A GAMMA RING WITH MINIMUM CONDITIONS

By

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Abstract. The aim of this note is to study the structure of a \( \Gamma \)-ring (not in the sense of Nobusawa) with minimum conditions. By ring theoretical techniques, we obtain various properties on the semi-prime \( \Gamma \)-ring and generalize Nobusawa’s result which corresponds to the Wedderburn-Artin Theorem in ring theory. Using these results, we have that a \( \Gamma \)-ring with minimum right and left conditions is homomorphic onto the \( \Gamma \)-ring \( \sum_{i=1}^{q} D_{n(i),m(i)}^{a(i)}, \) where \( D_{n(i),m(i)}^{a(i)} \) is the additive abelian group of the all rectangular matrices of type \( n(i) \times m(i) \) over some division ring \( D^{(i)} \), and \( \Gamma_o \) is a subdirect sum of the \( \Gamma_i, 1 \leq i \leq q \), which is a non-zero subgroup of \( D_{m(i),n(i)}^{a(i)}, \) of type \( m(i) \times n(i) \) over \( D^{(i)} \).

1. Introduction.

Nobusawa [8] introduced the notion of a \( \Gamma \)-ring \( M \) as follows: Let \( M \) and \( \Gamma \) be additive abelian groups. If for all \( a, b, c \in M \) and \( \alpha, \beta, \gamma \in \Gamma \), the conditions

\begin{align*}
N_1. & \quad aab \in M, \quad aab, c \in \Gamma \\
N_2. & \quad (a+b)c = aac + bac, \quad a(\alpha + \beta)b = aab + a\beta b, \quad a\alpha(b+c) = aab + aac \\
N_3. & \quad (aab)\beta c = a(\alpha \beta)b, \quad a\alpha(b\beta c) \\
N_4. & \quad x\gamma y = 0 \text{ for all } x, y \in M \text{ implies } \gamma = 0,
\end{align*}

are satisfied, then \( M \) is called a \( \Gamma \)-ring.

Barnes [1] weakened slightly defining conditions and gave the definition as follows:

If these conditions are weakened to

\begin{align*}
B_1. & \quad aab \in M \\
B_2. & \quad \text{same as } N_2 \\
B_3. & \quad (aab)\beta c = a\alpha(b\beta c),
\end{align*}

then \( M \) is called a \( \Gamma \)-ring.

In this paper, the former is called a $\Gamma$-ring in the sense of Nobusawa and the latter merely a $\Gamma$-ring.

Nobusawa [8] determined the structures of simple and semi-simple $\Gamma$-rings in the sense of Nobusawa with minimum right and left conditions as follows:

Using the notation introduced in [5], when $M$ is simple, as a ring,

$$\begin{pmatrix} R & \Gamma \\ M & L \end{pmatrix} \cong \begin{pmatrix} D_{m} & D_{m,n} \\ D_{n,m} & D_{n} \end{pmatrix}$$

where $D$ is a division ring ([8] Theorem 2); when $M$ is semi-simple, as a ring,

$$\begin{pmatrix} R & \Gamma \\ M & L \end{pmatrix} \cong \sum_{i=1}^{q} \begin{pmatrix} D_{m(i)}^{(i)} & D_{m(i), n(i)}^{(i)} \\ D_{n(i), m(i)}^{(i)} & D_{n(i)}^{(i)} \end{pmatrix}$$

where $D^{(i)}$, $1 \leq i \leq q$, are division rings ([8] Theorem 3).

Nobusawa's definitions are in the following: $M$ is simple if $a\Gamma b=0$ implies $a=0$ or $b=0$; $M$ is semi-simple if $a\Gamma a=0$ implies $a=0$.

In [2], we defined that a $\Gamma$-ring $M$ is prime if for any ideal $A$ and $B$ of $M$, $A\Gamma B=0$ implies $A=0$ or $B=0$; a $\Gamma$-ring $M$ is semi-prime if for any ideal $A$ of $M$, $A\Gamma A=0$ implies $A=0$.

When $M$ is a $\Gamma$-ring in the sense of Nobusawa, one can easily verify that $M$ is prime if and only if $a\Gamma b=0$ implies $a=0$ or $b=0$; $M$ is semi-prime if and only if $a\Gamma a=0$ implies $a=0$ ([1] Theorem 5). Thus, when $M$ is a $\Gamma$-ring in the sense of Nobusawa, Nobusawa's terms 'simple' or 'semi-simple' are equivalent to our 'prime' or 'semi-prime' respectively.

However, when $M$ is a $\Gamma$-ring (not in the sense of Nobusawa), they are quite different notations. Following Luh [7] we call a $\Gamma$-ring $M$ is completely prime if $a\Gamma b=0$ implies $a=0$ or $b=0$; $M$ is completely semi-prime if $a\Gamma a=0$ implies $a=0$. Then, the primeness cannot imply the completely primeness, even for a finite $\Gamma$-ring ([7] Example 3.1). The semi-prime $\Gamma$-ring is one without non-zero strongly-nilpotent ideal (Theorem 2.10 below), while the completely semi-prime $\Gamma$-ring is one without non-zero strongly-nilpotent element (Definition 2.2). The gap between the primeness and completely primeness and the gap between semi-primeness and completely semi-primeness are caused by lack of a multiplication: $\Gamma \times M \times \Gamma \rightarrow \Gamma$. In the following we do not treat completely prime $\Gamma$-rings nor completely semi-prime ones, but prime and semi-prime $\Gamma$-rings.

Also, it should be noticed that a semi-prime $\Gamma$-ring with minimum right condition cannot always have the minimum left condition, nor dim($\ell M$) can be equal to dim($M\alpha$) even if it has both minimum right and left conditions, while a semi-prime ring $R$ (an ordinary ring) with minimum right condition has the
minimum left condition, and \( \dim_\ell (R) = \dim (R_\ell) \) (The comments followed Theorem 3.23).

The main aims of this paper are to study the structure of the semi-prime \( \Gamma \)-ring with minimum right condition and to generalize Nobusawa's results to the prime and semi-prime \( \Gamma \)-rings with minimum conditions and to determine the structure of the \( \Gamma \)-ring with minimum conditions.

Using ring theoretical techniques, we obtain various fundamental results on \( \Gamma \)-rings with minimum right condition. Then, using these results, we have the analogues of the Wedderburn-Artin Theorem for simple (Definition 3.9 and Theorem 3.15 below) and semi-prime \( \Gamma \)-rings with minimum right and left conditions. Also, these converses are considered. Nobusawa's results are obtained as corollaries of our theorems. Consequently, the structure of a \( \Gamma \)-ring with minimum right and left conditions is determined.

For the following notions we refer to [2]: the right operator ring \( R \), the left operator ring \( L \), a right (left, two-sided) ideal of \( M \), a principal ideal \( \langle a \rangle \), \([N, \Phi] \), where \( N \subseteq M \) and \( \Phi \subseteq \Gamma \), but for the prime radical \( \Phi(M) \), a residue class \( \Gamma \)-ring, and the natural homomorphism to [3].

2. Strongly-nilpotent ideals.

DEFINITION 2.1. Let \( M \) be a \( \Gamma \)-ring and \( L \) be the left operator ring. Let \( S \) be a non-empty subset of \( M \) and denote \( S_i = \{a \in L \mid aS = 0\} \). Then \( S_i \) is a left ideal of \( L \), called an annihilator left ideal. Let \( T \) be a non-empty subset of \( L \) and denote \( T_r = \{x \in M \mid Tx = 0\} \). Then \( T_r \) is a right ideal of \( M \), called an annihilator right ideal. For singleton subsets we abbreviate this notation, for example, \( [a]_L = a_r \), where \( a \) is an element of \( L \).

DEFINITION 2.2. An element \( x \) of a \( \Gamma \)-ring \( M \) is nilpotent if for any \( \gamma \in \Gamma \) there exists a positive integer \( n = n(\gamma) \) such that \( (x\gamma)^n x = (x\gamma)(x\gamma)\cdots(x\gamma)x = 0 \). A subset \( S \) of \( M \) is nil if each element of \( S \) is nilpotent. An element \( x \) of a \( \Gamma \)-ring \( M \) is strongly-nilpotent if there exists a positive integer \( n \) such that \( (x\Gamma)^n x = (x\Gamma)(x\Gamma)\cdots(x\Gamma)x = 0 \). A subset of \( M \) is strongly-nil if each its element is strongly-nilpotent. \( S \) is strongly-nilpotent if there exists a positive integer \( n \) such that \( (S\Gamma)^n S = (S\Gamma)(S\Gamma)\cdots(S\Gamma)S = 0 \).

By definitions for a subset \( S \) of \( M \) we have the following diagram of implication:

\[ S \text{ is strongly-nilpotent} \Rightarrow S \text{ is strongly-nil} \Rightarrow S \text{ is nil}. \]
Lemma 2.3. The sum of a finite number of strongly-nilpotent right (left) ideals of a Γ-ring M is a strongly-nilpotent right (left) ideal.

Proof. The proof needs only be given for two strongly-nilpotent right ideals A, B. Suppose \((A\Gamma)^nA=(B\Gamma)^nB=0\). Now we have \((A+B)\Gamma(A+B)\Gamma^2\cdots\Gamma(A+B)\), with \(m+n+2\) brackets, so that \((A+B)\Gamma^{m+n+1}(A+B)\) is a sum of terms, each consisting of \(m+n+2\) factors which are either A or B. Such a term \(T\) contains either \(m+1\) factors A or \(n+1\) factors B. In the former case, \(T^{(A\Gamma)^nA}\) or \(T^{(A\Gamma)^nA}\); in the latter case, \(T^{(B\Gamma)^nB}\) or \(T^{(B\Gamma)^nB}\). Thus, \((A+B)\Gamma^{m+n+1}(A+B)=0\) and \(A+B\) is strongly-nilpotent.

Corollary 2.4. The sum of any set of strongly-nilpotent right (left) ideals of a Γ-ring M is a strongly-nilpotent right (left) ideal.

Proof. Each element \(x\) of the sum is in a finite sum of strongly-nilpotent right ideals of \(M\), which by Lemma 2.3 is strongly-nilpotent. Therefore \(x\) is strongly-nilpotent, and the sum is strongly-nil.

Lemma 2.5. The sum \(S(M)'\) of all strongly-nilpotent right ideals of a Γ-ring M coincides with the sum \(S(M)\) of all strongly-nilpotent left ideals and with the sum \(S(M)\) of all strongly-nilpotent ideals.

Proof. Let \(I\) be a strongly-nilpotent right ideal. The ideal \(I+M\Gamma I\) is strongly-nilpotent, because \((I+M\Gamma I)^n I+M\Gamma I^n I\) for \(n=1, 2, \ldots\). It follows \(I \subseteq S(M)\) and hence that \(S(M)' \subseteq S(M)\). But \(S(M) \subseteq S(M)'\) trivially, and hence \(S(M)=S(M)'\). Similarly, \(S(M)= S(M)'\).

When a Γ-ring M has the descending (or ascending) chain condition for right ideals, it is abbreviated to M has min-r condition (or max-r condition). The terms min-l condition or max-l condition on a Γ-ring M are likewise defined.

It is natural to ask whether \(S(M)\) is strongly-nilpotent. This is so when M has either the min-r or max-r conditions (min-l or max-l also serve). The case of max-r is trivial, because \(S(M)\) is a finite sum of strongly-nilpotent right ideals. When M has min-r condition, a strongly-nil right ideal is always strongly-nilpotent, which will be shown in the following theorem. We note that a non-strongly-nilpotent right ideal means the right ideal which is not strongly-nilpotent.

Theorem 2.6. Any non-strongly-nilpotent right ideal of a Γ-ring M with min-r condition contains an idempotent element.
A gamma ring with minimum conditions

Proof. Let $I$ be a non-strongly-nilpotent right ideal of $M$ and $I_1$ be minimal in the set of non-strongly-nilpotent right ideals which are contained in $I$. Then, $I_1=I_1I_1$, since $I_1I_1$ is not strongly-nilpotent. Let $S$ be the set of right ideals $S$ with properties (1) $SF_1\neq 0$ and (2) $S \subseteq I_1$.

The set $S$ is not empty ($I_1 \in S$) and we suppose that $S_1$ is a minimal member of $S$. Let $s \in S_1$, $\delta \in \Gamma$ with $s \delta I_1 \neq 0$. Then, $s \delta I_1 = S_1$, because $s \delta I_1 \subseteq S$. It follows that $a \in I_1$ exists with $s \delta a = s$. Then $a$ is not nilpotent, because if $a$ is nilpotent, $s = s \delta a = s \delta a \delta a = \cdots = (s \delta)(a \delta) \cdots (a \delta)a = 0$, a contradiction. Hence, $I$ cannot be a nil right ideal. This proves that if $I$ is a strongly-nil right ideal then $I$ is strongly-nilpotent, since if $I$ is strongly-nil then $I$ is nil.

Now $a \Gamma M \subseteq I_1$, and $a \Gamma M$ is not strongly-nilpotent, for $a$ is not nilpotent. Hence $a \Gamma M = I_1$, because of the minimal property of $I_1$. Likewise, $a \Gamma a \Gamma M = I_1$, and hence $a \in a \Gamma a \Gamma M$, so that $a = a \omega a_1$, where $a_1 \in a \Gamma M$. Note that $a \omega(a_1 - a \omega a_1) = 0$ and hence $a_1 - a \omega a_1 \in [a, \omega]_r \cap a \Gamma M$. Set $a_2 = a + a_1 - a \omega a_1$. Then, $a \omega a_2 = a \omega a_1 + a \omega a_1 = (a \omega a_1) a = a \omega a + a - a \omega a = a$. Also, $a \omega(a_1 - a \omega a_1) = (a + a_1 - a \omega a_1) a_2 = a \omega a_1 - a \omega a_1 a_2$. Moreover, $a_2$ is not nilpotent, because $a \omega a_2 = a$ and $a$ is not zero. It follows that $a \Gamma M = a_2 \Gamma M$, and that $[a_2, \omega]_r \cap a \Gamma M \subseteq [a, \omega]_r \cap a \Gamma M$. However, either $a_2 \omega a_1 = a_1 \omega a_2 a_1$, in which case $I$ contains the idempotent $a_1 \omega a_1$, or else $a_2 \omega a_1 \neq a_1 \omega a_2 a_1$, in which case $a_1 - a_2 \omega a_1 \in [a, \omega]_r$ and $a_1 - a \omega a_1 \in [a_2, \omega]_r$. In the latter case, $[a_2, \omega]_r \cap a \Gamma M \subseteq [a, \omega]_r \cap a \Gamma M$. This process is repeated, if necessary, beginning with $a_2$ instead of $a$, and obtaining $a_4$; etc. The process ceases because of the minimum condition and this proves that $I$ has an idempotent element.

Corollary 2.7. The sum $S(M)$ of all strongly-nilpotent ideals of the $\Gamma$-ring $M$ with $\min$-$r$ or $\max$-$r$ conditions, is a strongly nilpotent ideal.

Definition 2.8. When the sum $S(M)$ of all strongly-nilpotent ideals of $M$ is strongly-nilpotent, $S(M)$ is called the Wedderburn radical of $M$ (or the strongly-nilpotent radical) and denoted by $W$.

Definition 2.9. A $\Gamma$-ring $M$ is semi-prime if, for any ideal $U$ of $M$, $U \Gamma U = 0$ implies $U = 0$.

For a semi-prime $\Gamma$-ring we have the following theorem.

Theorem 2.10. ([3] Theorem 1, 2 and 3). If $M$ is a $\Gamma$-ring, the following conditions are equivalent:

(1) $M$ is semi-prime,
(2) If $a \in M$ and $a \Gamma M \Gamma a = 0$, then $a = 0$,
(3) If $\langle a \rangle$ is a principal ideal of $M$ such that $\langle a \rangle \cap \langle a \rangle = 0$, then $a = 0$,
(4) If $U$ is a right ideal of $M$ such that $U^2 = 0$, then $U = 0$,
(5) If $V$ is a left ideal of $M$ such that $V^2 = 0$, then $V = 0$,
(6) The prime radical of $M$, $\mathfrak{P}(M)$, is zero,
(7) $M$ contains no non-zero strongly-nilpotent ideals (right ideals, left ideals),
(8) The sum $S(M)$ of all strongly-nilpotent ideals of $M$ is zero.

**Theorem 2.11.** Let $M$ be a $\Gamma$-ring which has a Wedderburn radical $W$. Then the residue class $\Gamma$-ring $M/W$ is semi-prime.

**Proof.** Set $\overline{M} = M/W$ and suppose $N$ is a strongly-nilpotent ideal of $\overline{M}$, and suppose that $(\overline{N})^n = \overline{0}$. Let $N$ be the inverse image of $\overline{N}$ under the natural homomorphism $M \rightarrow \overline{M}$. Thus, $N = \{x \in M | x + W \in \overline{N}\}$. Clearly, $(N^r)^n N \subseteq W$ and hence $(N^r)^{sn+r+1} N = 0$, where $(W^r)^n W = 0$. Thus, $N \subseteq W$ and $\overline{N} = \overline{0}$. Hence, $\overline{M}$ is semi-prime.

If $M$ has min-r condition, then $M/W$ has min-r condition ([3] Lemma 1), Corollary 2.7 and Theorem 2.11 yield the following theorem.

**Theorem 2.12.** Let $M$ be a $\Gamma$-ring with min-r condition. Then the residue class $\Gamma$-ring $M/S(M)$ is a semi-prime $\Gamma$-ring with min-r condition, where $S(M)$ is the sum of all strongly-nilpotent ideals of $M$.

3. Semi-prime $\Gamma$-rings with min-r condition.

For a right ideal $I$ of a $\Gamma$-ring $M$, if there exists an idempotent element $l$ of the left operator ring $L$ such that $I = lM$, we say that $I$ has the idempotent generator $l$. The idempotent generator plays an important role in the following.

**Theorem 3.1.** Any non-zero right ideal in a semi-prime $\Gamma$-ring $M$ with min-r condition has an idempotent generator.

**Proof.** The result is first proved when the ideal is a minimal right ideal $A$. Since $M$ is semi-prime, $A^2 A \neq 0$. Then, there exist $\delta, \alpha \in \mathfrak{I}$ such that $a\delta A = A$. Thus, there exists $e \in A$ such that $a = a\delta e$. Then, $e = e\delta e$, since from $a = a\delta e = (a\delta e)\delta e$ we have $\delta (e - e\delta e) = 0$ which means $e - e\delta e = 0$, for the set $B = \{c \in A | a\delta c = 0\}$ is a right ideal contained properly in the minimal right ideal $A$ and is $(0)$. Since $e \in A$, $0 \neq e\delta M \subseteq A$ and hence $e\delta M = A$, where $[e, \delta]$ is an idempotent of $L$.

Let $I$ be any non-zero right ideal of $M$. Since $I$ contains one or more minimal right ideals, idempotent generators of the minimal right ideal(s) exist in $[I, \mathfrak{I}]$. Choose an idempotent $l \in [I, \mathfrak{I}]$ such that $l \cap I$ is as small as possible.
If \( l_r \cap I \neq 0 \), then \( l_r \cap I \ni l'M \), where \( l' \) is an idempotent of \( L \). Then, \( \ell' \in \ell'[M, \Gamma] \subseteq [I, \Gamma] \) and \( \ell'' = 0 \), for since \( \ell'M \subseteq l_r \), \( \ell''M = 0 \). Set \( m = l + l' - l' \) and then \( m \in [I, \Gamma] \), for \( [I, \Gamma] \) is a right ideal of \( L \). Clearly, \( m^2 = m \), because \( \ell'' = 0 \). Moreover, \( m \cap I \subseteq l_r \cap I \), since we have \( \ell m = l \) which implies \( m \subseteq l_r \), and \( \ell'' = 0 \) but \( m\ell'' = l' \neq 0 \) which implies \( l'M \subseteq l_r \) but \( l'M \not\subseteq m \). This contradicts the minimality of \( l_r \cap I \) and the contradiction arises from taking \( l_r \cap I \neq 0 \). Hence one has \( l_r \cap I = 0 \).

Now let \( x \in I \), then \( (x - lx) = 0 \), where \( x - lx \in I \), for \( lx \in I[\Gamma] \subseteq I \). It follows that \( I = lM \), for since \( l \in [I, \Gamma] \), \( lM \subseteq I[\Gamma] \subseteq I \).

**Corollary 3.2.** A semi-prime \( \Gamma \)-ring \( M \) with \( \text{min-}r \) condition has \( \text{max-}r \) condition.

**Proof.** The proof is analogous to that in ring theory but to tackle the situation that the generator does not exist in \( M \) but in \( L = \{M, \Gamma\} \). For the sake of completeness, we write it out.

Suppose that the non-empty set \( S \) of some right ideals in \( M \) has no maximal elements. Take an element \( J_i \) of \( S \), then by the assumption there exists \( J_i \in S \) such that \( J_i \supseteq J_{i+1} \). Repeating this process, we have an infinite sequence of right ideals:

\[
J_1 \supseteq J_2 \supseteq \ldots \supseteq J_n \supseteq \ldots
\]

Set \( N = \bigcup J_i \). Then, by Theorem 3.1 \( N = lM \), where \( l \) is an idempotent of \( L \). Thus, \( l \ell = l \ell \in lL = [M, \Gamma] = [N, \Gamma] = [\bigcup J_i, \Gamma] \) and hence there exists an integer \( m \) such that \( l \in \bigcup J_m \). Then, \( N = lM \subseteq \bigcup J_m \Gamma M \subseteq \bigcup J_m \), so that \( J_m = J_m \Gamma M \subseteq J_m \), a contradiction. Hence, every non-empty set of right ideals of \( M \) has a maximal element. Evidently, the \( \text{max-}r \) condition holds in \( M \).

**Lemma 3.3.** If a \( \Gamma \)-ring \( M \) is semi-prime, then the right operator \( R \) and the left operator \( L \) are semi-prime.

**Proof.** Suppose \( rRr = 0 \). Then \( MrM = 0 \). Theorem 2.10 (5) implies \( Mr = 0 \) and then \( r = 0 \). Thus, \( R \) is semi-prime. Similarly, it may be verified that \( L \) is semi-prime.

**Theorem 3.4.** Let \( T \) be any non-zero ideal of semi-prime \( \Gamma \)-ring \( M \) with \( \text{min-}r \) condition. Then \( T \) has a unique idempotent generator.

**Proof.** Let \( T = sM \), where \( s = \sum [e_i, \delta_i] \) is an idempotent, be the given ideal. Then \( s_i = T_i \) is a left ideal of the left operator ring \( L \) and \( T_i \cap [T, \Gamma] = 0 \), because \( (T_i \cap [T, \Gamma])^2 \subseteq T_i[T, \Gamma] = 0 \) and \( L \) is semi-prime (Lemma 3.3). Hence
s_t \cap [T, \Gamma] = 0. \) But for any \( \sum_i [x_i, \gamma_i] \in [T, \Gamma] \) \( (\sum_i [x_i, \gamma_i] - \sum_i [x_i, \gamma_i])s = 0 \) and hence \( \sum_i [x_i, \gamma_i] \in [T, \Gamma] \) which means that \( \sum_i [x_i, \gamma_i] = \sum_i [x_i, \gamma_i]s \). It follows that \( [T, \Gamma] = [T, \Gamma]s = sM \Gamma \) and \( s \) is a two-sided identity for the ring \( [T, \Gamma] \). The latter fact shows that \( s \) is unique.

**Definition 3.5.** Let \( M \) be a \( \Gamma \)-ring and \( L \) be the left operator ring. If there exists an element \( \sum_i [e_i, \delta_i] \in L \) such that \( \sum_i e_i \delta_i x = x \) for every element \( x \) of \( M \), then it is called that \( M \) has the left unity \( \sum_i [e_i, \delta_i] \).

It can be verified easily that \( \sum_i [e_i, \delta_i] \) is the unity of \( L \). Similarly we can define the right unity which is the unity of the right operator ring \( R \).

**Corollary 3.6.** A semi-prime \( \Gamma \)-ring \( M \) with min-\( r \) condition has a left unity.

**Proof.** In Theorem 3.4 set \( T = M \). Then, \( L = [M, \Gamma] = sM \Gamma \). Thus, \( s \) is the unity of \( L \). Then for any \( x \) of \( M \) \( [sx - x, \Gamma] = 0 \) and so \( (sx - x) \Gamma M \Gamma(sx - x) = 0 \). Since \( M \) is semi-prime \( sx - x = 0 \) or \( sx = x \).

By symmetry we have

**Corollary 3.7.** A semi-prime \( \Gamma \)-ring \( M \) with min-\( l \) condition has a right unity.

**Corollary 3.8.** Let \( T \) be any non-zero ideal of a semi-prime \( \Gamma \)-ring \( M \) with min-\( r \) condition. Then, the generating idempotent of \( T \) is the idempotent which lies in the center of \( L \).

**Proof.** Let \( T = (\sum_i [e_i, \delta_i])M \) and suppose the \( l \in L \). Since \( (\sum_i [e_i, \delta_i])l \in [T, \Gamma] \), we have \( (\sum_i [e_i, \delta_i])l = (\sum_i [e_i, \delta_i]l) \sum_i [e_i, \delta_i] = \sum_i [e_i, \delta_i]l \sum_i [e_i, \delta_i] = l \sum_i [e_i, \delta_i] \), for \( l \sum_i [e_i, \delta_i] \in L \). Therefore, \( \sum_i [e_i, \delta_i] \) is central in \( L \).

**Definition 3.9.** A \( \Gamma \)-ring \( M \) is said to be simple if \( M \Gamma M \neq 0 \) and \( M \) has no ideals other than 0 and \( M \).

**Corollary 3.10.** (1) Any non-zero ideal \( T \) of a semi-prime \( \Gamma \)-ring \( M \) with min-\( r \) condition is a semi-prime \( \Gamma \)-ring with min-\( r \) condition. (2) Any minimal ideals \( S \) of a semi-prime \( \Gamma \)-ring \( M \) with min-\( r \) condition is a simple \( \Gamma \)-ring.

**Proof of (1).** Let \( J \) be a right ideal of \( T \) (considered as a \( \Gamma \)-ring) \( (J \Gamma T \subseteq J) \). Let \( T = sM \), where \( s = \sum_i [e_i, \delta_i] \) is an idempotent. Since \( [J, \Gamma] \subseteq [T, \Gamma] \) Theo-
rem 3.4 implies \([J, \Gamma'] = [J, \Gamma']\). Thus, \(J\Gamma M = (J, \Gamma'] M = J\Gamma (sM) = J\Gamma T \subseteq J\) and hence \(J\) is a right ideal of \(M\). It is immediate that the \(\Gamma\)-ring \(T\) has no strongly-nilpotent right ideals and satisfies the min-\(r\) condition.

**Proof of (2).** Let \(T\) be any non-zero ideal of \(M\). Then, as shown in the proof of (1), a right ideal of \(T\) is a right ideal of \(M\). Now, we show that a left ideal \(Q\) of \(T\) is a left ideal of \(M\). Suppose that \(T = sM\), where \(s\) is an idempotent. Then, \(MTQ = [M, \Gamma]Q = [M, \Gamma](sQ) = ([M, \Gamma]Q) = [T, \Gamma]Q \subseteq Q\). So \(Q\) is a left ideal of \(M\). Therefore, an ideal of \(T\) is an ideal of \(M\).

Since \(S\) is a minimal ideal of \(M\), we deduce that \(S\) is a simple \(F\)-ring.

**Theorem 3.11.** If \(T\) is an ideal in a semi-prime \(\Gamma\)-ring \(M\) with min-\(r\) condition, then \(M = T \oplus [T, \Gamma']\). If \(M = T \oplus K\), where \(K\) is an ideal of \(M\), then \(K = [T, \Gamma']\).

**Proof.** Suppose that \(T = sM\), where \(s = \sum_i [e_i, \delta_i]\) is an idempotent, then \(M = sM \oplus (1_L - s)M\), where \(1_L\) denotes the left unity of \(M\). \([T, \Gamma](1_L - s)M = [T, \Gamma]s(1_L - s)M = [T, \Gamma](s - s)M = 0\). Hence, \((1_L - s)M \subseteq [T, \Gamma]\). Conversely, suppose that \([T, \Gamma]x = 0\) and \(x = x' + x''\), where \(x' \in T\), \(x'' \in (1_L - s)M\). Then, \(sx = sx' + sx'' = sx'\) and \(0 = [T, \Gamma]x = ([T, \Gamma]s)x = [T, \Gamma]sx' = [T, \Gamma]x'\). Since \(T \Gamma M \subseteq T\), \(T \Gamma M \Gamma x' = 0\) and hence \(x' \Gamma M \Gamma x' = 0\), which implies \(x' = 0\). Thus, \(x = x'' \in (1_L - s)M\) and then \([T, \Gamma]x \subseteq (1_L - s)M\). Hence \([T, \Gamma] = (1_L - s)M\) and \(M = T \oplus [T, \Gamma]\).

In the case when \(M = T \oplus K\), it follows that \(T \Gamma K = 0\) (since \(T \Gamma K \subseteq T \cap K\)) and hence \(K \subseteq [T, \Gamma]\). However \(T \oplus K = T \oplus [T, \Gamma]\) and hence \(K = [T, \Gamma]\).

We now prove the fundamental theorem on semi-prime \(\Gamma\)-rings with min-\(r\) condition.

**Theorem 3.12.** A semi-prime \(\Gamma\)-ring \(M\) with min-\(r\) condition has only a finite number of minimal ideals and is their direct sum.

**Proof.** Form \(M_1 \oplus M_2 \oplus \ldots \oplus M_n\) of minimal ideals \(M_i\) of \(M\). Because \(M\) has the max-\(r\) condition (Corollary 3.2), there is a sum \(S\) having maximal length \(q\). Suppose that \([S, \Gamma] \neq 0\). Then \([S, \Gamma]\) contains a minimal ideal, which can be added directly to \(S\), because \(S \cap [S, \Gamma] = 0\). This contradicts our supposition that \(S\) has maximal length of minimal ideals. Hence \([S, \Gamma] = 0\) and \(M = S \oplus [S, \Gamma] = S\), which proves that \(M\) is a direct sum of minimal ideals, \(M = M_1 \oplus M_2 \oplus \ldots \oplus M_q\), say.
By Corollary 3.10 and Theorem 3.12 we have

**Theorem 3.13.** A semi-prime $\Gamma$-ring with min-$r$ condition is a direct sum of a finite number of simple $\Gamma$-rings with min-$r$ condition.

**Definition 3.14.** A $\Gamma$-ring $M$ is prime if for all pairs of ideals $S$ and $T$ of $M$, $ST=0$ implies $S=0$ or $T=0$. A $\Gamma$-ring $M$ is left (right) primitive if (i) the left (right) operator ring of $M$ is a left (right) primitive ring, and (ii) $xFM=0$ ($Mx=0$) implies $x=0$. $M$ is a two-sided primitive $\Gamma$-ring (or simply a primitive $\Gamma$-ring) if both left and right primitive.

Luh proved the following theorem.

**Theorem 3.15** ([7] Theorem 3.6). For a $\Gamma$-ring $M$ with min-$l$ condition, the three conditions

1. $M$ is prime,
2. $M$ is primitive,
3. $M$ is simple

are equivalent.

Of course, Theorem 3.15 also holds when $M$ has min-$r$ condition instead of min-$l$ condition. Thus, we can replace the term 'simple' in Theorem 3.13 by 'prime' or 'primitive'.

We will prove further results on the one sided ideal structure of a semi-prime $\Gamma$-ring with min-$r$ condition.

**Lemma 3.16.** Let $I$ be a right ideal in a semi-prime $\Gamma$-ring $M$ with min-$r$ condition and $J_1$ be a right ideal contained in $I$. Then there exists a right ideal $J_2$ in $I$ such that $I=J_1 \oplus J_2$.

**Proof.** Taking $l \neq 0$, $J_i \neq 0$ and $l=IM$ and $J_1=sM$, where $l=\sum e_i \delta_i$, $s=\sum f_j \varepsilon_j$ are idempotents. Write $x \in I$ as $x=sx+(l-s)x$. The set $J_2 = \{x-sx \mid x \in I\}$ is a right ideal and $J_2 \subseteq I$. Clearly, $I=J_1 \oplus J_2$.

**Definition 3.17.** Idempotents $l_1, \ldots, l_k \in L$ are mutually orthogonal if $l_il_j=0$ for $i \neq j$.

The notation $l=l_1 \oplus \cdots \oplus l_k$ indicates that $l=l_1+\cdots+l_k$, where $l_1, \ldots, l_k$ are mutually orthogonal idempotents.

In Lemma 3.16 we can choose generating idempotents $s_1$ of $J_1$, $s_2$ of $J_2$, so
A gamma ring with minimum conditions

that \( l = s_1 \oplus s_2 \). The proof is given in the following.

Take \( l = lM \) and \( J_1 = sM \) as before, and set \( s_1 = sl \) and \( s_2 = l - sl \). Then \( ls = s \) since \( s \in \langle M, l \rangle \), and \( s = s^2 = s((sl)s = s_1 s \) so that \( J_1 = sM = s_1 (sM) \subseteq sM = lM \). Thus, \( J_1 = s_1 M \). However, \( J_2 = \{ x - sx \mid x \in I \} = \{ \langle a - sla \rangle a \in M \} = \{ (l - sl)a \mid a \in M \} = s_2 M \). We can easily verify that \( s_1, s_2 \) are idempotents and that \( l = s_1 \oplus s_2 \).

**Q.E.D.**

**Definition 3.18.** An idempotent of the left operator ring \( L \) is **primitive** if it cannot be written as a sum of two orthogonal idempotents.

Lemma 3.16 and subsequent comments imply that in a semi-prime \( \Gamma \)-ring with min-\( r \) condition an idempotent of \( L \) is primitive if and only if it generates a minimal right ideal.

**Lemma 3.19.** Let \( M \) be a semi-prime \( \Gamma \)-ring with min-\( r \) condition. Then any idempotent element \( l \) of the left operator ring \( L \) is a sum of mutually orthogonal primitive idempotents.

**Proof.** Let \( l = lM \) and \( M_1 \) be a minimal right ideal in \( I \). There exists a right ideal \( M'_1 \subseteq I \) such that \( l = M_1 \oplus M'_1 \) (by Lemma 3.16). Then, either \( M'_1 = 0 \), in which case \( l \) is primitive (\( l \) generates the minimal right ideal), or we choose generating idempotents \( s_1 \) of \( M_1 \); \( s'_1 \) of \( M'_1 \) such that \( l = s_1 \oplus s'_1 \) (by the above comment). Observe that \( s_1 \) is a primitive idempotent. If \( s'_1 \) is not primitive, this process may be applied to \( M'_1 = s'_1 M \), giving \( s'_1 = s_2 \oplus s'_2 \), where \( s_2 \) is primitive. Evidently, \( l = s_1 \oplus s_2 \oplus s'_2 \), and \( s'_1 M \supseteq s'_2 M \supseteq \cdots \) being strictly decreasing, must be stop after a finite number of terms. Then, \( l = s_1 \oplus \cdots \oplus s_k \), say, which each \( s_i \) is a primitive idempotent.

**Corollary 3.20.** Any non-zero right ideal in a semi-prime \( \Gamma \)-ring \( M \) with min-\( r \) condition is a direct sum of minimal right ideals.

**Proof.** Lemma 3.19 implies that \( l = lM = s_1 M \oplus \cdots \oplus s_k M \).

By symmetry, we have

**Corollary 3.21.** Any non-zero left ideal in a semi-prime \( \Gamma \)-ring with min-\( l \) condition is a direct sum of minimal left ideals.
Luh proved the following theorem.

**Theorem 3.22 ([6] Theorem 3.6).** Let \( M \) be a semi-prime \( \Gamma \)-ring and \( L \) and \( R \) be respectively the left and right operator rings of \( M \). If \( \epsilon \delta e = e \), where \( e \in M \), \( \delta \in \Gamma \), then the following statements are equivalent:

1. \( M \epsilon \) is a minimal left ideal of \( M \),
2. \( \epsilon \delta M \) is a minimal right ideal of \( M \),
3. \( [M, \Gamma][\epsilon, \delta] \) is a minimal left ideal of \( L \),
4. \( [\delta, \epsilon][\Gamma, M] \) is a minimal right ideal of \( R \),
5. \( [\epsilon, \delta][M, \Gamma] \) is a minimal right ideal of \( L \),
6. \( [\Gamma, M][\delta, \epsilon] \) is a minimal left ideal of \( R \),
7. \( [\epsilon, \delta][M, \Gamma][\epsilon, \delta] \) is a division ring,
8. \( [\delta, \epsilon][\Gamma, M][\delta, \epsilon] \) is a division ring.

Moreover, the division rings \( [\epsilon, \delta][M, \Gamma][\epsilon, \delta] \) and \( [\delta, \epsilon][\Gamma, M][\delta, \epsilon] \) are isomorphic if any of the above statements occurs.

Corollary 3.20 showed that every non-zero right ideal of a semi-prime \( \Gamma \)-ring \( M \) is a direct sum of minimal right ideals. This decomposition applies to \( M \) itself and gives a right dimension number for \( M \), considered as an \( R \)-module.

**Theorem 3.23.** Let \( M \) be a semi-prime \( \Gamma \)-ring with min-r condition and let \( M = I_1 \oplus \ldots \oplus I_m = J_1 \oplus \ldots \oplus J_n \), where \( I_i, J_j \) are minimal right ideals. Then, \( m = n \).

The proof is established by the quite similar fashion to that for an ordinary ring and so we omit it.

The integer \( m = n \) in Theorem 3.23 is called the right dimension of the semi-prime \( \Gamma \)-ring with min-r condition and denoted by \( \dim(M_R) \). One can define the left dimension of a \( \Gamma \)-ring in a similar manner. But it should be noticed that a semi-prime \( \Gamma \)-ring with min-r condition cannot always have the min-l condition. For example, let \( D \) be a division ring and \( M \) be the discrete direct sum of the division rings \( D_i = D \), \( i \in \mathbb{N} \) (the set of all natural numbers), and \( \Gamma \) be the set of all transposed elements of \( M \). Then, the \( \Gamma \)-ring \( M \) is semi-prime and \( \dim(\epsilon \delta M) = \infty \), while \( \dim(M_R) = 1 \). Even for a semi-prime \( \Gamma \)-ring with both min-r and min-l conditions, generally the right dimension cannot be equal to the left one. When \( M = D_{\times} \), the set of all matrices of type \( 2 \times 1 \) over a division ring \( D \), and \( \Gamma = D_{\times} \), \( \dim(M_R) = 2 \) and \( \dim(\epsilon \delta M) = 1 \).

When \( M \) is a semi-prime \( \Gamma \)-ring with min-r condition, we consider the left operator ring \( L \). Corollary 3.6 shows \( M \) has the left unity. Thus, by Lemma
A gamma ring with minimum conditions

3.19, \( 1_L = [e_1, \delta_1] + \cdots + [e_k, \delta_k] \), where \([e_1, \delta_1], \ldots, [e_k, \delta_k]\) are mutually orthogonal primitive idempotents. This implies that \( L = [e_1, \delta_1] L \oplus \cdots \oplus [e_k, \delta_k] L \), where \([e_1, \delta_1] L, \ldots, [e_k, \delta_k] L\) are minimal right ideals. Also, we have \( L = L[e_1, \delta_1] \oplus \cdots \oplus L[e_k, \delta_k] \), where \( L[e_1, \delta_1], \ldots, L[e_k, \delta_k]\) are minimal left ideals (Theorem 3.22). Thus, we have \( \dim(L_L) = \dim(L_L) \). By symmetry, when \( M \) is a semi-prime \( \Gamma \)-ring with \( \text{min-l} \) condition, for the right operator ring \( R \) we have \( \dim(R_R) = \dim(R_R) \).

4. Simple \( \Gamma \)-rings with \( \text{min-r} \) and \( \text{min-l} \) conditions.

Let \( I \) be an ideal of \( R \) such that \( 0 \subseteq I \subseteq R \). Then \( MI \) is an ideal of \( M \). Since \( M \) is simple, \( MI \) must be 0 or \( M \). If \( MI=M \), then \( R=[\Gamma, MI]=[\Gamma, M]=RI \subseteq I \), a contradiction. If \( MI=0 \), then \( I=0 \), also a contradiction. Thus, \( R \) has only ideals \( 0 \) and \( R \), and \( R^2 \neq 0 \), for \( MR^2=M[\Gamma, M]=[\Gamma, M]=MGM=M \neq 0 \). This proves \( R \) is simple. Similarly, it may be shown that \( L \) is simple.

If \( M \) is simple, then \( M \) is semi-prime. Indeed, for any ideal \( U \) of \( M \) we assume \( UFU=0 \). Since only ideals of \( M \) are \( 0 \) and \( M \), \( U=0 \) or \( U=M \). If \( U=M \), then \( MG=M \neq 0 \), a contradiction. Thus, \( U=0 \) and \( M \) is semi-prime.

DEFINITION 4.1. If \( M_i \) is a \( \Gamma_i \)-ring for \( i=1, 2 \), then an ordered pair \((\theta, \phi)\) of mappings is called a homomorphism of \( M_1 \) onto \( M_2 \) if it satisfies the following properties:

1. \( \theta \) is a group homomorphism from \( M_1 \) onto \( M_2 \).
2. \( \phi \) is a group homomorphism from \( \Gamma_1 \) onto \( \Gamma_2 \).
3. For every \( x, y \in M_1 \), \( r \in \Gamma_1 \), \( (x \gamma y) \theta = (x \theta) (\gamma \phi) (y \theta) \).

Furthermore, if both \( \theta \) and \( \phi \) are injections, then \((\theta, \phi)\) is called an isomorphism from the \( \Gamma_1 \)-ring \( M_1 \) onto the \( \Gamma_2 \)-ring \( M_2 \).

THEOREM 4.2. Let \( M \) be a simple \( \Gamma \)-ring with \( \text{min-r} \) and \( \text{min-l} \) conditions and \( \Gamma_0 = \Gamma / \kappa \), where \( \kappa = \{ \gamma \in \Gamma | M\gamma M = 0 \} \). Then, the \( \Gamma_0 \)-ring \( M \) is isomorphic onto the \( \Gamma \)-ring \( D_{n,m} \), where \( D_{n,m} \) is the additive abelian group of all rectangular matrices of type \( n \times m \) over a division ring \( D \), and \( \Gamma' \) is a non-zero subgroup of the additive abelian group \( D_{n,m} \) of all rectangular matrices of type \( m \times n \), and \( m = \dim(LM) \) and \( n = \dim(MR) \).

PROOF. Let \( e \delta M \), where \( e \delta e = e \), be a minimal right ideal of \( M \) (Theorem 3.1) and let \( D = [e \delta M \Gamma e, \delta] \); certainly \( D \) is a division ring (Theorem 3.22). Also,
\[ [e \delta M, \Gamma] = e \delta L \] is a minimal right ideal of \( L \) (Theorem 3.22). Since \( (e \delta M \Gamma e \delta) e \delta L = e \delta L \) (for \( 0 \neq (e \delta M \Gamma e \delta) e \delta L \)) we see that \( e \delta L \) is a vector space over \( D \) (a left \( D \)-space).

First we prove:

\[ l_1, \ldots, l_n \in e \delta L \text{ are linearly independent over } D \text{ if and only if } L = [M, \Gamma]. \]

Suppose \( L_1 + \cdots + L_n \) is not direct sum. Then, there exist \( a_1, \ldots, a_n \in L \), not all \( a_i \) zero, such that \( a_1 l_1 + \cdots + a_n l_n = 0 \). Set \( L_i = \{ a \in L [e, \delta] | a l_i \subseteq L_1 + \cdots + L_{i-1} + L_{i+1} + \cdots + L_n \} \), where we suppose that \( a_l l_i = 0 \). Then, \( 0 \neq a_I [e, \delta] \subseteq L_I \) and \( L_\delta = L [e, \delta] \), because \( L [e, \delta] \) is a minimal left ideal (Theorem 3.22). Hence, \( [e, \delta] \subseteq L [e, \delta] = L_I \) and then \( l_i = e \delta l_i = y_1 l_1 + \cdots + y_i l_i \cdots + y_n l_n \) where \( y_i \in L \). Then, \( l_i = (e \delta y_1 e \delta) l_i + \cdots + (e \delta y_{i-1} e \delta) l_{i-1} + (e \delta y_{i+1} e \delta) l_{i+1} + \cdots + (e \delta y_n e \delta) l_n \), which means that \( l_i, \ldots, l_n \) are linear dependent over \( D \).

Conversely, if \( L_1 + \cdots + L_n \) is a direct sum, then \( (e \delta L \Gamma) l_i + \cdots + (e \delta L \Gamma) l_n \) is a direct sum, which means \( l_i, \ldots, l_n \) are linearly independent over \( D \). Q.E.D.

Next, we prove:

\[ a_1 \delta_1 L + \cdots + a_k \delta_k L \text{ if and only if } a_1 \delta_1 M + \cdots + a_k \delta_k M. \]

Suppose \( a \delta_1 M + \cdots + a_k \delta_k M \) is a direct sum. If \( \sum I_i l_I = 0 \) with \( l_i \subseteq a_i \delta_i L \), then \( \sum I_i l_I x = 0 \) for all \( x \in M \), where \( l_i x \subseteq l_I M \subseteq [a_i \delta_i M, \Gamma] M \subseteq a_i \delta_i M \). Thus, \( l_I x = 0 \) for all \( x \in M \) and for all \( i \). Hence, \( l_I = 0 \) for every \( i \).

Conversely, assume that \( a \delta_1 L + \cdots + a_k \delta_k L \) is a direct sum. If \( \sum I_i x_i = 0 \), with \( x_i \subseteq a_i \delta_i M \), then \( \sum I_i [x_i, \gamma] = 0 \) for all \( \gamma \in \Gamma \), where \( [x_i, \gamma] \subseteq [x_i, \Gamma] \subseteq [a_i \delta_i M, \Gamma] = a_i \delta_i L \). It follows that \( [x_i, \gamma] = 0 \) for every \( \gamma \in \Gamma \) and every \( i \), and \( x_i \Gamma M \subseteq x_i = 0 \) for every \( i \). Since \( M \) is semi-prime, \( x_i = 0 \) for every \( i \). Thus, \( a \delta_1 M + \cdots + a_k \delta_k M \) is a direct sum. Q.E.D.

Thus, by (A), the comment (followed Theorem 3.23) on the dimensions of \( L \), (B) and Theorem 3.22, we have \( \dim (a \delta M, \Gamma) = \dim (L_L) = \dim (L_L) = \dim (M_R) \). Similarly, we can prove \( \dim (a \delta M) = \dim (a M) = \dim (a M) = \dim (R_R) \).

For \( a \in M \) define a mapping \( \rho_a \) of \([e \delta M, \Gamma]\) to \( e \delta M \) by \([x, \gamma] \rho_a = x \gamma a \), where \([x, \gamma] \subseteq [e \delta M, \Gamma] \). Set \( N = \{ \rho_a | a \in M \} \).

For \( \gamma \in \Gamma \) define a mapping \( \psi_\gamma \) of \( e \delta M \) to \([e \delta M, \Gamma]\) by \( x \psi_\gamma = [x, \gamma] \), where \( x \subseteq e \delta M \). Set \( A = \{ \psi_\gamma | \gamma \in \Gamma \} \).

Then one can easily verify that for all \( a, b \in M \) and \( \gamma, \delta \in \Gamma \)

\[ \rho_a + \rho_b = \rho_a + b, \quad \psi_\gamma + \psi_\delta = \psi_\gamma + \delta, \quad \text{and} \quad \rho_a \psi_\delta \rho_b = \rho_{a+b}. \]
A gamma ring with minimum conditions

thus \(N\) becomes a \(\Gamma_1\)-ring, where \(\Gamma_1 = A\).

Set \(\kappa = \{\gamma \in \Gamma | M \Gamma = 0\}\), then \(\kappa\) is a subgroup of \(\Gamma\). For any element \(\gamma \in \Gamma\) we define \(a \gamma b = a \gamma b\) (well defined), where \(\gamma = \gamma + \kappa\). Then we get a \(\Gamma_0\)-ring \(M\), where \(\Gamma_0 = \Gamma / \kappa\).

Let \(\rho\) be a mapping of \(M\) to \(N\) by \(\rho(a) = \rho_a, a \in M\), and let \(\phi\) be a mapping from \(\Gamma_0\) to \(A\) by \(\phi(\gamma) = \phi_{\gamma}\) (well defined), where \(\gamma + \kappa = \gamma\). Then \(\rho(a) = 0 \Rightarrow \rho_a = 0 \Rightarrow \rho_0M \Gamma a = 0 \Rightarrow \rho_0 = 0 \Rightarrow a \Gamma M = 0 \Rightarrow a = 0\), since \(M \rho_0 = M\), due to \(M\) being simple, and \(M\) is semi-prime. Also, \(\rho(\gamma) = 0 \Rightarrow \rho_{\gamma} = 0 \Rightarrow [\rho_0M, \gamma] = 0 \Rightarrow [M, \rho_0] = 0 \Rightarrow M_0M = 0 \Rightarrow \gamma = 0\), since \(M\) is simple.

Next, \(\rho(a \gamma b) = \rho(a \gamma b) = \rho_a \rho_{\gamma} = \rho(a) \rho(\gamma) \rho(b)\). Both, \(\rho\) and \(\phi\) are clearly surjections. Hence, the mapping \((\rho, \phi)\) is a isomorphism from the \(\Gamma_0\)-ring \(M\) onto the \(\Gamma_1\)-ring \(N\), where \(\Gamma_1 = A\).

Let \(\dim(\ell M) = m\) and \(\dim(M_0) = n\), and let \(D_{n,m}\) and \(D_{m,n}\) denote respectively the set of all matrices of type \(n \times m\) over \(D\) and that of all matrices of type \(m \times n\) over \(D\). Similarly, \(D_n\) and \(D_m\) are respectively the total matrix ring of type \(n \times n\) over \(D\) and that of type \(m \times m\) over \(D\).

Choose a basis \(l_1, \ldots, l_n\) of the vector space \(\ell \rho_0M, \Gamma\) and a basis \(u_1, \ldots, u_m\) of the vector space \(\rho_0M\).

For \(a \in M\) we have

\[
l_i a = l_i \rho_a = \alpha_{ij}u_j + \alpha_{jm}u_m; \quad i = 1, 2, \ldots, n.
\]

Now the correspondence

\[
\rho_a \mapsto (\alpha_{ij}); \quad 1 \leq i \leq n, \quad 1 \leq j \leq m
\]

is a group isomorphism from the additive abelian group \(N\) into the additive abelian group \(D_{n,m}\). Thus, \(\theta : a \mapsto (\alpha_{ij})\) is a group isomorphism of \(M\) into \(D_{n,m}\).

We show that this is an isomorphism onto \(D_{n,m}\):

Along the similar fashion described in the above, ring theory shows that elements of the left operator \(L\) are linear transformations of the vector space \([\rho_0M, \Gamma]\) and as a ring \(L\) is isomorphic onto \(D_n\), and elements of the right operator ring \(R\) are linear transformations of the vector space \(\rho_0M\) and \(R\) isomorphic onto \(D_m\). Since \(M\) is a left \(L\)-right \(R\)-bimodule, for any \(l \in L, x \in M, r \in R, l x r \in M\). Let \(l \mapsto (\alpha_{ij}) \in D_n, x \mapsto (\alpha_{ij}) \in D_{n,m}, r \mapsto (\tau_{ij}) \in D_m\). Then for any \(a \in [\rho_0M, \Gamma]\),

\[
a(l x r) = ((a(l x r))((\alpha_{ij})(\tau_{ij})) = a(\alpha_{ij})(\tau_{ij}),
\]

and hence, \(l x r \theta = (\alpha_{ij})(x)(\tau_{ij})\). Thus, \(LMR \subseteq M\) implies \((LMR) \theta \subseteq (M) \theta\), and so \(D_n(M) \theta D_m \subseteq (M) \theta\). It follows \(D_{n,m} \subseteq (M) \theta\), for \((M) \theta \subseteq D_{n,m}\). Hence, \((M) \theta = D_{n,m}\).

Q. E. D.
Shoji Kyuno

By the similar argument, we obtain that the additive abelian group $\Gamma_0$ is isomorphic onto a subgroup of $D_{m,n}$, and we denote the isomorphism by $\phi$.

We now prove $(a\tilde{\theta}b)\theta = a\theta\tilde{\theta}b\theta$:

Let $a\theta = (\alpha_{ij})$, $b\theta = (\beta_{ij})$. Then, for any $l \in [\delta M, \Gamma]$ we have

$$l(a\tilde{\theta}b) = ((l(\alpha_{ij})(\omega_{uv})) = (l(\beta_{ij}))(\omega_{uv}),$$

thus, $(a\tilde{\theta}b)\theta = (\alpha_{ij})(\omega_{uv})(\beta_{ij}) = a\theta\tilde{\theta}b\theta$.

Clearly, $D_{m,n}$ is a $\Gamma'$-ring, where $\Gamma'$ is $(\Gamma_0, \phi)$, which is a non-zero subgroup of $D_{m,n}$.

Therefore, the $\Gamma'$-ring $M$ is isomorphic onto the $\Gamma'$-ring $D_{m,n}$ and the proof is completed.

When $M$ is a $\Gamma$-ring in the sense of Nobusawa, $\kappa = 0$ and then $\Gamma_0 = \Gamma$, and furthermore since $\Gamma$ is a right $L$-left $R$-bimodule $D_{m,n}(\Gamma)\phi D_{m,n} \subseteq (\Gamma)\phi$. On the other hand, $(\Gamma)\phi \subseteq D_{m,n}$, and so $(\Gamma)\phi = D_{m,n}$, thus we have

**Corollary 4.3 ([8] Theorem 2).** A simple $\Gamma$-ring $M$ in the sense of Nobusawa with min-$r$ and min-$l$ conditions is isomorphic onto the $\Gamma'$-ring $D_{m,n}$, where $\Gamma' = D_{m,n}$.

We note that the term ‘simple’ in this corollary is the one given in Definition 3.9. However, as shown already, since $M$ has minimum condition, $M$ becomes prime (Theorem 3.15). Then, since $M$ is the prime $\Gamma$-ring in the sense of Nobusawa, $M$ is completely prime ([11] Theorem 5), which coincides with ‘$M$ is simple’ in Theorem 2 in Nobusawa [8].

5. $\Gamma$-rings with minimum right and left conditions.

First we consider the semi-prime $\Gamma$-ring with min-$r$ and min-$l$ conditions. Let $\Gamma' = \Gamma/\kappa$, where $\kappa = \{\gamma \in \Gamma | M_0 \gamma \Gamma = 0\}$, and $M = M_1 \oplus \cdots \oplus M_q$, where $M_1, \cdots, M_q$ are simple $\Gamma$-rings with min-$r$ and min-$l$ conditions (Theorem 3.13). Let $\kappa = \{\gamma \in \Gamma | M_i \gamma M_i = 0\}$, $1 \leq i \leq q$, then $\kappa = \kappa_1 \cap \cdots \cap \kappa_q$. Thus, $\Gamma' = \Gamma/\kappa$ is isomorphic to a subgroup of $\Gamma/\kappa_1 \oplus \cdots \oplus \Gamma/\kappa_q$. Set $\Gamma/\kappa_1 = \Gamma_i$. This means that $\Gamma_0$ is isomorphic to a subdirect sum of the $\Gamma_i$, $1 \leq i \leq q$. Theorem 4.2 implies that $M_i$ is isomorphic onto $D_{n_{i}(i), m_{i}(i)}$ over a division ring $D^{(i)}$ and $\Gamma_i$ is isomorphic to a non-zero subgroup of $D_{m_{i}(i), n_{i}(i)}$ over $D^{(i)}$. Thus, we have

$$M = \sum_{i=1}^{q} D_{n_{i}(i), m_{i}(i)}^{(i)}$$

(\text{direct sum}) and

$\Gamma_0 = \Gamma/\kappa$ is a subdirect sum of the $\Gamma_i$, where $\Gamma_i \subseteq D_{n_{i}(i), n_{i}(i)}^{(i)}$, $1 \leq i \leq q$, where the product of elements of $D_{n_{i}(i), n_{i}(i)}^{(i)}$ and of $D_{n_{j}(j), m_{j}(j)}^{(j)}$ is performed as usual if $i = j$.
and is 0 if \( i \neq j \).

Thus we have

**Theorem 5.1.** Let \( M \) be a semi-prime \( \Gamma \)-ring with min-\( r \) and min-\( l \) conditions. Then, the \( \Gamma \)-ring \( M \) is homomorphic onto the \( \Gamma \)-ring \( \sum_{i=1}^{q} D_{n(i), \mu(i)}^{(i)} \) where \( \Gamma_0 \) is a subdirect sum of the \( \Gamma_i, 1 \leq i \leq q \), which is a non-zero subgroup of \( D_{n(i), \mu(i)}^{(i)} \).

Theorem 2.12 and Theorem 5.1 yield the following corollary.

**Corollary 5.2.** Let \( M \) be a \( \Gamma \)-ring with min-\( r \) and min-\( l \) conditions. Then, the \( \Gamma \)-ring \( M \) is homomorphic onto the \( \Gamma \)-ring \( \sum_{i=1}^{q} D_{n(i), \mu(i)}^{(i)} \) where \( \Gamma_0 \) is a subdirect sum of the \( \Gamma_i, 1 \leq i \leq q \), which is a non-zero subgroup of \( D_{n(i), \mu(i)}^{(i)} \).

We consider the converse of the preceding comment to Theorem 5.1. First we prove the converse of Theorem 4.2.

**Theorem 5.3.** \( D_{n,m} \), \( D \) is a division ring, is a simple \( \Gamma \)-ring with min-\( r \) and min-\( l \) conditions, where \( \Gamma \) is a non-zero subgroup of \( D_{m,n} \) and \( [D, D_{n,m}] = D_m \) and \( [D_{n,m}, \Gamma] = D_n \).

**Proof.** Denote the elementary matrices by \( E_{ij} \in D_{n,m}, 1 \leq i \leq n, 1 \leq j \leq m; \ G_{st} \in D_m, 1 \leq s, t \leq m; \ H_{pq} \in D_n, 1 \leq p, q \leq n \). Let \( A=(a_{ij}) \) belong to \( D_{n,m} \), then \( A=\sum_{i,j} a_{ij} E_{ij} \).

The ideal generated by \( A \) contains \( H_{pq} A G_{st} = a_{st} E_{pt} \). If \( A \neq 0 \), then \( a_{st} \neq 0 \) for some \( (q, s) \) and the \( E_{pt} \) is in the ideal generated by \( A \). This is true for all \( p=1, \ldots, n; t=1, \ldots, m \), and hence the ideal is equal to \( D_{n,m} \), so that \( D_{n,m} \) is simple. To verify the min-\( r \) condition, observe that \( D_{n,m} \) is a right vector space of dimension \( nm \) over \( D \). Every right ideal \( J \) of \( D_{n,m} \) is a subspace, since \( A \in J \Rightarrow Ad = A(d E_m) \in J \), where \( E_m \) the identity matrix and \( d \in D \). The min-\( r \) condition holds. Similarly, the min-\( l \) condition holds.

**Theorem 5.4.** If \( M=M_1 \oplus \cdots \oplus M_q \), where \( M_1, \ldots, M_q \) are simple \( \Gamma \)-rings with min-\( r \) and min-\( l \) conditions, then \( M \) is a semi-prime \( \Gamma \)-ring with min-\( r \) and min-\( l \) conditions, where \( \Gamma \) is a subdirect sum of the \( \Gamma_i \)'s, \( M_i \Gamma M_j=0 \) (\( i \neq j \)) and \( M_i \Gamma_j M_i=0 \) (\( i \neq j \)).

**Proof.** Let \( S \) be a strongly-nilpotent ideal of \( M \) and let \( S_1, \ldots, S_q \) be its component ideals in \( M_1, \ldots, M_q \), respectively. If \( (S \Gamma)^n S=0 \) then \( (S_i \Gamma)^n S_i=0 \) for each \( i \). Since \( M_i \) is simple \( S_i=M_i \) or \( S_i=0 \). If \( S_i=M_i \) then \( (S_i \Gamma)^n S_i=M_i=0 \), a contradiction. Thus, \( S_i=0 \) and hence \( S=S_{1} \oplus \cdots \oplus S_q=0 \) and \( M \) is semi-prime.
To verify the min-\(r\) condition, suppose \(J^{(1)} \supseteq J^{(2)} \supseteq \cdots\) is a descending sequence of right ideals of \(M\). The components \(J_i^{(n)}\) in the \(\Gamma_i\)-ring \(M_i\) are a descending sequence in \(M_i\) \((J^{(1)} \supseteq J^{(2)} \supseteq \cdots \supseteq J_i^{(n)} \supseteq \cdots)\) and hence \(J_i^{(n)}\) is fixed for \(n \geq n(i)\), say. It followed that \(J^{(n)}\) is fixed for \(n \geq \max\{n(1), \ldots, n(q)\}\), and hence the min-\(r\) condition holds in \(M\). Similarly, the min-\(l\) condition can be verified.

We consider the \(\Gamma\)-rings in the sense of Nobusawa.

Let \(M\) be a \(\Gamma\)-ring in the sense of Nobusawa and \(M\) be semi-prime with min-\(r\) and min-\(l\) conditions. Let \(M = M_1 \oplus \cdots \oplus M_q\), where \(M_1, \ldots, M_q\) are simple \(\Gamma\)-rings with min-\(r\) and min-\(l\) conditions (Theorem 3.13). Let \(\Gamma_i = \Gamma/\kappa_i\), where \(\kappa_i = \{\gamma \in \Gamma | M_i \gamma M_i = 0\}\). We show that each \(\Gamma\)-ring \(M_i\) is the \(\Gamma_i\)-ring in the sense of Nobusawa. Since \(\Gamma M_i \Gamma \subseteq \Gamma_i\), \(\kappa_i\) is an ideal of \(\Gamma\). Indeed, \(M_i(\Gamma M_i \kappa_i) M_i = (M_i M_i \kappa_i) M_i = M_i M_i \kappa_i = 0\) and then \(\Gamma M_i \kappa_i \subseteq \kappa_i\). Similarly, \(\kappa_i M_i \Gamma \subseteq \kappa_i\). Hence, we can define a multiplication: \(\Gamma_i \times M_i \times \Gamma_i - \Gamma_i\) as follows:

For any \(\gamma, \delta \in \Gamma_i\), \(a \in M_i\), where \(\gamma = \gamma + \kappa_i\), \(\delta = \delta + \kappa_i\),

\[
\gamma a \delta = \gamma a \delta \quad \text{well defined.}
\]

Clearly, \(M_i \gamma M_i = 0\) implies \(\gamma = 0\).

Therefore, by Corollary 4.3, we have \(\Gamma_i = D_{m_{(1)}, \ldots, m_{(q)}}\). Since \(\kappa = 0\) and so \(\Gamma = M_i\), \(\Gamma\) is isomorphic to the subgroup of \(\sum_{i=1}^q D_{m_{(i)}, \ldots, m_{(i)}}\). Let this isomorphism be \(\phi\), then

\[
\gamma \phi = \gamma_1 + \cdots + \gamma_q, \quad \text{where } \gamma_i = \gamma + \kappa_i, \quad 1 \leq i \leq q.
\]

We show that the subgroup coincides with the group \(\sum_{i=1}^q D_{m_{(i)}, \ldots, m_{(i)}}\). Fix an element \(i\) of the index set \(\{1, 2, \ldots, q\}\). For any \(\sigma_i \in \Gamma_i = D_{m_{(i)}, \ldots, m_{(i)}}\), choose an element \(\sigma \in \Gamma\) such that \(\sigma_i = \sigma + \kappa_i\). Let \(\sigma \phi = \sigma_1 + \cdots + \sigma_q\), where \(\sigma_k = \sigma + \kappa_k, 1 \leq k \leq q\), and \(E_{i \ell}\) be the unit matrix of \(D_{m_{(i)}}\), and \(F_{i \ell}\) be the unit matrix of \(D_{n_{(i)}}\). Then, since \(\Gamma\) is the right \(\ell\)-left \(\Gamma\)-bimodule and \(D_{m_{(i)}} = [M_i, \Gamma_i] \subseteq \Gamma_i\) and \(D_{n_{(i)}} = [\Gamma_0, M_i] \subseteq R\), \(\sigma_i = E_{i \ell}(\sigma \phi) F_{i \ell} \in (\Gamma) \phi, 1 \leq i \leq q\). Now let \(i\) be free. Then, \(\sum_{i=1}^q \sigma_i \in (\Gamma) \phi\), where each \(\sigma_i\) is an arbitrary element of \(\Gamma_i\). This means \(\sum_{i=1}^q D_{m_{(i)}, \ldots, m_{(i)}} \subseteq (\Gamma) \phi\), and \((\Gamma) \phi = \sum_{i=1}^q D_{m_{(i)}, \ldots, m_{(i)}}\).

Thus, we have

\[
M = \sum_{i=1}^q D_{m_{(i)}, \ldots, m_{(i)}} \quad \text{and} \quad \Gamma = \sum_{i=1}^q D_{m_{(i)}, \ldots, m_{(i)}},
\]

which is Theorem 3 of Nobusawa [8].

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A gamma ring with minimum conditions

References


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