}(3, H^C) = 1$, hence $\beta \in G^c = \{\alpha \in F_4^c | \alpha E_i = E_i, \alpha F_i(1) = F_i(1), i = 1, 2, 3\}$, moreover $\beta \in (G^c)^r$ and $\beta | H^C = 1$. By Theorem 1.3.2, there exists $p \in S p(1, H^C)$ such that $\beta(m + a e) = m + (p a) e$, $m + a e \in H^C \oplus H^C e = C^r$, hence $\beta(M + a e) = M + p a$, $M + a \in C^r$, that is, $\beta = \phi(p, E)$. Hence $\alpha = \phi(1, A) \beta = \phi(1, A) \phi(p, E) = \phi(p, A)$. Therefore $\phi$ is onto. $\text{Ker} \phi = \{(1, E), (-1, -E)\} = Z_2$, Thus we have the required isomorphism.

**Lemma 2.3.4.** $\phi : S p(1, H^C) \times S p(3, H^C) \to (F_4)^r$ of Theorem 2.3.3 satisfies

1. $\gamma = \phi(1, E), \gamma e = \phi(j, j E), \gamma n = \phi(i, i E), \sigma = \phi(-1, I_i)$.

2. $\tau \phi(p, A) = \phi(\tau p, \tau A), \gamma c \phi(p, A) = \phi(\gamma c p, \gamma c A), \sigma \phi(p, A) = \phi(\sigma p, I_i A I_i)$.

**Theorem 2.3.5.** (1) $(F_4)^r \cong (S p(1) \times S p(3))/Z_2 \cong (F_4^c)^r$.

2. $(F_4^c)^r \cong (S p(1, R) \times S p(3, R))/Z_2 \times Z_2$.

3. $(F_4^c)^r \cong (S p(1) \times S p(3, 2))/Z_2 \cong (\gamma A)^r \sim (F_4^c)^r$.

**Proof.** (1) Let $\alpha \in (F_4)^r = (F_4^c)^r = (F_4^c)^r \subseteq (F_4^c)^r$. By Theorem 2.3.3, there exist $p \in S p(1, H^C), A \in S p(3, H^C)$ such that $\alpha = \phi(p, A)$. From the condition $\tau \alpha = \alpha \tau$, we have $\phi(\tau p, \tau A) = \phi(p, A)$ (Lemma 2.3.4). Hence

$$\tau p = p, \quad \tau A = A \quad \text{or} \quad \tau p = -p, \quad \tau A = -A.$$ 

The latter case is impossible (cf. Theorem 1.3.4). Therefore $p \in S p(1), A \in S p(3)$. Thus $(F_4)^r \cong \phi(S p(1) \times S p(3)) \cong (S p(1) \times S p(3))/Z_2$. $(F_4^c)^r = (\gamma A)^r = (\tau A)^r$.

(2) $F_4^c = (F_4^c)^r \cong (F_4^c)^{\tau c}$.
In fact, since $\gamma$ and $\gamma_c$ are conjugate in $G_2 \subset F_4$: $\delta \gamma = \gamma_c \delta$, $\delta \tau = \tau \delta$ (Proposition 1.2.3), $(F_4^c)^{\tau \gamma} \cong \alpha \rightarrow \delta \alpha \delta^{-1} \in (F_4^c)^{\gamma \gamma_c}$ gives an isomorphism. Let $\alpha \in ((F_4^c)^{\gamma \gamma_c})^\gamma = (\tau \gamma_c)^\gamma$, $\alpha = \phi(p, A)$, $p \in S \sigma p(1, H^c)$, $A \in S \sigma p(3, H^c)$. From $\tau \gamma_c \sigma = \sigma \gamma_c$, we have $\phi(\tau \gamma_c \sigma, \tau \gamma_c A) = \phi(p, A)$. Hence $(\tau \gamma_c)^\gamma \cong (S \sigma p(1, H^c) \times S \sigma p(3, H^c)) / \mathbb{Z}_2$ (cf. Theorem 1.3.5) $\cong (S \sigma p(1, R) \times S \sigma p(3, R)) / \mathbb{Z}_2 \times 2$. $(\phi(i, i E) = \gamma H)$.

(3) Define $\phi: S \sigma p(1, H^c) \times S \sigma p(1, 2, H^c) \rightarrow (F_4^c)^\gamma$ by $\phi(p, A) = \phi(p, \Gamma_1 A \Gamma_1^{-1})$. Let $\alpha \in (F_4^c)^{\gamma \gamma_c} \rightarrow (F_4^c)^{\gamma \gamma_c}$ by $\phi(p, A) = \phi(p, \Gamma_1 A \Gamma_1^{-1})$. Let $\alpha \in S \sigma p(1, 2, H^c), A \in S \sigma p(1, 2, H^c)$. From $\sigma \alpha = \alpha \sigma$, we have $\phi(p, \alpha \sigma) = \phi(p, A)$. Thus, as in (1), $(F_4^c)^{\gamma \gamma_c} \cong (S \sigma p(1, 2, H^c)) / \mathbb{Z}_2$.

$F_{(\alpha)} = (F_4^c)^{\gamma \gamma_c} \equiv (F_4^c)^{\gamma \gamma_c}$

because $\gamma \sim \gamma \sigma$ under $\delta \in F_1$: $\delta \gamma = \gamma \sigma \delta$, $\delta \tau = \tau \delta$ (Proposition 2.2.3). Now $(F_4^c)^{\gamma \gamma_c} \sim (\tau \gamma \sigma)^\gamma \equiv (\tau \gamma \sigma)^\gamma$.  

### 2.4. Subgroups of type $B_4$ of Lie groups of type $F_4$

Hereafter we use the following $C$-vector subspaces of $\mathfrak{g}^c$.

$\mathfrak{g}(2, \mathfrak{g}^c) = \{ X \in \mathfrak{g}^c | E_1 \times X = 0 \} = \{ X \in \mathfrak{g}^c | 4 E_1 \times (E_1 \times X) = 0 \}$

$$= \left\{ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & \xi_2 & \xi_1 \\ 0 & \xi_1 & 0 \end{array} \right\} \text{identify } \left\{ \begin{array}{ccc} \xi_2 & x \\ \xi_1 & \xi_0 \end{array} \right\}, \quad \xi_1, \xi_2 \in C, \quad x \in \mathfrak{g}^c, \quad \xi_2 \in \mathfrak{g}^c$$

$C_1 \mathfrak{g}^c = \{ \xi E_1 | \xi \in C \}$

$(\mathfrak{g}^c)_e = \{ X \in \mathfrak{g}^c | \sigma X = 0 \} = \mathfrak{g}(2, \mathfrak{g}^c) \oplus C_1 \mathfrak{g}^c$,

$(\mathfrak{g}^c)_o = \{ X \in \mathfrak{g}^c | \sigma X = -X \} = \{ X \in \mathfrak{g}^c | 2 E_1 \times X = 0 \}$

$$= \{ X \in \mathfrak{g}^c | E_1 \times X = 0, \quad (E_1, X) = 0 \}$$

$$= \left\{ \begin{array}{ccc} 0 & x_3 & \xi_2 \\ \xi_3 & 0 & 0 \\ x_2 & 0 & 0 \end{array} \right\} x_3, \xi_3 \in \mathfrak{g}^c$$

and $(\mathfrak{g}^c)_o = \{ X \in \mathfrak{g}^c | \sigma(X) = 0 \}$, $\mathfrak{g}(2, \mathfrak{g}^c)_o = \{ X \in \mathfrak{g}(2, \mathfrak{g}^c) | \sigma(X) = 0 \}$. $(\mathfrak{g}^c)_o$, $(\mathfrak{g}^c)_o$ are invariant under $\alpha \in (F_4^c)^{\sigma}$.

**Lemma 2.4.1.** $(F_4^c)^{\sigma} = (F_4^c)_{E_1}$.

**Proof.** Let $\alpha \in (F_4^c)^{\sigma}$. Then $\alpha E_2 \in \mathfrak{g}(2, \mathfrak{g}^c)$. In fact, $\alpha E_2 \alpha = \alpha(-F_4(1) \times F_4(1)) = -\alpha F_4(1) \times F_4(1) = -(F_4(1) \times F_4(1)) \sigma = x_2 \xi_2 E_2 + x_3 \xi_3 E_2 - F_4(\xi_2 \xi_3) \subset \mathfrak{g}(2, \mathfrak{g}^c)$. Similarly $\alpha E_3 \in \mathfrak{g}(2, \mathfrak{g}^c)$. Therefore $\alpha E_1 \alpha = E - \alpha E_2 - \alpha E_3$ has the form
E_1 + \xi_1 E_2 + \xi_1 E_3 + F_1(x). Then $0 = \alpha(E_1 \times E_1) = \alpha E_1 \times \alpha E_1 = (E_1 + \xi_1 E_2 + \xi_1 E_3 + F_1(x))^* = (\xi_1 - x \bar{x}) E_1 + \xi_1 E_2 + \xi_1 E_3 - F_1(x)$. This implies $\xi_1 = x = 0$. Thus we have $\alpha E_1 = E_1$. Conversely let $\alpha \in (F_4)_{\xi_1}$. Since $3^C = (3^C)_0 \oplus (3^C)_- \sigma$ and $(3^C)_\pm = \{X \in 3^C | E_1 = 0\} \oplus \{\xi E_1 | \xi \in C\}$, $(3^C)_\sigma = \{X \in 3^C | 2E_1 + X = X\}$ are invariant under $\alpha$, $\alpha \sigma X = \alpha \sigma (X_1 + X_2) = \alpha X_1 + \alpha X_2 = \alpha \sigma (X_1) + \sigma (\alpha X_2) = \sigma \alpha (X_1 + X_2) = \sigma \alpha X$ for $X = X_1 + X_2$, $X_1 \in (3^C)_\sigma$, $X_2 \in (3^C)_- \sigma$. Hence $\alpha \sigma = \sigma \alpha$, that is, $\alpha \in (F_4^C)^\sigma$.

**Lemma 2.4.2.** $(F_4^C)^\sigma / \text{Spin}(8, C) = (S^C)^\bullet$. In particular, the group $(F_4^C)^\sigma$ is connected.

**Proof.** We define a complex 8-dimensional sphere $(S^C)^\bullet$ by

$$(S^C)^\bullet = \{X \in 3^C | E_1 + X = 0, \text{tr}(X) = 0, (X, X) = 2\} = \left\{ \begin{pmatrix} \xi & x \\ \bar{x} & -\xi \end{pmatrix} \middle| \xi^2 + \bar{x}x = 1, \xi \in C, x \in \mathbb{C} \right\}.$$ 

The group $(F_4^C)^\sigma = (F_4^C)_{\xi_1}$ acts on $(S^C)^\bullet$ (Lemma 2.1.2, Proposition 2.1.3.(3)). We shall show that this action is transitive. To show this we prepare some elements of $(F_4^C)^\sigma$.

For $a \in \mathbb{C}$ such that $ad \neq 0$, define a $C$-linear transformation $\alpha(a)$ of $3^C$, $\alpha(a)X(\xi, x) = Y(\eta, y)$, by

$$
\begin{align*}
\eta_1 & = \xi_1, \\
\eta_2 & = \frac{1}{2} (\xi_2 + \xi_3) + \frac{1}{2} (\xi_2 - \xi_3) \cos 2\nu + (a, x_1) \frac{\sin 2\nu}{\nu}, \\
\eta_3 & = \frac{1}{2} (\xi_2 + \xi_3) - \frac{1}{2} (\xi_2 - \xi_3) \cos 2\nu - (a, x_1) \frac{\sin 2\nu}{\nu}, \\
y_1 & = x_1 - \frac{1}{2} (\xi_2 - \xi_3) a \frac{\sin 2\nu}{\nu} - 2(a, x_1) a \frac{\sin 5\nu}{\nu^5}, \\
y_2 & = x_1 \cos \nu - x_3 \frac{\sin 3\nu}{\nu}, \\
y_3 & = x_1 \cos \nu + x_3 \frac{\sin 3\nu}{\nu}
\end{align*}
$$

where $\nu \in C$, $\nu^3 = ad$. Then $\alpha(a) = \exp \tilde{A}(a)$ where $A(a) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -a & 0 \end{pmatrix} \in (\mathfrak{m}^C)^- \mathfrak{E}_1 = \{A \in (\mathfrak{m}^C)^- | \tilde{A} E_1 = 0\}$, hence $\tilde{A}(a) \in (\mathfrak{f}_4^C)^\sigma = \mathfrak{b}_4^C \oplus (\mathfrak{m}_8^C)^- \mathfrak{E}_1$ (Proposition 2.1.4). Therefore $\alpha(a) \in (F_4^C)^\sigma$.

Now let $X = \begin{pmatrix} \xi & x \\ \bar{x} & -\xi \end{pmatrix} \in (S^C)^\bullet$. Choose $a \in \mathbb{C}$ such that $(a, x) = 0$ and $ad = \ldots$
\( (\pi/4)^a \), then \( \alpha(a)X=X=\begin{pmatrix} 0 & x_1 \\ x_1 & 0 \end{pmatrix} \), \( x_1x_1=1 \). And then \( \alpha((\pi/4)x_1)X=E_2-E_3 \).

This shows the transitivity. The isotropy subgroup of \( (F_4^c)^a \) at \( E_2-E_3 \) is \( (F_4^c)_{E_2-E_3} = \{ \alpha \in F_4^c \mid \alpha E_i = E_i, \ i=1,2,3 \} \) and we know that it is isomorphic to \( \text{Spin}(8, C) \) as the universal covering group of \( SO(8, C) = SO(6^+) \) (cf. Principle of triality [8]). Thus we have the homeomorphism \( (F_4^c)^a/\text{Spin}(8, C) = (S^c)^a \).

**Theorem 2.4.3.** \( (F_4^c)^a \equiv \text{Spin}(9, C) \).

**Proof.** Since the group \( (F_4^c)^a \) is connected (Lemma 2.4.2), we can define a homomorphism \( \pi : (F_4^c)^a \to SO(9, C) = SO((V^c)^a) \) by \( \pi(\alpha) = \alpha \mid (V^c)^a \) where

\[
(V^c)^a = \{(2, (\xi x), A) \mid \xi \in C, \ x \in \mathbb{C} \}
\]

with the norm \( (X, Y)/2 = \xi^2 + x\bar{x} \). Ker \( \pi = \{1, \sigma = \text{Z}_2 \} \) (cf. Principle of triality [8]). Hence \( \pi \) induces a monomorphism \( d\pi : (\mathbb{I}^c)^a \to \mathbb{S} \mathfrak{g}(9, C) \). Since \( \text{dim}_{\mathbb{C}}(\mathbb{I}^c)^a = \text{dim}_{\mathbb{C}}(\mathbb{I}^c)^a \mathbb{C} \mathbb{C} = 28+8 = 36 = \text{dim}_{\mathbb{C}}(\mathbb{S} \mathfrak{g}(9, C), d\pi \) is onto, hence \( \pi \) is also onto (Lemma 0.6). Thus \( (F_4^c)^a/\text{Z}_2 \equiv SO(9, C) \). Therefore \( (F_4^c)^a \) is isomorphic to \( \text{Spin}(9, C) \) as the universal covering group of \( SO(9, C) \).

**Theorem 2.4.4.**

1. \( (F_4)^a \equiv \text{Spin}(9) = (F_4_{(4,20)})^a \).
2. \( (F_4_{(4,20)})^a \equiv \text{spin}(4, 5) \).
3. \( (F_4_{(4,20)})^a \equiv \text{(2a)}^a \).

**Proof.**

1. \( (F_4)^a = ((F_4^c)^a)^a = ((F_4^c)^a)^a \) is connected (Lemma 0.7) because \( (F_4^c)^a = \text{Spin}(9, C) \) (Theorem 2.4.3) is simply connected. Since \( (F_4^c)^a \) acts on \( (V^c)^a \), the group \( (F_4)^a = ((F_4^c)^a)^a \) acts on

\[
V^a = \{(2, (\xi x), A) \mid \xi \in \mathbb{R}, \ x \in \mathbb{C} \}
\]

with the norm \( (X, Y)/2 = \xi^2 + x\bar{x} \). We can define a homomorphism \( \pi : (F_4)^a \to SO(9) = SO(V^a) \) by \( \pi(\alpha) = \alpha \mid V^a \). Ker \( \pi = \{1, \sigma = \text{Z}_2 \} \). Since \( \text{dim}(\mathbb{I}^c)^a = \text{dim}(\mathbb{S} \mathfrak{g}(9)) \), \( \pi \) is onto. Thus \( (F_4)^a/\text{Z}_2 \equiv SO(9) \). Therefore \( (F_4)^a \) is isomorphic to \( \text{Spin}(9) \) as the universal covering group of \( SO(9) \). \( (F_4_{(4,20)})^a = (\tau \sigma)^a = (\tau)^a \).

(Remark) In the proof of Lemma 2.4.3, if we know that \( F_4^c \) is simply connected, the connectedness of \( (F_4^c)^a \) is trivial (Lemma 0.7). But the simply connectedness of \( F_4^c \) is usually follows from the simply connectedness of \( F_4 \) and the fact that \( (F_4)^a = \text{Spin}(9) \) ([8]). To avoid a circular argument we took the way like Lemma 2.4.2, Theorem 2.4.3).
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(2) As in (1), \((F_4(\mathbb{C}))^\sigma=\langle(F_4^i)^\sigma\rangle^T\) is connected. The group \((F_4(\mathbb{C}))^\sigma\) acts on 

\[ V^{1,3}=\mathcal{Z}(\mathbb{S}(\mathfrak{h}),\mathfrak{g}^\sigma)=\left\{ X=\begin{pmatrix} \xi & x' \\ \bar{x} & -\bar{\xi} \end{pmatrix} \right| \xi\in\mathbb{R}, x'\in\mathbb{G}^\sigma \] 

with the norm \((X,X)/2=\xi^2+x'\bar{x}'\). We can define a homomorphism \(\pi: (F_4(\mathbb{C}))^\sigma\to O(4,5)_0=O(V^{1,3})\) by \(\pi(\alpha)=\alpha|V^{1,3}\). Ker \(\pi=\{1,\sigma\}=\mathbb{Z}_2\). As similar to (1), 

\((F_4(\mathbb{C}))^\sigma/Z_2'=O(4,5)_0\). Therefore \((F_4(\mathbb{C}))^\sigma\) is denoted by \(\text{spin}(4,5)\) (not simply connected) as a double covering group of \(O(4,5)_0\).

(3) \((F_4(\mathbb{C}))'\approx (F_4(\mathbb{C}))^\sigma\)

because \(\sigma\sim\sigma'\) under \(\delta\in F_4: \delta\sigma=\sigma\delta, \delta\tau=\tau\delta\) (Proposition 2.2.3). As in (1), \((F_4(\mathbb{C})^\sigma)^\sigma=(\sigma\sigma')^\sigma\) is connected. The group \((\sigma\sigma')^\sigma\) acts on 

\[ V^{8,1}=\mathcal{Z}(\mathbb{S}(\mathfrak{h}),\mathfrak{g}^\sigma)=\left\{ X=\begin{pmatrix} \xi & ix \\ ix & -\xi \end{pmatrix} \right| \xi\in\mathbb{R}, x\in\mathbb{G} \] 

with the norm \((X,X)/2=\xi^2-x\bar{x}'\). We can define a homomorphism \(\pi: (\sigma\sigma')^\sigma\to O(8,1)_0=O(V^{8,1})\) by \(\pi(\alpha)=\alpha|V^{8,1}\). As similar to (1), \((\sigma\sigma')^\sigma/Z_2\approx O(8,1)_0\). Therefore \((\sigma\sigma')^\sigma\) is isomorphic to \(\text{Spin}(8,1)\) as the universal covering group of \(O(8,1)_0\).

3.1. Lie groups of type \(E_6\).

The universal connected linear Lie groups of type \(E_6\) are obtained as

\[ E_6^c=E_6(\mathcal{Z}(3,\mathfrak{g}^c))=\{\alpha\in\text{iso}_c(\mathcal{Z}(3,\mathfrak{g}^c))|\det\alpha X=\det X\}, \]

\[ E_6=\{\alpha\in\text{iso}_c(\mathcal{Z}(3,\mathfrak{g}^c))|\det\alpha X=\det X, \langle\alpha X,\alpha Y\rangle=\langle X,Y\rangle, \} \]

\[ E_6(\mathfrak{h}^c)=E_6(\mathcal{Z}(3,\mathfrak{h}^c))=\{\alpha\in\text{iso}_c(\mathcal{Z}(3,\mathfrak{h}^c))|\det\alpha X=\det X, \} \]

\[ E_6(\mathfrak{h}^0)=E_6(\mathcal{Z}(3,\mathfrak{h}^0))=\{\alpha\in\text{iso}_c(\mathcal{Z}(3,\mathfrak{h}^0))|\det\alpha X=\det X, \langle\alpha X,\alpha Y\rangle=\langle X,Y\rangle, \} \]

\[ E_6(\mathfrak{h}^{-14})=E_6(\mathcal{Z}(3,\mathfrak{h}^{-14}))=\{\alpha\in\text{iso}_c(\mathcal{Z}(3,\mathfrak{h}^{-14}))|\det\alpha X=\det X, \langle\alpha X,\alpha Y\rangle=\langle X,Y\rangle, \} \]

\[ E_6(\mathfrak{h}^{-26})=E_6(\mathcal{Z}(3,\mathfrak{h}^{-26}))=\{\alpha\in\text{iso}_c(\mathcal{Z}(3,\mathfrak{h}^{-26}))|\det\alpha X=\det X, \} \]

where \(\langle X,Y\rangle=(\tau X,Y), \langle X,Y\rangle_7=(\tau Y,X)\) and \(\langle X,Y\rangle_8=(\tau\sigma X,Y)\). (Similarly the group \(E_6(\mathcal{Z}(3,\mathfrak{h}^c))\) is defined). \(E_6^c, E_6, E_6(\mathfrak{h}^{-14})\) are simply connected (see Appendix).

The Lie algebra \(e_6^c\) of the Lie group \(E_6^c\) has the following structure.

PROPOSITION 3.1.1 ([2]). \(e_6^c=f_6^c\oplus\mathcal{Z}(3,\mathfrak{g}^c)\).
where \( \mathfrak{F}(3, \mathbb{C})_h = \{ T \in \mathfrak{F}(3, \mathbb{C}) | \text{tr}(T) = 0 \} \) and for \( T \in \mathfrak{F}(3, \mathbb{C})_h \), \( \bar{T} \) is the \( C \)-linear transformation of \( \mathfrak{F}^0 \) defined by \( \bar{T}X = T \cdot X \).

### 3.2. Involutions of Lie groups of type \( E_6 \)

**Lemma 3.2.1.** If \( \alpha \in E_6^c \) then \( \alpha^{-1} \in E_6^c \).

**Proof.** ([4]). \( \alpha^{-1}(Y \times Y) = \alpha^{-1}(Y \times Y) = (aY \times aY) \times (aY \times aY) \) (Lemma 2.1. 1). \( \alpha(Y) \alpha(Y) = \alpha((Y \times Y) \times (Y \times Y)), Y \in \mathfrak{F}^c \). Put \( Y = X \times X \), \( X \in \mathfrak{F}^c \), then \( \alpha^{-1}((Y \times Y) \times (Y \times Y)) = (aY \times aY) \times (aY \times aY) \).

(1) Case \( \det X \neq 0 \). We have \( \alpha^{-1}X \times \alpha^{-1}X = \alpha(X \times X) \). Hence 3 det \( \alpha^{-1}X = (\alpha^{-1}X, \alpha^{-1}X \times \alpha^{-1}X) = (\alpha^{-1}X, \alpha(X \times X)) = (X, X \times X) = 3 \) det \( X \). Consider \( \alpha^{-1} \) instead of \( \alpha \), then we have also det \( \alpha^{-1}X = \det X \).

(2) Case \( \det X = 0 \). If det \( \alpha^{-1}X \neq 0 \), we can use the result of (1). \( 0 = \det X = \det \alpha^{-1}X = \det \alpha^{-1}X \) (result of (1)) \( \neq 0 \), a contradiction. Thus \( \det \alpha^{-1}X = 0 \), hence \( \det \alpha^{-1}X = \det X \) is also valid.

We define an involutive automorphism \( \lambda \) of \( E_6^c \) by

\[
\lambda(\alpha) = \alpha^{-1}, \quad \alpha \in E_6^c \quad \text{Lemma 3.2.1).}
\]

Note that \( \lambda \) induces involutive automorphisms of \( E_6, E_6(\mathbb{C}), E_6(\mathbb{C}), E_6(-2g), E_6(-2g) \) and \( E_6(3, \mathbb{H}) \). As in \( G^c, E_6^c \) has involutive automorphisms \( \tilde{r}, \tilde{r_7}, \) and \( \tilde{r_9} \).

**Theorem 3.2.2.** \( (E_6^c)^{\tilde{r}} = E_6, (E_6^c)^{\tilde{r_7}} = E_6(\mathbb{C}), (E_6^c)^{\tilde{r_9}} = E_6(-2g), (E_6^c)^{\tilde{r_7}} = E_6(-2g), (E_6^c)^{\tilde{r_9}} = E_6(-2g) \).

**Proof.** As for \( E_6(\mathbb{C}), E_6(-2g) \), these are direct results of Lemma 1.2.1.(2). \( E_6, E_6(\mathbb{C}), E_6(-2g) \) are nothing but their definitions.

The Lie algebras of the Lie groups of type \( E_6 \) are as follows.

**Proposition 3.2.3.**

(1) \( e_6 = \{ \phi \in e_6^c | -\tau^t \phi \sigma = \phi \} = f_4 \oplus i\mathfrak{F}(3, \mathbb{C})_b \),

(2) \( e_6(\mathbb{C}) = \{ \phi \in e_6^c | 2 \tau \phi = \phi \} = f_4(\mathbb{C}) \oplus i\mathfrak{F}(3, \mathbb{C})_b \).

(3) \( e_6(-14) = \{ \phi \in e_6^c | -\tau^t \phi = \phi \} = f_4(-14) \oplus i\mathfrak{F}(3, \mathbb{C})_b \).

(4) \( e_6(-2g) = \{ \phi \in e_6^c | \tau \sigma = \sigma \} = f_4(-2g) \oplus i\mathfrak{F}(1, 2, \mathbb{C})_b \).

(5) \( e_6(-2g) = \{ \phi \in e_6^c | \tau \phi = \phi \} = f_4 \oplus i\mathfrak{F}(3, \mathbb{C})_b \).

**Proof.** The involutive automorphisms of \( e_6^c \) induced by \( \gamma, \sigma, \lambda, \tau \) are

\[
\gamma \gamma = \gamma \gamma + \gamma, \quad \sigma \sigma = \sigma \sigma + \sigma, \quad \lambda(\phi) = \delta - \bar{\delta}, \quad \tau \phi = \tau \phi + \tau.
\]
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for \( \delta + \bar{\delta} \in 3^{\mathbb{C}} \otimes 3^{\mathbb{C}}(3, \mathbb{C})_{\delta} = \mathbb{C} \). From this, Proposition is clear (Lemma 2.2.1).

In addition to \( \gamma, \gamma C, \gamma H \subseteq G \subseteq F \subseteq E \), \( \sigma, \sigma' \subseteq F \subseteq E \), we define one more involutive element \( \rho \subseteq E \), \( \rho : 3^{\mathbb{C}} \rightarrow 3^{\mathbb{C}} \) by

\[
\rho X = \begin{pmatrix}
-\xi_1 & ix_1 & -ix_2 \\
ix_2 & -ix_1 & -\xi_3 \\
ix_3 & ix_3 & \xi_1 \\
\end{pmatrix} = \bar{P}XP, \\
P = \begin{pmatrix}
i & 0 & 0 \\
0 & i & 0 \\
0 & 0 & 1 \\
\end{pmatrix}.
\]

**Proposition 3.2.3.** (1) \( \gamma \) and \( \rho \) are conjugate in \( E \); \( \delta \gamma = \rho \delta \); \( \delta \in E \).

(2) \( \sigma \) and \( \gamma \rho \) are conjugate in \( E \); \( \delta \sigma = \gamma \rho \delta \); \( \delta \in E \).

(3) \( \sigma \) and \( \gamma H \rho \) are conjugate in \( E \); \( \delta \sigma = \gamma H \rho \delta \); \( \delta \in E \).

**Proof** will be given in 3.5.12.

### 3.3. Subgroups of type \( F_4 \) of Lie groups of type \( E_6 \)

**Theorem 3.3.1.** (1) \( (E_6^{\mathbb{C}})^1 = F_4^{\mathbb{C}} \).

(2) \( (E_{4(-26)})^1 = F_4 = (E_6)^1 \).

(3) \( (E_{6(56)})^1 = (E_6)^1 \).

(4) \( (E_{6(-14)})^1 = (E_6)_{18} \).

**Proof.** (1) It is results of Proposition 2.1.3.(1)-(3).

(2) \( (E_{6(-26)})^1 = (E_6)^1 = (F_4)^1 \) (result of (1)) = \( F_4 \) (Theorem 2.2.2). \( (E_6)^1 = (\tau \chi)^1 = (\lambda)^1 \).

(3) \( (E_{6(12)})^1 = (\chi)^1 = (\chi)^1 = (F_4)^1 = (E_6)^1 \) (Theorem 2.2.2). \( (E_6)^1 = (\tau \chi)^1 = (\tau \chi)^1 = (\chi)^1 \).

(4) \( (E_{6(-14)})^1 = (\tau \chi)^1 = (\tau \chi)^1 = (\chi)^1 = (F_4)^1 = (E_6)^1 \) (Theorem 2.2.2).

To prove this, define \( \delta : 3^{\mathbb{C}} \rightarrow 3^{\mathbb{C}} \) by

\[
\delta X = \begin{pmatrix}
\xi_1 & ix_2 & ix_2 \\
ix_2 & -\xi_1 & -ix_1 \\
ix_2 & -\xi_1 & -ix_1 \\
\end{pmatrix} = DXD, \\
D = \begin{pmatrix}
1 & 0 & 0 \\
0 & i & 0 \\
0 & 0 & i \\
\end{pmatrix}
\]

(see Proposition 3.6.5), then \( \delta \subseteq E_6 \), \( \delta' = \sigma \), \( \delta \sigma = \sigma \delta \), \( \delta \tau = \tau \delta \). (Hereafter, this \( \delta \) will be denoted by \( \sqrt{\sigma} \)). Now \( (\tau)^1 = \alpha \rightarrow \delta \sigma \in (\tau \sigma)^1 \) gives an isomorphism.
3.4. Subgroups of type $C_i$ of Lie groups of type $E_6$.

We consider the Jordan algebra $\mathfrak{J}(4, \mathbb{H}^C) = \{P \in M(4, \mathbb{H}^C) | P^* = P\}$ with the Jordan multiplication $P^* Q = (PQ + QP)/2$ and the inner product $(P, Q) = \text{tr}(P^* Q)$. We define $g : \mathfrak{J}^c = \mathfrak{J}(3, \mathbb{H}^C) \oplus (\mathbb{H}^C)^3 \rightarrow \mathfrak{J}(4, \mathbb{H}^C)_h = \{P \in \mathfrak{J}(4, \mathbb{H}^C) | \text{tr}(P) = 0\}$ by

$$g(M + a) = \begin{pmatrix} \frac{1}{2} \text{tr}(M) & ia \\ ia^* & M - \frac{1}{2} \text{tr}(M) \end{pmatrix}, \quad M + a \in \mathfrak{J}^c. $$

**Lemma 3.4.1.** $g : \mathfrak{J}^c \rightarrow \mathfrak{J}(4, \mathbb{H}^C)_h$ is a $C$-linear isomorphism and satisfies

$$g(X \cdot gY) = g((X \times Y) + \frac{1}{4}(gX, gY)E), \quad X, Y \in \mathfrak{J}^c.$$

**Proof.**

$$g((M + a) \times (N + b)) = g((M - a) \times (N - b))$$

$$= g(M \times N - \frac{1}{2}(a^*b + b^*a) + \frac{1}{2}(aN + bM))$$

$$= \begin{pmatrix} \frac{1}{2} \text{tr}(M \times N) - \frac{1}{2}(a, b) \\ \frac{i}{2}(aN + bM) \\ \frac{i}{2}(aN + bM)^* \\ M \times N - \frac{1}{2}(a^*b + b^*a) - \frac{1}{2}(\text{tr}(M \times N) - (a, b))E \end{pmatrix}$$

$$= g(M + a) \cdot g(N + b) - \frac{1}{4}(M, N) - \frac{1}{2}(a, b))E$$

$$= g(M + a) \cdot g(N + b) - \frac{1}{4}(g(M + a), N + b)E.$$ 

Thus the first formula is shown. Take the trace of both sides, then we have the second formula.

**Theorem 3.4.2.** $(E_6^C)^t \cong Sp(4, C)/\mathbb{Z}_s, \ Z_s = \{E, -E\}$.

**Proof ([4]).** We define $\phi : Sp(4, \mathbb{H}^C) \rightarrow (E_6^C)^t$ by

$$\phi(A)X = g^{-1}(A(gX)A^*), \quad X \in \mathfrak{J}^c.$$

We have to prove $\phi(A) \equiv (E_6^C)^t$. Denote $\alpha = \phi(A)$ and put $Z = aX$.

$$3 \text{det}X = 3 \text{det}Z = (Z \times Z, Z) = (g(\gamma(Z \times Z)) , gZ)$$

$$= (gZ \cdot gZ - \frac{1}{4}(\gamma Z, Z)E, gZ) = (gZ \cdot gZ - \frac{1}{4}(gZ, gZ)E, gZ)$$
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\begin{align*}
&= (A(gX)A^* A(gX)A^* A(gX)A^* A(gX)A^* A(gX)A^*) - \frac{1}{4} (A(gX)A^* A(gX)A^* A(gX)A^*) \\
&= (gX \cdot gX - \frac{1}{4} (gX, gX) E, gX) = (gX, gX - \frac{1}{4} (gX, X) E, gX) \\
&= (g(X \times X), gX) = (X \times X, X) = 3 \det X, \\
&= (\gamma \alpha X, \alpha Y) = (g(\alpha X), g(\alpha Y)) = (A(g \alpha X) A^*, A(g \alpha Y) A^*) = (gX, gY) = (X, Y) \\
&= (\alpha \gamma X, \gamma Y), \text{ hence } \gamma \alpha = \alpha \gamma.
\end{align*}

Thus \( \alpha \in (E, c)^i \). We shall show \( \phi \) is onto. To show this we prepare

\textbf{Lemma 3.4.3.} Any element \( P \in \mathfrak{S}(4, H^c) \) such that \( P^2 = P \), \( \text{tr}(P) = 1 \) can be transformed to \( E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{S}(4, H^c) \) by a certain \( A \in S_p(4, H^c) : APA^* = E_1. \)

\textbf{Proof} is similar to Lemma 2.3.1.

Now, for \( \alpha \in (E, c)^i \), \( (g(\alpha E))^2 = g(\alpha E) + \frac{3}{4} E. \) In fact, \( (g(\alpha E))^2 = g(\gamma(\alpha E \times \alpha E)) \)
\[
+ \frac{1}{4}(\gamma \alpha E, \alpha E)E = g(\gamma \alpha^{-1}(E \times E)) + \frac{1}{4}(\alpha \gamma E, \alpha E)E = g(\alpha \gamma E) + \frac{1}{4}(\gamma E, E)E = g(\alpha E)
\]
\[
+ \frac{3}{4} E. \] Put \( P = \frac{1}{4}(2g(\alpha E) + E). \) Then \( P \in \mathfrak{S}(4, H^c), P^2 = \frac{1}{16}(4g(\alpha E))^2 + 4g(\alpha E) + E = \frac{1}{4}(2g(\alpha E) + E) = P \) and \( \text{tr}(P) = 1. \) Hence there exists \( A \in S_p(4, H^c) \) such that
\[
P = AE_1A^* \quad \text{(Lemma 3.4.3).}
\]
Then \( \phi(A)E = g^{-1}(A(g \alpha E) A^*) = g^{-1}(A(2E_1 - \frac{1}{2} E) A^*) = g^{-1}(2P - \frac{1}{2} E) = g^{-1}(g(\alpha E)) = \alpha E. \) Put \( \beta = \phi(A)^{-1} \alpha, \) then \( \beta E = E, \) hence \( \beta \in F(4, c) \) (Proposition 2.1.3. (4)), moreover \( \beta \in (F(4, c)) \). By Theorem 2.3.3, there exist \( \rho \in S_p(1, H^c), D \in S_p(3, H^c) \) such that
\[
\beta(M + \alpha) = DMD^* + \rho aD^*, \quad M + \alpha \in \mathfrak{S}(c).
\]
Put \( B = \text{diag}(\rho, D) \in S_p(4, H^c), \) then \( \beta = \phi(B). \) In fact,
\[
\phi(B)(M + \alpha) = g^{-1}(B(g(M + \alpha)) B^*)
\]
\[
= g^{-1} \begin{pmatrix} \rho & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} \frac{1}{2} \text{tr}(M) & ia \\ ia^* & M - \frac{1}{2} \text{tr}(M)E \end{pmatrix} \begin{pmatrix} \rho & 0 \\ 0 & D^* \end{pmatrix}
\]
\[
= g^{-1} \begin{pmatrix} \frac{1}{2} \text{tr}(M) & i\rho aD^* \\ iDa^* \rho & DMD^* - \frac{1}{2} \text{tr}(M)E \end{pmatrix}
\]
\[
= DMD^* + \rho aD^* = \beta(M + \alpha).
\]
Hence $\alpha = \phi(A)\beta = \phi(A)\phi(B) = \phi(AB)$, $AB \in S(p(4,H^c))$. Therefore $\phi$ is onto. Ker $\phi = \{E, -E\} = Z_2$. Thus we have the required isomorphism.

**Lemma 3.4.4.** $\phi: S(p(4,H^c)) \to E_8^c$ of Theorem 3.4.2 satisfies

1. $\gamma = \phi(I_1), \gamma_c = \phi(jE), \gamma_H = \phi(iE), \sigma = \phi(I_2)$.
2. $\tau \phi(A)\gamma = \gamma \phi(\tau A)\gamma = \phi(I_1(\tau A)I_1)$, $\tau \phi(A)\gamma_c = \phi(\tau_c A)$, $\tau \phi(A)\gamma_H = \phi(\tau H)A_1)$, $\tau \phi(A)\gamma = \phi(\gamma_c A)$, $\tau \phi(A)\sigma = \phi(I_2AI_2)$.

**Proof.** It follows from $\tau g(\tau X) = g(\gamma X) = I_1(gA)I_1$, $g(\gamma X) = g(\gamma_c(g X) = I_2(gX)I_1$, $g(\gamma H) = g(\gamma H) = I_2(gX)I_1$, $g(\sigma X) = I_2(gX)I_1$, $X \in Z^c$.

**Theorem 3.4.5.** (1) $(E_6)^{17} \cong S^{(4)}/Z_2(\cong (E_6)^{17})$.
(2) $(E_6)^{17} \simeq (\gamma \gamma_c)^{17} \cong S^{(4,R)}/Z_2 \times 2 \cong (\tau \gamma c)^{17} \simeq (E_8^c)^{17}$.
(3) $(E_6)^{17} \cong S^{(4)}/Z_2 \times 2 \cong (\tau \gamma c)^{17} \simeq (E_8^c)^{17}$.
(4) $(E_6)^{17} \cong S^{(2)}/Z_2 \times 2 \cong (\tau \gamma c)^{17} \simeq (E_8^c)^{17}$

**Proof.** (1) Let $\alpha \in (E_6)^{17} = (\tau \gamma c)^{17}$, $\alpha = \phi(A)$, $A \in S^{(4,H^c)}$ (Theorem 3.4.2).

From $\gamma \psi = \alpha, \psi = \phi(A)$ (Lemma 3.4.4). Hence $\tau A = A$ or $\tau A = -A$. The latter case is impossible. In fact, put $A = iB$, then $BB^* = -E, B \in M(4,H)$, a contradiction. Therefore $A \in S^{(4)}$. (2) $(E_6)^{17} \cong S^{(4)}/Z_2$. $(E_6)^{17} = (\gamma \gamma c)^{17} = (\tau \gamma c)^{17}$.

(2) $\tau \gamma c$ under $\delta \in G_2 \subset F_4 \subset E_6, \delta \gamma = \gamma \delta, \delta \tau = \tau \delta$ (Proposition 1.2.3). Let $\alpha \in (E_6)^{17}, \alpha = \phi(A), A \in S^{(4,H^c)}$. From $\gamma \gamma c \alpha = \gamma c \gamma = \alpha$, we have $\phi(\gamma c A) = \phi(A)$. Thus $(\gamma \gamma c)^{17} = (S^{(4,H^c)} / (iE)S^{(4,H^c)}) / Z_2$ (cf. Theorem 1.3.5) $\cong S^{(4,R)}/Z_2 \times 2$. $(\psi iE) = \gamma_H$.

$E_6^{(17)} = (E_6)^{17} = (E_6)^{17}$

because $\gamma \beta \gamma c$ under $\delta \in G_2 \subset F_4 \subset E_6, \delta \gamma = \gamma \delta, \delta \tau = \tau \delta$ (Proposition 1.2.3). Now $(E_6)^{17} = (\gamma \gamma c)^{17} = (\gamma \gamma c)^{17}$.

(3) Define $\phi: S^{(1,3,H^c)} \to (E_6)^{17}$ by $\phi(A) = \phi(I_1AI_1^{-1})$. Let $\alpha \in (E_6)^{17} = (\gamma \gamma c)^{17}$, $\alpha = \phi(A), A \in S^{(1,3,H^c)}$. From $\tau = \alpha$, we have $\phi(\tau A) = \phi(A)$. Hence $\tau A = A$ or $\tau A = -A$. The latter case is impossible. In fact, there exists no $A \in M(4,H)$ such that $A^*I_1A = -I_1$ because the signature of both sides are different. Therefore $A \in S^{(1,3)}$. (4) $(E_6)^{17} \cong S^{(1,3)}/Z_2$. $(E_8^c)^{17} = (\tau \gamma c)^{17}$.

(4) Define $\phi: S^{(2,2,H^c)} \to (E_6)^{17}$ by $\phi(A) = \phi(I_1AI_1^{-1})$. Let $\alpha \in (E_6)^{17} = (\gamma \gamma c)^{17}$, $\alpha = \phi(A), A \in S^{(2,2,H^c)}$. From $\tau A = \alpha^{-1} \tau = \alpha$, we have $\phi(\tau A) = \phi(A)$. Therefore $\phi$ is onto. Ker $\phi = \{E, -E\} = Z_2$. Thus we have the required isomorphism.
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Hence \((E_{4\times 11})^T \equiv (Sp(2,2) \cap i\begin{pmatrix} 0 & I' \\ I' & 0 \end{pmatrix})/Sp(2,2)/\mathbb{Z}_2 \approx Sp(2,2)/\mathbb{Z}_2 \times 2\). (The explicit form of \(\rho = \phi \left( \begin{pmatrix} 0 & I' \\ I' & 0 \end{pmatrix} \right) : \mathfrak{z}_c \to \mathfrak{z}_c\) is

\[
\rho_\alpha X = \begin{pmatrix}
-\xi_1 & ex_1 & -ie \bar{x}_1 \\
e \bar{x}_1 & -\xi_1 & -ie x_1 \\
ix_1 & i \bar{x}_1 & \xi_1
\end{pmatrix} = P_\epsilon XP_\epsilon, \quad P_\epsilon = \begin{pmatrix}
ie & 0 & 0 \\
0 & ie & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

because \(\gamma \sim \gamma \sigma\) under \(\delta = F_i \subset E_6: \delta \gamma = \gamma \sigma \delta, \delta \sigma = \sigma \delta\) (Proposition 2.3.3). Now \((E_{4\times 11})^T \sim (\tau \gamma \sigma)^T = (\tau \lambda \sigma)^T\).

3.5. Subgroups of type \(C, \Theta A_3\) of Lie groups of type \(E_6\).

Let \(k : M(3, \mathbb{H}^C) \to \{P \in M(6, \mathbb{H}^C) | JP = \bar{P}J\}\) be the algebraic \(C\)-isomorphism (resp. \(k : (\mathbb{H}^C)^3 \to \{P \in M(2, 6, \mathbb{H}^C) | JP = \bar{P}J\}\) be the \(C\)-linear isomorphism) defined by

\[
k((a+b)j) = \left( \begin{pmatrix} a \\ -b \\ \bar{a} \end{pmatrix} \right), \quad a, b \in \mathbb{H}^C
\]

and denote the inverse \(k^{-1}\) of \(k\) by \(h\).

**Lemma 3.5.1.** \(\det(kM) = (\det M)^2, \; M \in \mathfrak{z}(3, \mathbb{H}^C)\).

**Proof.** Since we know that the determinant of a skew-symmetric matrix \(S\) is square of a polynomial with respect to its components \(s_{ij}\), we can easily calculate as

\[
\det \left( \begin{array}{cccccc}
0 & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\
-s_{12} & 0 & s_{23} & s_{24} & s_{25} & s_{26} \\
-s_{13} & -s_{23} & 0 & s_{34} & s_{35} & s_{36} \\
-s_{14} & -s_{24} & -s_{34} & 0 & s_{45} & s_{46} \\
-s_{15} & -s_{25} & -s_{35} & -s_{45} & 0 & s_{56} \\
-s_{16} & -s_{26} & -s_{36} & -s_{46} & -s_{56} & 0
\end{array} \right)
= (s_{12}s_{34}s_{56} - s_{11}s_{35}s_{46} + s_{13}s_{35}s_{46})^2
\]

Note that \((kM)j \in M(6, C^C)\) is skew-symmetric and use the above formula, then

\[
\det(kM) = \det((kM)j) \quad (m, n \in C^C)\]
On the other hand, \( \det M \) is
\[
\begin{pmatrix}
\xi & m_j + n_j & m_j + n_j \\
\xi & m_j + n_j & m_j + n_j \\
m_j + n_j & \xi & m_j + n_j \\
m_j + n_j & \xi & m_j + n_j \\
-\xi & 0 & 0 \\
-\xi & 0 & 0
\end{pmatrix}
\]
\[
= \det \begin{pmatrix}
0 & \xi & n_1 & m_1 & n_2 & m_2 \\
-\xi & 0 & m_3 & n_1 & m_2 & n_2 \\
0 & -\xi & m_3 & n_1 & m_2 & n_2 \\
-\xi & 0 & m_3 & n_1 & m_2 & n_2 \\
0 & -\xi & m_3 & n_1 & m_2 & n_2 \\
0 & -\xi & m_3 & n_1 & m_2 & n_2
\end{pmatrix}
\]
\[
= \begin{pmatrix}
\xi \xi + (m_j + n_j)(m_j + n_j) \\
-\xi & 0 & 0 \\
0 & -\xi & 0 \\
0 & 0 & -\xi \\
0 & 0 & 0
\end{pmatrix}
\]
\[
= \text{the interior part of the above bracket.}
\]

**Lemma 3.5.2.** The group \( E_6(\mathbb{Z}(3, H^C)) \) is connected.

**Proof.** The group \( (E_6(\mathbb{Z}(3, H^C)))^r \) is \( \{ \alpha \in E_6(\mathbb{Z}(3, H^C)) | \langle \alpha M, \alpha N \rangle = \langle M, N \rangle \} \) is connected. The outline of the proof is as follows (see [7]). In the homogeneous space \( (E_6(\mathbb{Z}(3, H^C)))^r / F_4(\mathbb{Z}(3, H)) \cong EV^H \) \( \{ X \in \mathbb{Z}(3, H^C) | \det M = 1, \langle M, M \rangle = 3 \} \), \( F_4(\mathbb{Z}(3, H)) = Sp(3)/Z_2 \) and \( EV^H \) are connected, hence \( (E_6(\mathbb{Z}(3, H^C)))^r \) is also connected. (In reality, \( (E_6(\mathbb{Z}(3, H^C)))^r = SU(6)/Z_2 \). And \( (E_6(\mathbb{Z}(3, H^C)))^r \) is a maximal compact subgroup of \( E_6(\mathbb{Z}(3, H^C)) \). Therefore the group \( E_6(\mathbb{Z}(3, H^C)) \) is connected.

**Proposition 3.5.3.** \( E_6(\mathbb{Z}(3, H^C)) \cong SU^*(6, C^C)/Z_2 \).

**Proof.** We define \( \phi : SU^*(6, H^C) \to E_6(\mathbb{Z}(3, H^C)) \) by
\[
\phi(A)M = k^{-1}(A(kM)A^*) = (hA)M(hA)^*, \quad M \in \mathbb{Z}(3, H^C).
\]
We have to prove \( \phi(A) \in E_6(\mathbb{Z}(3, H^C)) \). In fact, \( (\det(\phi(A)M))^r = \det(k(\phi(A)M)) \) (Lemma 3.5.1) \( = \det(A(kM)A^*) = \det(kM) = (\det M)^r \) (Lemma 3.5.1). Therefore \( \det(\phi(A)M) = \pm \det M \). Since \( SU^*(6, C^C) \) is connected (Proposition 0.2), the sign of \( (\det(\phi(A)M)) \) is constant with respect to \( A \). Hence \( (\det(\phi(A)M)) = \det M \), that is, \( \phi \) is well-defined. \( \Ker \phi = \{ E, -E \} = Z_2 \). Hence \( \phi \) induces a monomorphism \( \psi : \mathfrak{su}^*(6, C^C) \to e_6(\mathbb{Z}(3, H^C)) \). Since the Lie algebra \( e_6(\mathbb{Z}(3, H^C)) \) has the structure \( e_6(\mathbb{Z}(3, H^C)) = f_4(\mathbb{Z}(3, H^C)) \mathbb{C} \mathbb{Z}(3, H^C) \) (cf. Proposition 3.1.1) and \( \dim e_6(\mathbb{Z}(3, H^C)) = 21 + 14 = 35 = \dim \mathfrak{su}^*(6, H^C) \), \( d\phi \) is onto, hence \( \phi \) is also onto (Lemma 0.6) because \( E_6(\mathbb{Z}(3, H^C)) \) is connected (Lemma 3.5.2). Thus we have the required
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PROPOSITION 3.5.4. \((E_8^c)^c \cong (Sp(1, C) \times SU^*(6, C^c))/\mathbb{Z}_2\).

**Proof.** We define \(\phi: Sp(1, H^c) \times SU^*(6, C^c) \rightarrow (E_8^c)^c\) by
\[
\phi(p, A)(M+a) = k^{-1}(A(k M)A^*) + \varphi^{-1}(k a)A^{-1})
\]
\[
= (hA)M(hA)^* + \varphi(a)(hA)^{-1}, \quad M + a \in \mathbb{Z}(3), H^c \otimes (H^c)^c = \mathbb{Z}^c.
\]
We have to prove \(\phi(p, A) \in (E_8^c)^c\).

**Assertion 3.5.5.** \(\phi(p, A)^{-1} = \phi(p, A^*)\).

**Proof.** \(2\text{tr}(\phi(p, A)(M+a), N+b) = 2(M+a, \phi(p, A)(N+b)) = (k(M+a), k(\phi(p, A)(N+b)))\)

(where the inner product \((X, Y)\) in \(M(6, C^c)\) (resp. \(M(2, 6, C^c)\)) is defined by \(\frac{1}{2} \text{tr}(X^*Y + Y^*X)\))
\[
= (k(M + ka, A(k N)A^* + (k(p b))A^{-1}) = (k M, A(k N)A^*) + 2(k a, (k(p b))A^{-1})
\]
\[
= (A^*(k M) A, k N + 2((k(\overline{p} a))A^{-1}), k b) = (A^*(k M) A + (k(\overline{p} a))A^{-1}, k N + k b)
\]
\[
= (k(\phi(\overline{p}, A^*)(M + a)), k(N + b)) = 2\chi(p, A^*)(M + a, N + b).
\]
This shows \(\phi(p, A) = \phi(\overline{p}, A^*)\), hence \(\phi(p, A)^{-1} = \phi(p, A^*)\).

**Assertion 3.5.6.** \(\phi(p, A) \in (E_8^c)^c\).

**Proof.** Put \(\alpha = \phi(p, A)\) and we shall show \(\alpha^{-1}(X \times Y) = \alpha X, \alpha Y, X, Y \in \mathbb{Z}^c\).
Recall
\[
(M + a) \times (N + b) = \left( M \times N - \frac{1}{2}(a^*b + b^*a) \right) - \frac{1}{2}(a N + b M)
\]
Now \(\alpha^{-1}(M \times N) = \alpha M \times \alpha N\) is nothing but \(\det \alpha M = \det M\)(Lemma 2.1.2, Proposition 3.5.3).
\[
(\alpha a )^*(\alpha b) = (p a(hA)^{-1})^*(p b(hA)^{-1}) = (hA)^{-1}a^*b(hA)^{-1} = \phi(p, A^*)(\alpha^*b) = \phi(p, A^*)(\alpha^*b)
\]
\[
= \phi(\overline{p}, A^*)(\alpha^*b)\] (Assertion 3.5.5) = \(\alpha^{-1}(a b)\),
\[
(a \alpha)^*(a N) = (p a(hA)^{-1})(hA)N(hA)^* = p a N(hA)^* = \phi(p, A^*)(a N)
\]
\[
= \phi(\overline{p}, A^*)(a N)\] (Assertion 3.5.5) = \(\alpha^{-1}(a N)\).
This shows \(\alpha \in E_8^c\). Clearly \(\gamma \phi(p, A) = \phi(p, A)\gamma\). Thus Assertion 3.5.6 is shown.

We return to the proof of Proposition 3.5.4. Obviously \(\phi\) is a homomor-
We shall show \( \phi \) is onto. Let \( \alpha \in (E_0^c)^\gamma \). Since the restriction of \( \alpha \) to \( (S^c)^\gamma = S(3, H^c) \) belongs to \( E_0(S(3, H^c)) \), there exists \( A \in SU^*(6, C^c) \) such that

\[ \alpha M = h^{-1}(A(kM)A^*), \quad M \in S(3, H^c) \] (Proposition 3.5.3).

Put \( \beta = \phi(1, A)^{-1} \alpha \), then \( \beta | S(3, H^c) = 1 \). Hence \( \beta \in (G_0^c)^\gamma \) and \( \beta | H^c = 1 \). By Theorem 3.1.2, there exists \( \rho \in Sp(1, H^c) \) such that \( \beta = \phi(p, E) \). Hence \( \alpha = \phi(1, A) \beta = \phi(1, A) \phi(p, E) = \phi(p, A) \). Therefore \( \phi \) is onto. \( \ker \phi = \{(1, E), (-1, -E)\} = \mathbb{Z}_2 \). Thus we have the required isomorphism.

**Lemma 3.5.7.** \( \phi: Sp(1, H^c) \times SU^*(6, C^c) \to E_0^c \) of Proposition 3.5.4 satisfies

1. \( \gamma = \phi(-1, E), \quad \gamma_c = \phi(j, J), \quad \gamma_n = \phi(i, iI), \quad \sigma = \phi(-1, I_3) \).
2. \( \tau \phi(p, A) \gamma = \phi(\tau p, \tau A), \quad \gamma_c \phi(p, A) \gamma_c = \phi(\gamma_c p, -JAJ) \).

**Theorem 3.5.8.** \( (E_0^c \to \gamma)^\gamma \equiv (Sp(1) \times SU^*(6)) / \mathbb{Z}_2 \equiv (E_0^c \gamma) \).

**Proof.** Let \( \alpha \in (E_0^c \to \gamma) = (\gamma)^\gamma, \quad \alpha \phi(p, A), \quad p \in Sp(1, H^c), \quad A \in SU^*(6, C^c) \) (Proposition 3.5.4). From \( \tau \alpha = \alpha \tau \), we have \( \phi(\tau p, \tau A) = \phi(p, A) \) (Lemma 3.5.7). Hence \( (E_0^c \gamma)^\gamma \equiv (Sp(1) \equiv (SU^*(6)) / \mathbb{Z}_2 \) (cf. Theorem 1.3.4). \( (E_0^c \gamma)^\gamma = (\gamma)^\gamma \).

**Theorem 3.5.9.** (1) \( (E_0^c \gamma)^\gamma \equiv (Sp(1, C) \times SL(6, C)) / \mathbb{Z}_2, \quad Z_2 = \{(1, 1), (-1, -1, -E)\} \).

(2) \( (E_0^c \gamma)^\gamma \equiv (Sp(1, R) \times SL(6, R)) / \mathbb{Z}_2 \times 2 \).

**Proof.** (1) Since \( f': SL(6, C) \to SU^* (6, C^c), \quad f'(A) = \gamma A - \gamma JAJ \) where \( \gamma = \frac{1}{2}(1 + i) \), is an isomorphism (Proposition 0.2), \( \phi': Sp(1, H^c) \times SL(6, C) \to (E_0^c \gamma), \quad \phi'(p, A) = \phi(p, f'A) \) induces the required isomorphism (Proposition 3.5.4).

(3) \( E_0^c \gamma \equiv (E_0^c \gamma)^\gamma \) because \( \gamma \sim \gamma_c \) under \( \delta \in G_0 \subset E_0 \subset E_c: \delta \gamma = \gamma_c \delta, \delta \gamma = \tau \gamma \delta \) (Proposition 1.2.3). Let \( \alpha \in (\tau \gamma_c)^\gamma, \quad \alpha = \phi'(p, A), \quad p \in Sp(1, H^c), \quad A \in SL(6, C) \). From \( \tau \gamma_c = \alpha \tau \gamma_c \), we have \( \phi'(\tau \gamma_c p, \tau A) = \phi'(p, A) \) (\( \tau A = \gamma f'(A) \) and Lemma 3.5.7). Hence \( (E_0^c \gamma)^\gamma \equiv (\tau \gamma_c)^\gamma \equiv (Sp(1, H^c) \times SL(6, R) \cup iSp(1, H^c) \times (-iI) SL(6, R)) / \mathbb{Z}_2 \) (cf. Theorem 1.3.5) \( \equiv (Sp(1, R) \times SL(6, R)) / \mathbb{Z}_2 \times 2 \). (\( \phi'(i, -iI) = \gamma_c \).

**Lemma 3.5.10.** Since \( f : SU(6, C^c) \to SU^*(6, C^c), \quad f(A) = \gamma A - \gamma JAJ \) where \( \gamma = \frac{1}{2}(1 + i) \), is an isomorphism (Proposition 0.2), \( \phi: Sp(1, H^c) \times SU(6, C^c) \to (E_0^c \gamma), \quad \phi(p, A) = \phi(p, fA) \) is also an isomorphism. Now this \( \phi \) satisfies

1. \( \gamma = \phi(-1, E), \quad \gamma_c = \phi(j, J), \quad \gamma_n = \phi(i, iI), \quad \rho = \phi(1, I_4') \) where \( I_4' = \text{diag}(-1, 1, -1, 1, 1, 1) \).
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(2) \(\tau \phi(p, A) = \phi(p, -JAJ), \quad \psi(p, A) = \phi(p, -JA^*A), \quad \gamma \phi(p, A) \gamma_c = \phi(\gamma_c p, -JAJ), \quad \gamma \phi(p, A) \gamma_n = \phi(\gamma_n p, IA), \quad \sigma \phi(p, A) \sigma = \phi(p, I^*A),\)

Proof. It is clear from \(\tau f(A) = f(-JAJ), \quad (f A)^* = f(-JA^*A), \quad f(J) = J, \ f(I_2) = I_2\) and Lemma 3.5.7. \(\rho = \phi(1, I^*_n)\) is obtained by the direct calculation.

Theorem 3.5.11. (1) \((E_6)^c Y \cong (Sp(1) \times SU(6))/Z_2 \cong (E_6^{c+1})^c Y\).

(2) \((E_4)^c Y \cong (Sp(1) \times SU(2, 4))/Z_2 \cong (\tau \lambda \gamma \sigma)^c Y \cong (E_6)^c Y\).

(3) \((E_6)^c Y \cong (Sp(1, R) \times SU(3, 3))/Z_2 \times 2\).

Proof. (1) Let \(\alpha \in (E_6)^c Y = (\tau \lambda)^c Y, \ \alpha = \phi(p, A), \ p \in Sp(1, H^c), \ A \in SU(6, C^c)\).

From \(\tau \lambda \alpha = \alpha \tau \lambda\), we have \(\phi(\tau p, \tau A) = \phi(p, A)\) (Lemma 3.5.10). Thus \((E_6)^c Y \cong (Sp(1) \times SU(6))/Z_2\) (cf. Theorem 1.3.4). \((E_6^c Y) = (\tau \lambda)^c Y\).

(2) Define \(\varphi : Sp(1, H^c) \times SU(2, 4, C^c) \rightarrow (E_6)^c Y\) by \(\varphi(p, A) = \phi(1, I^*_n) \tau \lambda \gamma \sigma = \alpha\). Let \(\alpha \in (E_6^c Y) = (\tau \lambda)^c Y, \ \alpha = \varphi(p, A), \ p \in Sp(1, H^c), \ A \in SU(2, 4, C^c)\). From \(\tau \lambda \alpha \varphi(p, A) = \varphi(p, A)\). Thus \((E_6^c Y) \cong (Sp(1) \times SU(2, 4))/Z_2\) (cf. Theorem 1.3.4).

\(E_6^{c+1} = (E_6^c)^c Y \cong (E_6^c Y)^c Y \cong (\tau \lambda)^c Y\)

because \(\gamma \sim \gamma \sigma\) under \(\delta \in F_4 \subset E_6: \delta \gamma = \gamma \sigma \delta, \delta \tau \lambda = \tau \lambda \delta\) (Proposition 2.2.3). Now \((E_6^c Y) \cong (\tau \lambda)^c Y \cong (\tau \lambda)^c Y\).

(3) \((E_6^{c+1}) = (E_6^c)^c Y \cong (E_6^c Y)^c Y \cong (\tau \lambda)^c Y\)

because \(\gamma \sim \gamma \mu\) under \(\delta \in G_2 \subset E_6: \delta \gamma = \gamma \mu \delta, \delta \tau \lambda = \tau \lambda \delta\) (Proposition 1.2.3). Let \(SU(3, 3, K) = \{A \in M(6, K) : A^* IA = I, \ \det A = 1\}, \ I = \text{diag}(1, -1, 1, -1, 1, 1), \ K = C, C^c\) and define \(\varphi : Sp(1, H^c) \times SU(3, 3, C^c) \rightarrow (E_6)^c Y\) by \(\varphi(p, A) = \phi(1, I^*_n) \tau \lambda \gamma \sigma = \alpha\). Let \(\alpha \in (E_6^c Y) = (\tau \lambda)^c Y, \ \alpha = \varphi(p, A), \ p \in Sp(1, H^c), \ A \in SU(3, 3, C^c)\). From \(\tau \lambda \alpha \varphi(p, A) = \varphi(p, A)\). Thus \((E_6^c Y) \cong (\tau \lambda)^c Y \cong (Sp(1, R) \times SU(3, 3))/Z_2 \times 2\) (cf. Theorem 1.3.5). (\(\varphi(j^c, j^c) = \gamma \sigma\)).

3.5.12. Proposition 3.2.3. (1) \(\gamma \sim \rho\). (2) \(\sigma \sim \gamma \rho\). (3) \(\alpha \sim \gamma \mu \rho\).

Proof. (1) Since \(I^*_n \cong I_2\) under a certain \(D_i \in SU(6), \ \rho = \phi(1, I^*_n) \sim \gamma \sigma = \phi(1, I_2)\) under \(\delta = \phi(1, D_i) \in (E_6)^c Y \) (Theorem 3.5.11.(1)). Furthermore \(\gamma \sigma \sim \gamma\) in \(F_4 \subset E_6\) (Proposition 2.2.3.(1)). Consequently \(\rho \sim \gamma\) in \(E_6\).

(2) As is shown in (1), \(\rho \sim \gamma \sigma\) under \(\delta \in G_2 \subset E_6\), hence \(\rho \gamma \sigma = \gamma \sigma = \gamma \rho \in E_6\).

(3) \(\gamma \sim \gamma\) under \(\delta \in G_2 \subset F_4 \subset E_6\) (Proposition 1.2.3). This \(\delta\) satisfies \(\delta(0) = 0\), hence \(\rho \sim \rho\). Therefore \(\gamma \rho \sim \gamma\) under \(\delta \in E_6\). Thus \(\gamma \rho \sim \gamma \rho \sim \rho\) (result of (1)) in \(E_6\).
Theorem 3.5.13. \((E_{a(-14)})^\sim \langle \tau \lambda_{\eta} \rho \rangle^\sim \approx (S \rho(1, R) \times SU(5, 1))/\mathbb{Z}_2\).

Proof. \(E_{a(-14)} = (E_e^\circ)^{\lambda_e} \approx (E_e^\circ)^{\lambda_e \eta \rho}\)

because \(\sigma \sim \eta \rho \eta \) under \(\delta \in E_a: \delta \sigma = \eta \rho \delta, \delta \tau = \tau \delta\) (Proposition 3.2.3). Put \(I' = I' = \text{diag}(-1, -1, -1, 1, -1)\) and \(SU(5, 1) = \{ A \in M(6, K) \mid A * I' A = I', \det A = 1 \}, \ K = C, \ C^\circ\). Define \(\varphi : S \rho(1, H^\circ) \times SU(5, 1, C^\circ) \to (E_e^\circ)^{\lambda_e \eta \rho}\) by \(\varphi(p, A) = \phi(p, \Gamma_\eta A \Gamma_\eta^{-1})\) where \(\Gamma_\eta = \text{diag}(i, i, i, i, i)\). Let \(\alpha \in (\tau \lambda_{\eta} \rho \eta)^\sim, \alpha = \varphi(p, A), \ p \in S \rho(1, H^\circ), \ A \in SU(5, 1, C^\circ)\). From \(\tau \lambda_{\eta} \rho \tau \lambda_{\eta} \rho \eta = \alpha\), we have \(\varphi(\tau \lambda_{\eta} \rho, \tau A) = \varphi(p, A)\). Thus \((E_{a(-14)})^\sim \langle \tau \lambda_{\eta} \rho \rangle^\sim \approx (S \rho(1, 'H) \times SU(5, 1))/\mathbb{Z}_2\) (cf. Theorem 3.4.5.(3)) \(\approx (S \rho(1, R) \times SU(5, 1))/\mathbb{Z}_2\).

3.6. Subgroups of type \(C \oplus D_2\) of Lie groups of type \(E_8\).

Lemma 3.6.1. For \(\alpha \in (E_e^\circ)^{\rho}\), there exists \(\xi \in C^* = C - \{0\}\) such that \(\alpha E_i = \xi E_i\).

Proof. Note that for \(\alpha \in (E_e^\circ)^{\rho}\) we have \(\{\alpha, \alpha^{-1} \in (E_e^\circ)^{\rho}\}\). As in Section 2.4, \((E_e^\circ)^{\rho}\) is not invariant under \(\sigma \in (E_e^\circ)^{\rho}\), hence \(\alpha E_i, \alpha^{-1} E_i, \ a \in E, \ 'a \alpha E_i, \ 'a^{-1} E_i \in \mathfrak{S}(2, G^\circ)\) as in Lemma 2.4.1. Suppose that \(\alpha E_i\) and \(\alpha^{-1} E_i \in \mathfrak{S}(2, G^\circ)\). Then \(\alpha E_i = \alpha^{-1} E_i \in \mathfrak{S}(2, G^\circ)\), and \(\xi E_i + \eta E_i + F_i(x) = \alpha E_i = \alpha^{-1} E_i \in \mathfrak{S}(2, G^\circ)\). Thus \(\xi E_i + \eta E_i + F_i(x) = \alpha E_i = \alpha^{-1} E_i \in \mathfrak{S}(2, G^\circ)\). This implies \(\xi E_i = \eta E_i = 0\). Hence \(\alpha E_i = 0\), a contradiction. Therefore \(\alpha E_i \not\in \mathfrak{S}(2, G^\circ)\) or \(\alpha^{-1} E_i \not\in \mathfrak{S}(2, G^\circ)\).

(1) Case \(\alpha E_i \not\in \mathfrak{S}(2, G^\circ)\). We can put \(\alpha E_i = \xi E_i + \eta E_i \neq 0\). Then \(0 = \alpha E_i = \xi E_i + \eta E_i + F_i(x)\), \(\xi \neq 0\). Hence \(\alpha E_i = \xi E_i\), \(\xi \neq 0\).

(2) Case \(\alpha^{-1} E_i \not\in \mathfrak{S}(2, G^\circ)\). Similarly as above, there exists \(\eta \in C^*\) such that \(\alpha^{-1} E_i = \eta E_i\). Then \(\alpha E_i = \eta E_i\) (put \(\xi = \eta^{-1}\).

Lemma 3.6.2. If \(\alpha \in (E_e^\circ)^{\rho}\) then \(\alpha, \alpha^{-1} \in (E_e^\circ)^{\rho}\).

Proof. Put \(\alpha E_i = \xi E_i, \xi \in C^*\) (Lemma 3.6.1). Then \(\xi = (\xi E_i, E_i) = (\xi E_i, E_i)\)
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$(E_1, \alpha E_1) = (E_1, E_1) = 1$.

**Lemma 3.6.3.** $((E_6^c)^*)_{E_1}/\text{Spin}(9, C) \simeq (S^C)^\circ$. In particular, the group $((E_6^c)^*)_{E_1}$ is connected.

**Proof.** We define a complex 9-dimensional sphere $(S^C)^\circ$ by

$$(S^C)^\circ = \{ X \in \mathbb{C}^9 \mid 4E_1 \times (E_1 \times X) = X, (E_1, X, X) = 1 \}$$

$$= \left\{ \left( \begin{array}{c} \xi \\ \eta \\ x \end{array} \right) \right| \xi \eta - x \bar{x} = 1, \xi, \eta \in \mathbb{C}, x \in \mathbb{C} \}.$$

The group $((E_6^c)^*)_{E_1}$ acts on $(S^C)^\circ$ (Lemma 3.6.2). We shall show that this action is transitive. To show this we prepare some elements of $((E_6^c)^*)_{E_1}$.

(1) For $d \in \mathbb{S}^C$, put $D_{22} = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & d & 1 \end{array} \right)$ and define a $C$-linear transformation $\delta_{22}(d)$ of $\mathbb{C}^9$ by $\delta_{22}(d)X = D_{22} XD_{22}^*$, $X = X(\xi, x) \in \mathbb{C}^9$, explicitly

$$\delta_{22}(d)X = \left( \begin{array}{ccc} \xi_1 & x_3 & \xi_4 + x_2 \\ \bar{\xi}_3 & \xi_2 & \bar{\xi}_4 + x_1 \\ d\bar{x}_3 + x_2 & \xi_4d + \bar{x}_1 & \xi_4d + 2(d, \bar{x}_1) + \bar{x}_1 \end{array} \right).$$

Then $\delta_{22}(d) \in ((E_6^c)^*)_{E_1}$. Similarly $\delta_{22}(d) \in ((E_6^c)^*)_{E_1}$ can be defined.

(2) For $\theta \in C^*$, define a $C$-linear transformation $\delta(\theta)$ of $\mathbb{C}^9$ by

$$\delta(\theta)X = \left( \begin{array}{ccc} \xi_1 & \theta x_3 & \theta^{-1}\bar{x}_4 \\ \theta \bar{x}_3 & \theta^2 \xi_2 & x_1 \\ \theta^{-1}x_3 & \bar{x}_1 & \theta^{-2}\xi_4 \end{array} \right) = D_\theta XD_\theta^*, \quad D_{\theta} = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \theta^{-1} \end{array} \right).$$

Then $\delta(\theta) \in ((E_6^c)^*)_{E_1}$.

Now let $X = \left( \begin{array}{ccc} \xi \\ \bar{x} \\ \eta \end{array} \right) \in (S^C)^\circ$. If $\xi \neq 0$ (resp. $\eta \neq 0$), operate $\delta_{22}(-\bar{x}/\xi)$ (resp. $\delta_{22}(-x/\eta)$) on $X$, then $X$ is transformed to a diagonal form. In the case of $\xi = \eta = 0$, choose $d \in \mathbb{S}^C$ such that $(d, \bar{x}) \neq 0$, then $\delta_{22}(d)\left( \begin{array}{ccc} 0 \\ \bar{x} \\ 0 \end{array} \right) = \left( \begin{array}{ccc} 0 \\ \bar{x} \\ 2(d, \bar{x}) \end{array} \right)$, hence it is reduced to the first case. Thus $X$ is transformed to a diagonal form $\xi E_4 + \text{re} \eta E_3, \xi \eta = 1$. Moreover choose $\theta \in C$ such that $\theta^2 = \eta$ and operate $\delta(\theta)$, then it can be transformed to $E_4 + E_3$. This shows the transitivity. The isotropy subgroup of $((E_6^c)^*)_{E_1}$ at $E_4 + E_3$ is $((E_4^c)^*)_{E_1} = (F_4^c)^o$ (Proposition 2.1.3.(4)) = $\text{Spin}(9, C)$ (Theorem 2.4.3). Thus we have the homeomorphism $((E_6^c)^*)_{E_1}/\text{Spin}(9, C) \simeq (S^C)^\circ$. 

PROPOSITION 3.6.4. \(((E^c_4)^*)_{E_1} \cong \text{Spin}(10, C)\).

PROOF. Since the group \(((E^c_4)^*)_{E_1} \) is connected (Lemma 3.6.3), we can define a homomorphism \(\pi : ((E^c_4)^*)_{E_1} \rightarrow \text{SO}(10, C) = \text{SO}(V^c)^o\) by \(\pi(\alpha) = \alpha|V^c\) where

\[
(V^c)^o = \mathbb{S}(2, \mathbb{C})^o = \{X \in \mathbb{S}(3, \mathbb{C}) \mid 4E_1 \times (E_1 \times X) = X\}
\]

with the norm \((E_1, X, X)\). Ker \(\pi = \{1, \sigma\} = Z_2\). Hence \(\pi\) induces a homomorphism \(d\pi : ((E^c_4)^*)_{E_1} \rightarrow \text{SO}(10, C)\). Since \(((E^c_4)^*)_{E_1} = (1, \sigma) \mathbb{F}(\mathbb{S}(3, \mathbb{C})_{E_1} = (1, \sigma) \mathbb{F}(2, \mathbb{C})^o\) (Proposition 3.1.1) and \(\dim_c((E^c_4)^*)_{E_1} = 36 + 9\) (Theorem 2.4.3) = \(45 = \dim_c \mathbb{S}(10, C)\), \(d\pi\) is onto, hence \(\pi\) is onto. Thus \(((E^c_4)^*)_{E_1} / Z_2 \cong \text{SO}(10, C)\). Therefore \(((E^c_4)^*)_{E_1}\) is isomorphic to \(\text{Spin}(10, C)\) as the universal covering group of \(\text{SO}(10, C)\).

PROPOSITION 3.6.5. \((E^c_4)^*\) has a subgroup \(\phi(C^*)\) which is isomorphic to the group \(C^*\). Where \(\phi(\theta), \theta \in C^*\), is the \(C\)-linear transformation of \(\mathbb{S}^c\) defined by

\[
\phi(\theta)X = \begin{pmatrix}
\theta x_1 & \theta x_3 & \theta x_2 \\
\theta x_3 & \theta^{-2} x_2 & \theta^{-2} x_1 \\
\theta x_2 & \theta^{-2} x_1 & \theta^{-2} x_3
\end{pmatrix} = S_\theta X S_\theta, \quad S_\theta = \begin{pmatrix}
\theta^2 & 0 & 0 \\
0 & \theta^{-1} & 0 \\
0 & 0 & \theta^{-1}
\end{pmatrix}.
\]

LEMMA 3.6.6. The groups \(\phi(C^*)\) and \(\text{Spin}(10, C)\) commute in \((E^c_4)^*\) element-wisely.

PROOF. The restrictions of \(\phi(\theta), \theta \in C^*\), to \(\mathbb{S}_1^c, \mathbb{S}(2, \mathbb{C}), \mathbb{S}(3, \mathbb{C})\) are constant mappings and \(\beta \in \text{Spin}(10, C)\) leaves invariant these spaces. From this we see that \(\phi(\theta)\) and \(\beta\) are commutative.

THEOREM 3.6.7. \((E^c_4)^* = (C^* \times \text{Spin}(10, C))/Z_4, \quad Z_4 = \{(1, 1), (-1, \sigma), (i, \sigma \sqrt{-1}), (-i, \sqrt{-1})\}\).

PROOF ([7]). We define \(\phi : C^* \times \text{Spin}(10, C) \rightarrow (E^c_4)^*\) by

\[
\phi(\theta, \beta) = \phi(\theta)\beta.
\]

Then \(\phi\) is a homomorphism (Lemma 3.6.6). We shall show \(\phi\) is onto. For \(\alpha \in (E^c_4)^*\), there exists \(\theta \in C^*\) such that

\[
\alpha E_1 = \theta^i E_1 \quad \text{(Lemma 3.6.1)}.
\]

Put \(\beta = \phi(\theta)^{-1} \alpha\), then \(\beta E_1 = E_1\), hence \(\beta \in \text{Spin}(10, C)\) (Proposition 3.6.4). Therefore \(\alpha = \phi(\theta)\beta = \phi(\theta, \beta)\), that is, \(\phi\) is onto. Ker \(\phi = \{(1, 1), (-1, \phi(-1)), (i, \phi(-i)), (-i, \phi(i))\} = Z_4\), \((\phi(-1) = \alpha, \phi(i) = \sqrt{-1}\) (Proposition 3.2.3.(1)). In fact, let \(\phi(\theta)\beta = 1, \theta \in C^*, \beta \in \text{Spin}(10, C)\). Operate on \(E_1\), then \(\theta^i = 1\), hence \(\theta = \pm 1, \pm i\).
Therefore \( \text{Ker } \phi = \mathbb{Z}_2 \). Thus we have the required isomorphism.

**Theorem 3.6.8.**

1. \( (E_{c(-26)})^* \cong R^* \times \text{Spin}(9, 1) \).
2. \( (E_{c(0)})^* \cong (R^* \times \text{spin}(5, 5)) \times \mathbb{Z}_2 \).

**Proof.**

1. For \( \alpha \in (E_{c(-26)})^* \) there exists \( \xi \in R^* = \{ \xi \in R \mid \xi > 0 \} \) such that \( \alpha E_1 = \xi E_1 \). In fact, \( \xi E_1 = \alpha E_1 (\xi \in \mathbb{C}^* \text{ (Lemma 3.6.1)}) = r \xi E_1 \), hence \( r \xi = \xi \), that is, \( \xi \in \mathbb{R}^* = \mathbb{R} - \{0\} \). Moreover \( \xi > 0 \). (Although it follows from the connectedness of \( (E_{c(-26)})^* \) (Lemma 0.7) we will give here a direct proof). As in Lemma 2.4.1 we have

\[
\alpha E_3 = \gamma_3 E_3 + \eta_3 E_3 + F_i(y), \quad \gamma_3, \eta_3 \geq 0, \quad y \in \mathbb{C},
\]

\[
\alpha^{-1} E_3 = \xi_3 E_3 + \zeta_3 E_3 + F_i(z), \quad \xi_3, \zeta_3 \geq 0, \quad z \in \mathbb{C}.
\]

Suppose \( \xi < 0 \). Then from \( \zeta_3 E_3 + \eta_3 E_3 + F_i(y) = \alpha^{-1} E_3 = 2 \alpha E_3 = \alpha E_3 \times \alpha E_3 = \alpha E_3 \times \alpha E_3 = 2 \xi E_3 \times (\eta_3 E_3 + \eta_3 E_3 + F_i(y)) = \xi \eta_3 E_3 + \xi \eta_3 E_3 - \xi F_i(y) \) we have \( \gamma_3 = \gamma_3 = 0 \). Hence \( \alpha E_3 = F_i(y) \). Again from \( 0 = \alpha^{-1} \langle E_3 \times E_3 \rangle = \alpha E_3 \times \alpha E_3 = F_i(y) \times F_i(y) = -y E_1 \) we have \( y = 0 \), a contradiction.

Now \( (E_{c(-26)})^* \) is connected (Lemma 0.7) because \( (E_{c(0)})^* \) is simply connected. The group \( (E_{c(-26)})^* \) acts on

\[
V_{6,1} = \langle \mathbb{R}^2, \mathbb{C}^* \rangle \cup \{X = \left[ \begin{array}{c} x \\ \eta \end{array} \right] \mid \xi, \eta \in \mathbb{R}, \quad x, \eta \in \mathbb{C} \}
\]

with the norm \( (E_1, X, Y) = \bar{x} \eta - x \bar{\eta} \). We can define a homomorphism \( \pi : (E_{c(-26)})^* \to O(9, 1) \), \( \pi = \pi = \mathbb{Z}_2 \). As similar to Proposition 3.6.4, \( \pi \) is onto. Thus \( (E_{c(-26)})^* \to O(9, 1) \). Therefore \( (E_{c(-26)})^* \) is isomorphic to \( \text{Spin}(9, 1) \) as the universal covering group of \( O(9, 1) \). Let \( \phi : \mathbb{R}^* \to (E_{c(-26)})^* \) be the restriction of \( \phi : C^* \to (E_{c(0)})^* \) defined in Lemma 3.6.5. Now \( \phi : \mathbb{R}^* \times \text{Spin}(9, 1) \to (E_{c(-26)})^* \), \( \phi(\theta, \beta) = \phi(\theta) \beta \), gives the required isomorphism (cf. Theorem 3.6.7).

(2) As in (1), for \( \alpha \in (E_{c(0)})^* \), \( \alpha E_1 = \xi E_1 \), \( \xi \in \mathbb{R}^* \). In this case there exists surely \( \alpha \in (E_{c(0)})^* \) such that \( \alpha E_1 = -E_1 \). In fact, \( \rho \in \text{Theorem 3.4.5.} \) is the required one. Now the connected group \( (E_{c(0)})^* \), as in (1), acts on

\[
V_{8,1} = \langle \mathbb{R}^4, \mathbb{C}^* \rangle \cup \{X = \left[ \begin{array}{c} \xi \\ \eta \\ x' \\ \eta' \end{array} \right] \mid \xi, \eta, x', \eta' \in \mathbb{R}, \quad x', \eta' \in \mathbb{C} \}
\]

with the norm \( (E_1, X, Y) = \bar{x} \eta - x' \bar{\eta}' \). We can define a homomorphism \( \pi : (E_{c(0)})^* \to O(5, 5) \), \( \pi = \pi = \mathbb{Z}_2 \). As similar to (1) we have \( (E_{c(0)})^* / \mathbb{Z}_2 \to O(5, 5) \). Therefore \( (E_{c(0)})^* \) is denoted by \( \text{spin}(5, 5) \) (not simply...
connected) as a double covering group of $O(5, 5)$. Put \((E_{6(c)})^a = \{a \in (E_{6(c)})^a \mid \alpha E_1 = \xi^E_1, \xi > 0\}\). By the use of $\phi$ in (1), we see that $\phi: R^8 \times \text{spin}(5, 5) \to \langle r_6(\phi) \rangle_{\phi}$, $\phi(\theta, \beta) = \phi(\theta)\beta$, is an isomorphism (cf. Theorem 3.6.7). Thus $(E_{6(c)})^a = ((E_{6(c)})_{\phi})_{\phi} \approx (R^8 \times \text{spin}(5, 5)) \times 2$.

**Theorem 3.6.9.**

1. $(E_6)^a \cong (U(1) \times \text{Spin}(10))/Z_4 \cong (E_{6(-14)})^a$.
2. $(E_{6(c)})^a \cong (U(1) \times \text{spin}(6, 4))/Z_4$.
3. $(E_{6(-14)})^a \sim (\tau \lambda r)^{a} \cong (U(1) \times \text{spin}(8, 2))/Z_4$.

**Proof.**

(1) For $\alpha \in (E_6)^a$, $\xi E_1 = \alpha E_1$ $\xi C$ (Lemma 3.6.1) $= \tau^{a(-1)} \xi E_1 = (\xi \tau^a)^{-1} E_1$, hence $\xi (\tau \xi) = 1$, that is, $\xi \in U(1) = \{\xi \in C \mid \xi (\tau \xi) = 1\}$. $(E_6)^a_{E_6} = ((E_6^c)^a)_{E_6}$ is connected as in Theorem 3.6.8. (1). The group $((E_6)^a)_{E_6}$ acts on $V^{20} = \{X \in \mathbb{C}^6 \mid 2E_1 \times X = -\tau X\} = \left\{\left(\begin{array}{c} \xi \\ x \\ \bar{x} \\ \tau \xi \end{array}\right) \mid \xi \in C, x \in \mathbb{R}\right\}$ with the norm $\langle X, X \rangle_{/2} = \xi (\tau \xi) + x \bar{x}$. We can define a homomorphism $\pi: ((E_6)^a)_{E_6} \to SO(10) = SO(V^{19})$. Ker $\pi = \{1, \sigma\} = Z_2$. Since $(E_6)^a_{E_6} = (1, \gamma) \oplus (5, \xi_6, E_6)^a$ (Proposition 3.1.1) and $(E_6)^a_{E_6} = (1, \gamma) \oplus (3, \xi_6, E_6)^a$, dim $(E_6)^a_{E_6} = 36 + 9 = 45 = \dim \mathfrak{so}(10)$, hence $\pi$ is onto. Thus $((E_6)^a)_{E_6}/Z_2 \cong SO(10)$. Therefore $((E_6)^a)_{E_6}$ is isomorphic to Spin(10) as the universal covering group of SO(10). (In reality $(E_6)^a)_{E_6} = (E_6)_{E_6}$.) Thus $\psi: U(1) \times \text{Spin}(10) \to (E_6)^a$, $\psi(\theta, \beta) = \psi(\theta)\beta$ where $\psi(\theta)$ is one defined in Lemma 3.6.5, induces the required isomorphism. $(E_{6(-14)})^a = (\tau \lambda r)^{a} \sim (\tau \lambda)^{a}$.

(2) For $\alpha \in (E_{6(c)})^a = (\tau \lambda r)^{a}$, $\alpha E_1 = \xi E_1$, $\xi \in U(1)$. The connected group $((E_{6(c)})^a)_{E_6}$, as in Theorem 3.6.8. (1), acts on $V^{a, 4} = \{X \in \mathbb{C}^6 \mid 2E_1 \times X = -\tau X\} = \left\{\left(\begin{array}{c} \xi \\ x \\ \bar{x} \\ \tau \xi \end{array}\right) \mid \xi \in C, x \in \mathbb{C}\right\}$ with the norm $\langle X, X \rangle_{/2} = \xi (\tau \xi) + x \bar{x}'$. As in (1), we have $((E_{6(c)})^a)_{E_6}/Z_2 \cong O(6, 4) = O(V^{a, 4})$. Therefore $((E_{6(c)})^a)_{E_6}$ is denoted by $\text{spin}(6, 4)$ (not simply connected) as a double covering group of $O(6, 4)$. Thus $\psi: U(1) \times \text{spin}(6, 4) \to (E_{6(c)})^a$, $\psi(\theta, \beta) = \psi(\theta)\beta$, induces the required isomorphism.

(3) $E_{6(-14)} = (E_6^{c})^{a \theta} \cong (E_6^{c})^{a \tau \lambda}$ because $\sigma \sim \sigma'$ under $\delta \in F_{1} \subseteq E_6$: $\delta \sigma = \sigma \delta$, $\delta \tau \lambda = \tau \lambda \delta$ (Proposition 2.2.3). For $\alpha \in ((E_6^{c})^{a \theta})^a = (\tau \lambda r)^a$, $\alpha E_1 = \xi E_1$, $\xi \in U(1)$. The connected group $((\tau \lambda r)^a)_{E_6}$, acts on $V^{a, 4} = \{X \in \mathbb{C}^6 \mid 2E_1 \times X = \tau X\} = \left\{\left(\begin{array}{c} \xi \\ x \\ \bar{x} \\ \tau \xi \end{array}\right) \mid \xi \in C, x \in \mathbb{C}\right\}$ with the norm $\langle X, X \rangle_{/2} = \xi (\tau \xi) - x \bar{x}$. As in (1), we have $((\tau \lambda r)^a)_{E_6}/Z_2 \cong$
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$O(8,2)_b = O(V^8, 2)$. Therefore $((\tau \lambda \sigma')^\rho)_{E_1}$ is denoted by $\text{spin}(8,2)$ (not simply connected) as a double covering group of $O(8,2)_b$. Thus $\phi : U(1) \times \text{spin}(8,2) \to (\tau \lambda \sigma')^\rho$, $\phi(\theta, \beta) = \phi(\theta)\beta$, induces the required isomorphism.

**Theorem 3.6.10.** $((E_{6(2)}^*)^\rho \cong (U(1) \times \text{spin}^*(10))/Z \cong (\tau \lambda \gamma^\rho)^\rho \cong (E_{6(-14)})^\rho$.

**Proof.**

$E_{6(2)}^* = (E_6^*)^{2\rho} \cong (E_6^*)^{2\rho}$

because $\gamma \sim \rho$ under $\delta \in E_6 : \delta \gamma = \rho \delta$, $\delta \tau \lambda = \tau \delta \lambda$ (Proposition 3.2.3). As in Theorem 3.6.9, for $\alpha \in ((E_6^*)^{2\rho})^\rho = (\tau \lambda \rho)^\rho$, $\alpha E_1 = \xi E_1$, $\xi \in U(1)$ and the group $((\tau \lambda \rho)^\rho)_{E_1}$ is connected. The group $((\tau \lambda \rho)^\rho)_{E_1}$ acts on

$$(V^*)^\rho = \mathbb{Z}(2, \mathbb{G}^*) = \left\{ X = \begin{pmatrix} \xi_2 & x \\ \xi_3 & \xi_4 \end{pmatrix} \mid \xi_2, \xi_3, \xi_4 \in C, x \in \mathbb{G}^* \right\}$$

with the norm $(E_1, X, X) = (\xi_2, x\xi_3 - x\xi_2)$ and the inner product $\langle X, Y \rangle_\rho$. Here

$$\langle X, Y \rangle_\rho = \begin{pmatrix} \xi_2 & x \\ \xi_3 & \xi_4 \end{pmatrix} \begin{pmatrix} \eta_2 & y \\ \eta_3 & \eta_4 \end{pmatrix} = (\tau \begin{pmatrix} -\xi_2 & -i x \\ i \xi i & \xi_3 \end{pmatrix}, \eta_3)$$

$$= -(\xi_2, \eta_3 + (\xi_3, \eta_3) + 2(i x Y, x) = -(\xi_2, \eta_3 + (\tau \xi_2, \tau \eta_3))$$

where $\xi = (\xi_2, \xi_3)$, $\eta = (\eta_2, \eta_3)$ and $S = \text{diag}(-1, 1, 2i J, 2i J, 2i J, 2i J) \in M(10, \mathbb{C})$. By the following coordinate transformation

$$\xi_2 = is_2 + s_2, \quad \xi_3 = is_2 - s_2, \quad \xi_4 = it_2 + t_2, \quad \xi_5 = it_2 - t_2,$$

we have $s_2^2 - s_2, \quad -(\xi_2, \eta_2 + (\xi_3, \eta_3) + 2i (\tau s_2, \tau t_2)$. Hence $(E_1, X, X) = -(s, x)E(\begin{pmatrix} s \\ x \end{pmatrix})$ and $\langle X, Y \rangle_\rho = (\tau s, \tau t)E(\begin{pmatrix} s \\ x \end{pmatrix})$ where $s = (s_2, s_3), t = (t_2, t_3)$. This shows that we have an isomorphism

$$\{ a \in \text{iso}(V^*)^{\rho} \}; (E_1, aX, aX) = (E_1, X, X), \langle aX, aY \rangle_\rho = \langle X, Y \rangle_\rho$$

$$\cong \{ A \in M(10, \mathbb{C}) \}; A \overrightarrow{= E, JA = (\tau A) J} = O(10) = O^*(10) = O^*(V^*)^{\rho}.$$

Thus we can define a homomorphism $\pi : ((\tau \lambda \rho)^\rho)_{E_1} \to SO^*(10) = O(10)_b$ by $\pi(\alpha) = a | (V^*)^{\rho}$. Ker $\pi = \{ 1, \sigma \} = Z_2$. As similar to Theorem 3.6.9 $((\tau \lambda \rho)^\rho)_{E_1} / Z_2 = SO^*(10)$. Therefore $((\tau \lambda \rho)^\rho)_{E_1}$ is denoted by $\text{spin}^*(10)$ (not simply connected) as a double covering group of $SO^*(10)$. And $\phi : U(1) \times \text{spin}^*(10) \to (\tau \lambda \rho)^\rho$, $\phi(\theta, \beta) = \tau(\theta)\beta$, induces the required isomorphism as in Theorem 3.6.9.

$E_{6(-14)} = (E_6^*)^{2\rho} \cong (E_6^*)^{2\rho}$

because $\sigma \sim \rho$ under $\delta \in E_6 : \delta \sigma = \gamma \rho \delta$, $\delta \tau \lambda = \tau \delta \lambda$ (Proposition 3.2.3). To determine the group $((E_6^*)^{2\rho})^\rho)_{E_1} = ((\tau \lambda \gamma^\rho)^\rho)_{E_1}$, consider the space $(V^*)^{\rho} = \mathbb{Z}(2, \mathbb{G}^*)$ with
the norm \((E_1, X, X)=\xi_2\xi_3-x\xi\) and the inner product \(<X, Y>_{\tau_\rho}=(\tau_\rho X, Y)\) as above, where \(S'_{\tau}=\text{diag}(-1, 1, 2iJ, 2iJ, -2iJ, -2iJ)\) and \(M(10, C)\). Since \(F\) and \(-F\) are conjugate in \(O(2)\) (see Proposition 0.4), by a suitable coordinate transformation, we have \(<X, Y>_{\tau_\rho}=\langle\tau s, \tau s'\rangle(2iJ)\). This shows 

\[
\{a\in\text{Iso}_{\tau}(V^\rho)\}((E_1, aX, \alpha X)=(E_1, X, X), \langle aX, \alpha Y\rangle_{\tau_\rho}=\langle X, Y\rangle_{\tau_\rho} \cong O^*(10).
\]

Hence by the same arguments just as before we have the isomorphism \((\tau\lambda\gamma\rho)^*\cong(U(1)\times\text{spin}^*(10))/\mathbb{Z}_4.

### Appendix

The Cartan decompositions of the exceptional universal linear Lie groups of type \(G_2\), \(F_4\) and \(E_6\) are given as follows.

\[
\begin{align*}
G_2 & : \text{simply connected compact Lie group of type } G_2, \\
G_2 & \cong G_2 \times \mathbb{R}^{14}, \\
G_2 & \cong (Sp(1) \times Sp(1))/\mathbb{Z}_2 \times \mathbb{R}^8, \\
F_4 & : \text{simply connected compact Lie group of type } F_4, \\
F_4 & \cong F_4 \times \mathbb{R}^{42}, \\
F_4 & \cong (Sp(1) \times Sp(3))/\mathbb{Z}_2 \times \mathbb{R}^{48}, \\
E_6 & : \text{simply connected compact Lie group of type } E_6, \\
E_6 & \cong E_6 \times \mathbb{R}^{18}, \\
E_6 & \cong Sp(4)/\mathbb{Z}_2 \times \mathbb{R}^{48}, \\
E_6 & \cong (Sp(1) \times SU(6))/\mathbb{Z}_2 \times \mathbb{R}^{48}, \\
E_6 & \cong (U(1) \times Spin(10))/\mathbb{Z}_2 \times \mathbb{R}^{22}, \\
E_6 & \cong F_4 \times \mathbb{R}^{48}.
\end{align*}
\]

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