PSEUDO-UMBILICAL SUBMANIFOLDS OF A SPACE FORM $N^{n+p}(C)$

By

Sun Huafei

Abstract. Let $M$ be an $n$-dimensional pseudo-umbilical submanifold in an $(n+p)$-dimensional space form $N^{n+p}(C)$. In this paper, we obtain some generalizations of B. Y. Chen in [1].

§1. Introduction.

Let $N^{n+p}(C)$ be an $(n+p)$-dimensional space form with constant sectional curvature $C$, and $M$ an $n$-dimensional submanifold in $N^{n+p}(C)$. Let $h$ be the second fundamental form of the immersion and $\xi$ the mean curvature vector, we denote by $\langle \cdot, \cdot \rangle$ the scalar product in $N^{n+p}(C)$. If there exists a function $\lambda$ on $M$ such that

$$\langle h(X,Y), \xi \rangle = \lambda(X,Y) \tag{1.1}$$

for all tangent vectors $X,Y$ on $M$, then $M$ is called a pseudo-umbilical submanifold in $N^{n+p}(C)$ (cf. [1]). It is clear that $\lambda \geq 0$. B. Y. Chen [1] proved:

(1) Let $M$ be an $n$-dimensional compact pseudo-umbilical submanifold in $N^{n+p}(C)$. Then

$$\int_M [nH\Delta H + n(C + H^2)S - \left(2 - \frac{1}{p}\right)S^2 - n^2 H^2 C] dv \leq 0,$$

where $S$, $H$ and $dv$ denote the square of the length of $h$, the mean curvature of $M$ and the volume element of $M$, respectively. (2) Let $M$ be an $n$-dimensional compact pseudo-umbilical submanifold in $N^{n+p}(C)$. If

$$nH\Delta H + n(C + H^2)S - \left(2 - \frac{1}{p}\right)S^2 - n^2 H^2 C \leq 0,$$

then the second fundamental form is parallel and $S$ is constant.

In this paper, we obtain the following generalizations of (1) and (2).
THEOREM 1. Let $M$ be an $n$-dimensional compact pseudo-umbilical submanifold in $N^{n+p}(C)$. Then
\[ \int_M \left[ n(C + H^2)S - \frac{3}{2} S^2 - n^2 H^2 C \right] dv \leq 0, \text{ for } p > 1 \]
and
\[ \int_M \left[ n(C + 4H^2)S - \frac{3}{2} S^2 - n^2 H^2 C - \frac{5}{2} n^2 H^4 \right] dv \leq 0, \text{ for } p > 2. \]

THEOREM 2. Let $M$ be an $n$-dimensional compact pseudo-umbilical submanifold in $N^{n+p}(C)$. If
\[ nH\Delta H + n(C + H^2)S - \frac{3}{2} S^2 - n^2 H^2 C \geq 0, \text{ for } p > 1 \]  
\( (1.2) \)

or
\[ nH\Delta H + n(C + 4H^2)S - \frac{3}{2} S^2 - n^2 H^2 C - \frac{5}{2} n^2 H^4 \geq 0, \text{ for } p > 2, \]  
\( (1.3) \)

then the second fundamental form is parallel and $S$ is constant. In particular, if the equality of (1.2) holds and $C = 1$, then $M$ is totally geodesic or $n = 2$ and $M$ is a veronese surface in $S^4$ (1) and if the equality of (1.3) holds and $C = 1$, then $M$ is totally geodesic, where $S^4(1)$ denotes the 4-dimensional unit sphere.

If $H = 0$ and $C = 1$, then Theorem 2 was proved jointly by A. M. Li and J. M. Li in [2].

§2. Local formulas.

We shall make use of the following convention on the ranges of indices:

\[ A, B, \ldots, = 1, \ldots, n, n+1, \ldots, n+p; i, j, \ldots, = 1, \ldots, n; \alpha, \beta, \ldots, = n+1, \ldots, n+p. \]

We choose a local field of orthonormal frames $e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+p}$ in $N^{n+p}(C)$. Such that, restricted to $M$ the vectors $e_1, \ldots, e_n$ are tangent to $M$ and $\{\omega_A\}$ is the field of dual frames. Then the structure equations of $N^{n+p}(C)$ are given by
\[ d\omega_A = -\sum_B \omega_B \wedge \omega_A, \omega_A + \omega_{BA} = 0, \]  
\( (2.1) \)
\[ d\omega_{AB} = -\sum_C \omega_{AC} \wedge \omega_{CB} + \frac{1}{2} \sum_{CD} K_{ABCD} \omega_C \wedge \omega_D, \]  
\( (2.2) \)
\[ K_{ABCD} = C(\delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC}). \]  
\( (2.3) \)
Restricting these forms to $M$, we have

$$\omega_\alpha = 0, \omega_{\alpha i} = \sum_j h^\alpha_{ij} \omega_j, h^\alpha_{ij} = h^\alpha_{ji}. \quad (2.4)$$

$$d\omega_\alpha = -\sum_j \omega_{\alpha j} \wedge \omega_j, \quad (2.5)$$

$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2} \sum_{kl} R_{iklj} \omega_k \wedge \omega_l, \quad (2.6)$$

$$R_{ijkl} = K_{ijkl} + \sum_\alpha (h^\alpha_{ij} h^\alpha_{kl} - h^\alpha_{ik} h^\alpha_{jl}), \quad (2.7)$$

$$d\omega_{\alpha \beta} = -\sum_\gamma \omega_{\alpha \gamma} \wedge \omega_{\beta \gamma} + \frac{1}{2} \sum_{ij} R_{ij\alpha\beta} \omega_i \wedge \omega_j, \quad (2.8)$$

$$R_{i\beta j\alpha} = \sum_k (h^\alpha_{ik} h^\beta_{\alpha j} - h^\alpha_{\alpha k} h^\beta_{ij}). \quad (2.9)$$

$h^\alpha_{ij}$ and $h^\alpha_{kl}$ are given by

$$\sum_k h^\alpha_{ik} \omega_k = dh^\alpha_{ij} - \sum_l h^\alpha_{ij} \omega_l - \sum_k h^\beta_{ij} \omega_k - \sum_\beta h^\beta_{ij} \omega_\beta, \quad (2.10)$$

and

$$\sum_l h^\alpha_{ljk} \omega_l = dh^\alpha_{ij} - \sum_l h^\alpha_{ij} \omega_l - \sum_l h^\alpha_{ik} \omega_l - \sum_\beta h^\beta_{ij} \omega_\beta, \quad (2.11)$$

respectively, where

$$h^\alpha_{ij} = h^\alpha_{ji}, \quad (2.12)$$

$$h^\alpha_{ij} - h^\alpha_{ji} = \sum_m h^\alpha_{ik} R_{mkl} + \sum_m h^\beta_{im} R_{mkl} + \sum_m h^\beta_{ij} R_{lkm}. \quad (2.13)$$

We call $h = \sum_{\alpha i} h^\alpha_{ij} \omega_i \omega_j e_\alpha$ the second fundamental form of the immersed manifold $M$. We denote the square of the length of $h$ by $S = \sum_{\alpha i} (h^\alpha_{ij})^2$. $\zeta = \frac{1}{n} \sum_\alpha \text{tr} H_\alpha e_\alpha$ and $H = \|h\| = \left(\frac{1}{n} \sum_\alpha (\text{tr} H_\alpha)^2\right)^{1/2}$ denote the mean curvature vector and the mean curvature of $M$, respectively. Here tr is the trace of the matrix $H_\alpha = (h^\alpha_{ij})$. Now, let $e_{n+p}$ be parallel to $\zeta$. Then we get

$$\text{tr} H_{n+p} = nh, \text{tr} H_\alpha = 0, \alpha \neq n + p. \quad (2.14)$$

The Laplacian $\Delta h^\alpha_{ij}$ of the second fundamental form $h^\alpha_{ij}$ is defined by $\Delta h^\alpha_{ij} = \sum_k h^\alpha_{ijk}$. By a simple calculation we have (cf. [1])

$$\frac{1}{2} \Delta S = \sum_{\alpha i} (h^\alpha_{ij})^2 + \sum_{\alpha i} (h^\alpha_{ij}) \Delta h^\alpha_{ij}$$
\[
\sum_{i\neq k\neq l\alpha}(h_{ij})^2 + \sum_{i\neq j\neq k\neq l\alpha}h_{ij}h_{kl}^2 + \sum_{i\neq j\neq k\neq l\alpha}h_{ij}^2 R_{ijkl} + \sum_{i\neq j\neq k\neq l\alpha}h_{ij}^2 h_{kl} R_{ijkl}
\]
\[
= \sum_{i\neq j\neq k\neq l\alpha}(h_{ij})^2 + nH \Delta H + n(C + H^2)S - n^2 H^2 C - \sum_{\alpha\beta}(trH_{\alpha}H_{\beta})^2
\]
\[
+ \sum_{\alpha\beta}tr(H_{\alpha}H_{\beta} - H_{\beta}H_{\alpha})^2.
\]

(2.15)

§3. **Proofs of Theorems.**

From (1.1) and (2.14) we get
\[
\sum_{\alpha}trH_{\alpha}h_{ij}^2 = n\lambda \delta_{ij}, H^2 = \lambda
\]
and
\[
h_{ij}^{\alpha\beta} = H \delta_{ij}.
\]

(3.1)

In order to prove our Theorems, we need the following Lemma 1 which can be proved by diagonalizing the matrix \((trH_{\alpha}H_{\beta})\) and using the inequality \(tr(H_{\alpha}H_{\beta} - H_{\beta}H_{\alpha})^2 \geq -2trH_{\alpha}H_{\beta}^2\) ([1]), and Lemma 2.

**Lemma 1** [2]. Let \(H_i (i \geq 2)\) be symmetric \((n \times n)\)-matrices, \(S_i = trH_i^2\) and \(S = \sum_{i} S_i\). Then
\[
\sum_{i\neq j}(trH_i H_j - H_j H_i)^2 - \sum_{i}(trH_i H_i)^2 \geq -\frac{3}{2} S^2
\]
and the equality holds if and only if all \(H_i = 0\) or there exist two of \(H_i\) different from zero. Moreover, if \(H_1 \neq 0, H_2 \neq 0, H_i = 0 (i \neq 1, 2)\), then \(S_1 = S_2\) and there exists an orthogonal \((n \times n)\)-matrix \(T\) such that
\[
TH_1T = \sqrt{\frac{S_1}{2}} \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix},
TH_2T = \sqrt{\frac{S_2}{2}} \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]

**Lemma 2.** When \(p > 2\),
\[
\sum_{\alpha\beta}(trH_{\alpha}H_{\beta} - H_{\beta}H_{\alpha})^2 - \sum_{\alpha\beta}(trH_{\alpha}H_{\beta})^2 \geq -\frac{3}{2} S^2 + 3nH^2 S - \frac{5}{2} n^2 H^4
\]

**Proof.** Using (2.14) and (3.1), when \(p > 2\), we have
\[
\sum_{\alpha\beta}(trH_{\alpha}H_{\beta} - H_{\beta}H_{\alpha})^2 - \sum_{\alpha\beta}(trH_{\alpha}H_{\beta})^2
\]
\[
= \sum_{\alpha\beta}(tr(H_{\alpha}H_{\beta} - H_{\beta}H_{\alpha})^2 - \sum_{\alpha\beta}(trH_{\alpha}H_{\beta})^2
\]
\[
(3.3)
\]

Applying Lemma 1 to (3.3) we have
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\[ \sum_{\alpha \beta} tr(H_{\alpha}H_{\beta} - H_{\beta}H_{\alpha})^2 - \sum_{\alpha \beta} (trH_{\alpha}H_{\beta})^2 \geq -\frac{3}{2} \left( \sum_{\alpha \neq \alpha + p} (trH_{\alpha}^2)^2 \right) - (trH_{\alpha + p}^2)^2 \]
\[ = -\frac{3}{2} (S - trH_{\alpha + p}^2)^2 - (trH_{\alpha + p}^2)^2 \]
\[ = -\frac{3}{2} (S - nH^2)^2 - n^2 H^4 \]
\[ = -\frac{3}{2} S^2 + 3nH^2 S - \frac{5}{2} n^2 H^4. \]

This completes the proof of Lemma 2.

Using (3.1) we can get
\[ \sum_{i \neq j} (h_{ij}^2) \geq \sum_{i \neq j} (h_{ij}^{n+p})^2 = n \sum_i (\nabla_i H)^2. \] (3.4)

It is obvious that
\[ \frac{1}{2} \Delta H^2 = H \Delta H + \sum_i (\nabla_i H)^2. \] (3.5)

Therefore, using Lemma 1, (3.4) and (3.5) when \( p > 1 \) by (2.15) we have
\[ \frac{1}{2} \Delta S \geq \sum_{i \neq j} (h_{ij}^{n+p})^2 + nH \Delta H + n(C + H^2)S - n^2 H^2 C - \frac{3}{2} S^2 \]
\[ \geq n \sum_i (\nabla_i H)^2 + nH \Delta H + n(C + H^2)S - \frac{3}{2} S^2 - n^2 H^2 C \]
\[ = \frac{1}{2} n \Delta H^2 + n(C + H^2)S - \frac{3}{2} S^2 - n^2 H^2 C \] (3.6)

Since \( M \) is compact, form (3.6) we have
\[ \int_M [n(C + H^2)S - \frac{3}{2} S^2 - n^2 H^2 C]dV \leq 0. \]

On the other hand, from the first inequality of (3.6), we know that if
\[ nH \Delta H + n(C + H^2)S - \frac{3}{2} S^2 n^2 H^2 C \geq 0 \] (3.7)

and \( M \) is compact, then the second fundamental form \( h_{ij}^a \) is parallel and \( S \) is constant. In particular, if the equality of (3.7) holds and \( C = 1 \), then we see that the equality of (3.2) holds. So by Lemma 1 (3.7) implies that all \( H_a = 0 \) (i.e. \( M \) is totally geodesic) or there exist two of \( H_a \) different from zero. In this case, by Lemma 1 we may therefore assume that
\[ H_{n+1} = f \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad H_{n+2} = g \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad f, g \neq 0. \]
Hence we have
\[ trH_{n+1} = trH_{n+2} = 0. \] (3.8)

Using (3.8) we find that \( \sum_a trH_a h^{\alpha}_{ij} = 0 \) and \( H = 0 \) identically. So by Lemma 1 the equality of (3.7) implies that \( M \) is totally geodesic or \( n = 2 \) and \( M \) is a veronese surface of \( S^4(1) \).

On the other hand, when \( p > 2 \) using Lemma 2, (3.4) and (3.5) from (2.15) we get
\[
\frac{1}{2} \Delta S \geq \sum_{\alpha} (h^{\alpha}_{ij})^2 + nH \Delta H + n(C + H^2)S - n^2 H^2 C - \frac{3}{2} S^2 + 3nH^2 S - \frac{5}{2} n^2 H^4
\]
\[
\geq \frac{1}{2} n \Delta H^2 + n(C + 4H^2)S - \frac{3}{2} S^2 - n^2 H^2 C - \frac{5}{2} n^2 H^4 \] (3.9)

Thus, when \( M \) is compact by (3.9) we obtain
\[
\int_M \left[ n(C + 4H^2)S - \frac{3}{2} S^2 - n^2 H^2 C - \frac{5}{2} n^2 H^4 \right] dV \leq 0.
\]

From the first inequality of (3.9), we see that if
\[
nH \Delta H + n(C + 4H^2)S - \frac{3}{2} S^2 - n^2 H^2 C - \frac{5}{2} n^2 H^4 \geq 0. \] (3.10)
then the second fundamental form \( h^{\alpha}_{ij} \) is parallel and \( S \) is constant. In particular, for \( p > 2 \) when the equality of (3.10) holds and \( C = 1 \), by Lemma 1 we find that the equality of (3.2) holds and \( M \) is totally geodesic. This completes the proofs of Theorems.

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References