A GEOMETRIC MEANING OF THE RANK OF HERMITIAN SYMMETRIC SPACES

Dedicated to Professor I. Mogi on his 60th birthday

By

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§ 1. Introduction.

Let \( M \) be a Kaehler manifold and denote by \( H \) the holomorphic sectional curvature of \( M \). We say that \( H \) is \( \delta \)-pinched if there exists a positive constant \( c \) such that

\[ \delta c \leq H \leq c. \]

In this paper, we shall prove the following

THEOREM. Let \( M \) be a compact irreducible Hermitian symmetric space of rank \( r \). Then the holomorphic sectional curvature of \( M \) is \( \frac{1}{r} \)-pinched.

Although it is possible to verify the result for each Hermitian symmetric space one by one by using the curvature tensors given by E. Calabi and E. Vesentini [1], we shall given here a systematic proof.

§ 2. Preliminaries.

We begin by constructing a compact Hermitian symmetric space. For details, see e.g. [3].

Let \( \mathfrak{g} \) be a complex simple Lie algebra and \( \mathfrak{h} \) a Cartan subalgebra of \( \mathfrak{g} \). The dual space of the complex vector space \( \mathfrak{h} \) is denoted by \( \mathfrak{h}^* \). An element \( \alpha \) of \( \mathfrak{h}^* \) is called a root of \((\mathfrak{g}, \mathfrak{h})\) if there exists a non-zero vector \( X_\alpha \) in \( \mathfrak{g} \) such that

\[ [H, X_\alpha] = \alpha(H) X_\alpha \quad \text{for} \quad H \in \mathfrak{h}. \]

We denote by \( \Delta \) the set of all non-zero roots of \((\mathfrak{g}, \mathfrak{h})\) and put \( \mathfrak{g}_\alpha = CX_\alpha \). Then we have a direct sum decomposition:

\[ \mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha. \]

Since the Killing form \( K \) of \( \mathfrak{g} \) is non-degenerate on \( \mathfrak{h} \times \mathfrak{h} \), for each \( \xi \in \mathfrak{h}^* \) we can

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define \( H_\xi \in \mathfrak{h} \) by
\[
K(H, H_\xi) = \xi(H) \quad \text{for} \quad H \in \mathfrak{h}.
\]
Put \( \mathfrak{h}_0 = \sum_{\alpha \in \Delta} R\mathfrak{h}_\alpha \). Then the dual space \( \mathfrak{h}_0^* \) of \( \mathfrak{h}_0 \) can be considered as a real subspace of \( \mathfrak{h}_0^* \). Define an inner product \( (, ) \) on \( \mathfrak{h}_0^* \) by
\[
(\xi, \eta) = K(H_\xi, H_\eta) \quad \text{for} \quad \xi, \eta \in \mathfrak{h}_0^*.
\]
For each \( \alpha \in \Delta \) we choose a basis \( E_\alpha \) so that \( \{H_\alpha \ (j=1, \ldots, 1), E_\alpha \ (\alpha \in \Delta)\} \)
forms Weyl's canonical basis of \( \mathfrak{g} \). Then we have \([E_\alpha, E_{-\alpha}] = H_\alpha\), and a Lie algebra \( \mathfrak{g} \) defined as follows is a compact real form of \( \mathfrak{g}^* \):
\[
\mathfrak{g} = \sum_{\alpha \in \Delta} R\sqrt{-1}H_\alpha + \sum_{\alpha \in \Delta} R(E_\alpha + E_{-\alpha}) + \sum_{\alpha \in \Delta} R\sqrt{-1}(E_\alpha - E_{-\alpha}).
\]
We denote by \( \{\alpha_i, \ldots, \alpha_l\} \) the fundamental root system of \( \mathfrak{g} \) with respect to a linear ordering in \( \mathfrak{g}_0^* \) (so that \( \dim \mathfrak{c}_0 = l \)).

Now we fix a simple root \( \alpha_i \) \( (i=1, \ldots, l) \). For simplicity, we put \( A_\alpha = E_\alpha + E_{-\alpha} \) and \( B_\alpha = \sqrt{-1}(E_\alpha - E_{-\alpha}) \). We define a subset \( \Delta_i \) of \( \Delta \), a subalgebra \( \mathfrak{t}_i \) of \( \mathfrak{g} \) and a subspace \( \mathfrak{m}_i \) of \( \mathfrak{g} \) by
\[
\Delta_i = \{\alpha = \sum_{i} m_\alpha \alpha_i; \ m_i \geq 1\},
\]
\[
\mathfrak{t}_i = \sum_{\alpha \in \Delta_i} R\sqrt{-1}H_\alpha + \sum_{\alpha \in \Delta_i} (RA_\alpha + RB_\alpha),
\]
\[
\mathfrak{m}_i = \sum_{\alpha \in \Delta_i} (RA_\alpha + RB_\alpha),
\]
where \( \Delta^+ \) denotes the set of all positive roots.

Let \( G \) be the simply connected Lie group with Lie algebra \( \mathfrak{g} \) and \( K_i \) the connected Lie subgroup of \( G \) with algebra \( \mathfrak{t}_i \). Let \( \pi \) denote the natural projection of \( G \) onto a compact homogeneous space \( M_i = G/K_i \) and put \( o = \pi(K_i) \). Then we can identify the vector space \( \mathfrak{m}_i \) with the tangent space \( T_o(M_i) \) of \( M_i \) at \( o \). It is easily seen that there exists a unique \( G \)-invariant Riemannian metric \( g \) on \( M_i \) such that \( g = -K|m_i| \times m_i \) at \( o \). It is known that a compact Riemannian homogeneous space \( M_i \) obtained as above from a pair \( (\mathfrak{g}, \alpha_i) \) of a complex simple Lie algebra \( \mathfrak{g} \) and a simple root \( \alpha_i \) becomes a Hermitian symmetric space if and only if the coefficient \( m_i \) of \( \alpha_i \) in every \( \alpha \in \Delta_i \) is equal to 1 and the center \( \mathfrak{g}(\mathfrak{t}_i) \) of \( \mathfrak{t}_i \) is 1-dimensional, and that every compact irreducible Hermitian symmetric space can be obtained in this way.

Hereafter we assume that \( M_i \) is a Hermitian symmetric space. Then it is known that there exists an element \( Z_o \) in \( \mathbb{g}(\mathfrak{t}_i) \) such that the complex structure of \( M_i \) at \( o \) is given by \( I = \text{ad} Z_o |_{\mathfrak{m}_i} \) and \( IA_\alpha = B_\alpha, IB_\alpha = -A_\alpha \) for \( \alpha \in \Delta_i \). Since \( Z_o \in \mathfrak{g}(\mathfrak{t}_i) \), we have
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(1) \( I \cdot \text{Ad}(k) = \text{Ad}(k) \cdot I \quad \text{for} \quad k \in K_t \).

Let \( \theta^a, \theta^{-a} \) be the dual forms of \( E_a, E_{-a} \). Then we have at \( a \)

(2) \( g = 2 \sum_{a \in d_1} \theta^a \theta^{-a} \),

since \( K(E_a, E_{-a}) = -1 \). The norm of \( X \in \mathfrak{m}_t \) is denoted by \( |X| \).

§ 3. Proof of Theorem.

First we state a fundamental lemma without proof.

**Lemma (E. Cartan).** Let \( a \) and \( a' \) be two maximal abelian subspaces of \( \mathfrak{m}_t \). Then

(i) there exists an element \( k \) in \( K_t \) such that \( \text{Ad}(k)a = a' \), and

(ii) \( \mathfrak{m}_t = \bigcup_{k \in K_t} \text{Ad}(k)a \).

The rank \( r \) of \( M_t \) as a symmetric space is, by definition, the common dimension of maximal abelian subspaces of \( \mathfrak{m}_t \). By a theorem of Harish-Chandra ([2], Lemma 8), there exist \( r \) roots \( \delta_1, \ldots, \delta_r \) in \( \Delta_t \) such that none of \( \delta_i \pm \delta_j \) belong to \( \Delta \), which are called strongly orthogonal roots. Thus the space \( \mathfrak{a}_a \) spanned by \( A_{\delta_1}, \ldots, A_{\delta_r} \) over \( R \) is a maximal abelian subspace of \( \mathfrak{m}_t \). We denote by \( R \) the curvature tensor of \( (M_t, g) \). Then we have the following formula due to E. Cartan:

\[
R(X, Y)Z = -[\lbrack X, Y \rbrack, Z] \quad \text{for} \quad X, Y, Z \in \mathfrak{m}_t.
\]

Put \( S = \{ X \in \mathfrak{m}_t \mid |X| = 1 \} \). Then, for \( X \in S \), the holomorphic sectional curvature \( H(X) \) of the plane section spanned by \( X \) and \( IX \) is given by

(3) \[
H(X) = g(R(X, IX)IX, X)
= -g([\lbrack X, IX \rbrack, IX], X)
= |X, IX|^2.
\]

We assert that the range of the function \( H \) on \( S \) coincides with that of \( H \) on \( S \cap \mathfrak{a}_a \). In fact, Lemma implies that, for every \( H \in S \), there exists an element \( k \) in \( K_t \) such that \( \text{Ad}(k)X \in S \cap \mathfrak{a}_a \). Therefore from (1) and (3) we have

\[
H(\text{Ad}(k)X) = |\lbrack \text{Ad}(k)X, I \text{Ad}(k)X \rbrack|^2
= |\lbrack \text{Ad}(k)X, \text{Ad}(k)IX \rbrack|^2
= |\text{Ad}(k)[X, IX]|^2
= |X, IX|^2.
\]
which proves our assertion.

Let \( X = \sum_{j=1}^{n} x_j A_{\delta_j} \in S \cap a_0 \). Then by (2) we have

\[
1 = |X|^2 = \sum_{j=1}^{n} x_j x_j g(E_{\delta_j} + E_{-\delta_j}, E_{\delta_j} + E_{-\delta_j})
\]

\[
= 2 \sum_{j=1}^{n} x_j^2,
\]

and

\[
[X, IX] = \left[ \sum_{j=1}^{n} x_j A_{\delta_j}, \sum_{k=1}^{n} x_k B_{\delta_k} \right]
\]

\[
= \sum x_j^2 [A_{\delta_j}, B_{\delta_j}]
\]

\[
= \sum x_j^2 [E_{\delta_j} + E_{-\delta_j}, \sqrt{-1}(E_{\delta_j} - E_{-\delta_j})]
\]

\[
= -2\sqrt{-1} \sum x_j^2 [E_{\delta_j}, E_{-\delta_j}]
\]

\[
= -2\sqrt{-1} \sum x_j^2 H_{\delta_j}.
\]

Hence

\[
|\{X, IX\}|^2 = 4 \sum x_j^2 |H_{\delta_j}|^2
\]

\[
= 4 \sum x_j^2 \langle \delta_j, \delta_j \rangle.
\]

But by a theorem of C. C. Moore ([3], p. 362) we have \( \langle \delta_1, \delta_1 \rangle = \cdots = \langle \delta_r, \delta_r \rangle \). Thus the range of \( H \) is given by

\[
4r \left( \frac{1}{2r} \right)^2 \langle \delta_1, \delta_1 \rangle \leq H \leq 4 \left( \frac{1}{2} \right)^2 \langle \delta_1, \delta_1 \rangle,
\]

since \( \sum x_j^2 = \frac{1}{2} \). Therefore our theorem is proved.

§ 4. Remark.

Let \((M_1, g_1)\) be a compact irreducible Hermitian symmetric space of rank \( r_1 \) and \( H_1 \) the holomorphic sectional curvature of \((M_1, g_1)\), \( \lambda = 1, \cdots, n \). Assume that \( \max H_1 = \cdots = \max H_n \). Then a compact Hermitian symmetric space \((M_1 \times \cdots \times M_n, g_1 \times \cdots \times g_n)\) of rank \( r_1 + \cdots + r_n \) is \( \frac{1}{r_1 + \cdots + r_n} \)-pinched

References


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