GRAPHS AND FINITE DISTRIBUTIVE PARTIAL LATTICES

By
Juhani Nieminem

Abstract

The Hasse diagram graph of a finite distributive partial lattice is characterized by means of prime convexes.

Median graphs constitute a well known and widely studied class of graphs; see for example the papers [1] and [2] and the references therein. They constitute a subclass of the Hasse diagram graphs of distributive partial lattices. In this paper we give a characterization for the Hasse diagram graphs $G$ of finite distributive partial lattices by means of prime convexes of $G$. This characterization generalizes that of Mulder and Schrijver for median graphs reprinted in [1, Theorem 2.2].

A meetsemilattice $S$ is a partial lattice if for any two elements $a, b$ having an upper bound in $S$ also the element $a \vee b$ belongs to $S$. Clearly every finite meetsemilattice is a partial lattice. A partial lattice $S$ is distributive if its every subset $\{s|s \leq k\}$ is a distributive lattice. A finite distributive partial lattice $S$ can be embedded in the distributive lattice $\mathcal{I}(S)$ of ideals of $S$, where the join of two ideals $I$ and $J$ is $I \vee J = \{s|s \leq i \vee j, i \in I \text{ and } j \in J\}$. By using this lattice we see that one shortest path joining two points $a$ and $b$ of the Hasse diagram graph $S$ contains the point $a \wedge b$, and if $a > b$, then every point $c$, $a \geq c \geq b$, is on some shortest $a-b$ path.

The graphs $G = \langle V, X \rangle$ considered here are finite, connected and undirected without loops and multiple lines. The points of $G$ constitute the set $V$ and its lines the set $X$. A pointset $A \subseteq V$ of $G$ is called a convex if $A$ contains all points of any shortest $a-b$ path (of any $a-b$ geodesic) for every two points $a, b \in A$. The intersection of two convexes is also a convex and thus the least convex containing a given pointset $B$ of $G$ is $\bigcap \{C|C \text{ is a convex and } B \subseteq C\}$. This set is briefly denoted by $\langle B \rangle$. A convex $A \neq V$ is called prime if the set $V \setminus A$ is also a convex. The sets $\emptyset$ and $V$ are trivial prime convexes. A graph $G$ has the prime convex intersection property (is a prime convex intersection graph) if its every...
convex $A$ is the intersection of all prime convexes containing $A$. By [1, Theorem 2.2], every median graph is a prime convex intersection graph. The class of prime convex intersection graphs is rather wide: for example every complete graph belongs to this class.

Let $a, b, c \in V$. A point $t$ satisfying the distance conditions $d(a, b) = d(a, t) + d(t, b), \ d(b, c) = d(b, t) + d(t, c)$ and $d(a, c) = d(a, t) + d(t, c)$ is a median of the points $a, b$ and $c$. A graph is a median graph if its all three points have exactly one median.

If $A$ is a subset of a set $U$, then $\bar{A} = U \setminus A$ is its complement in $U$.

When proving the main theorem of this note we need two auxiliary results which we prove first.

**Lemma 1.** A connected graph $G$ is a prime convex intersection graph if and only if for any nonempty convex $A$ and any point $x, x \in A$, there is a prime convex $P$ separating $A$ and $x$, i.e. $A \subset P$ and $x \in \overline{P}$.

**Proof.** If $G$ is a prime convex intersection graph, $A$ its nonempty convex and $x$ its point such that $x \in A$, there is a prime convex $P$ separating $A$ and $x$, because otherwise $A$ cannot be represented as an intersection of prime convexes of $G$. Conversely, if there is a prime convex separating any convex $A$ and any point $x$ of the lemma then $G$ is a prime convex intersection graph. Indeed, if there is a nonempty convex $A$ which cannot be expressed as the intersection of prime convexes, then the intersection contains a point $x$ not belonging to $A$. By assumption there is a prime convex $P$ separating $A$ and $x$, and thus the intersection cannot contain the point $x$, and the lemma follows.

**Lemma 2.** The convex $\langle a, b \rangle$ of a prime convex intersection graph $G$ consists of points on $a$-$b$ geodesics for every pair $a, b \in V$.

**Proof.** Let $a$ and $b$ be a pair of points such that the convex $\langle a, b \rangle$ contains at least one point $v$ which is not on any $a$-$b$ geodesic. This implies the existence of two points $x$ and $z, x$ is on an $a$-$b$ geodesic and $z$ is on another $a$-$b$ geodesic, such that no point $x_1, \ldots, x_m$ of an $x$-$z$ geodesic $x = x_0, x_1, \ldots, x_m, x_{m+1} = z$ is on any $a$-$b$ geodesic. Clearly $a$ and $b$ can be chosen such that every convex $\langle u, w \rangle$ with $d(u, w) < d(a, b)$ is the set of all points on $u$-$w$ geodesics. We may assume further that $d(a, b) \geq d(x, b), \ d(z, b) \geq d(x, b)$, and that $x$ and $z$ are as near to $b$ as possible. Let us consider the point $x_1$. Because $d(a, x) < d(a, b)$, the convex $\langle a, x \rangle$ consists of points on $a$-$x$ geodesics, and thus $x_1 \in \langle a, x \rangle$. By Lemma 1, the prime convex intersection property of $G$ implies now the existence of a prime
convex $P$ separating $\langle a, x \rangle$ and $x_1: \langle a, x \rangle \subset P$ and $x_1 \in \overline{P}$. Because $x_1 \in \langle a, b \rangle$, we have $x_1, b \in \overline{P}$. Let $x = b_0, b_1, b_2, \ldots, b_{k-1}, b_k = b$ be the points of an $x$-$b$ geodesic. Because $x$ and $z$ are as near to $b$ as possible, $d(z, b) \geq d(x, b)$ and $d(x_1, z) \geq d(x_1, x) = 1$, then a $b_i$-$x_1$ geodesic goes over $x, i = 1, \ldots, k$. This implies that there is no prime convex separating $\langle a, x \rangle$ and $x_1$, which is a contradiction. Thus the assumption is false and the convex $\langle a, b \rangle$ consists of points on $a$-$b$ geodesics for every pair $a, b \in V$, and the lemma follows.

Now we can present the characterization theorem of this note.

**Theorem.** A connected graph $G$ is isomorphic to the Hasse diagram graph of a finite distributive partial lattice if and only if the following two conditions hold:

(i) $G$ is a prime convex intersection graph;

(ii) $\bigcap \{P \mid P \in \mathcal{K}\} \neq \emptyset$ or $\mathcal{K} = \emptyset$ for the collection $\mathcal{K}$ of all nontrivial prime convexes in $G$ having the following property: if $P_1 \in \mathcal{K}$, there are $P_2, P_3, \ldots, P_n \in \mathcal{K}$ such that $P_1 \cap P_j \neq \emptyset$ and $P_1 \cap P_2 \cap \cdots \cap P_n = \emptyset$.

**Proof.** Mulder and Schrijver proved that a connected graph $G$ is a median graph if and only if $G$ is a prime convex intersection graph and its prime convexes satisfy the Helly property [1, Theorem 2.2]. The condition (ii) above is nothing but a weakened Helly property for prime convexes of $G$.

Assume first that $G$ is the Hasse diagram graph of a finite distributive partial lattice $S$.

(i) Let $x \in S$. The element corresponding $x$ in the ideal lattice $I(S)$ of $S$ is $[x]$. Because $I(S)$ is distributive, one $[x] - [x]$ geodesic goes over the element $[x] \wedge [x] = [x \wedge x]$. Thus, if the distance $d([x], [x]) = n$ in $I(S)$, then $d(x, x) = 2n$ in $S$, because the $x - x \wedge x - x$ path always belongs to $S$. In particular, if $C$ is a convex of the Hasse diagram graph of $I(S)$, then the set $\{x \mid [x] \in C \text{ in } I(S)\} = C_*$ is a convex in $S$. Moreover, if $C$ is a prime convex in $I(S)$, then $C_*$ is a prime convex in $S$. Let $A$ be a nonempty convex of $G$, $x$ a point of $G$ with $x \in A$ and $A^*$ the least convex of the graph $G(I(S))$ of $I(S)$ with the property: $[x] \in A^*$ in $G(I(S))$ if $x \in A$ in $G$. Clearly, $[x] \in A^*$ in $G(I(S))$. Because $I(S)$ is a distributive lattice, the graph $G(I(S))$ is a median graph and has thus the prime convex intersection property. Hence there is a prime convex $C$ in $G(I(S))$ separating $A^*$ and $[x]$, which implies that the prime convex $C_*$ separates $A$ and $x$ in $G$. By Lemma 1, this proves that $G$ has the prime convex intersection property, and thus (i) holds for $G$.

(ii) Assume that the collection $\mathcal{K}$ of the theorem is nonempty. We prove
that least element 0 of $S$ belongs to $\cap \{ \bar{P} | P \in \mathcal{X} \}$, from which the assertion follows. In fact, we prove the assertion for $n=3$; the proofs are the same for other values of $n$ and hence they are omitted. Let $P_1, P_2, P_3 \in \mathcal{X}$ be three prime convexes of $G$ such that $P_1 \cap P_2 \neq \phi$ and $P_1 \cap P_2 \cap P_3 = \phi$. The sets $P_1 \cap P_2, P_1 \cap P_3$ and $P_2 \cap P_3$ are convexes of $G$, and because $S$ is finite, every one of them has a least element, and let them be $a \in P_1 \cap P_2, b \in P_1 \cap P_3$ and $c \in P_2 \cap P_3$. Assume that $0 \in \cap \{ \bar{P} | P \in \mathcal{X} \}$, which means that 0 belongs to at least one set of $\mathcal{X}$, say to $P_1$. Because $0, a, b \in P_1$, then also $a \land b \land c \in P_1$. The relation $a, c \in P_2$ implies that $a \land c \in P_2$. On the other hand, $a \geq a \land c \geq a \land b \land c$, where $a, a \land b \land c \in P_1$, and thus $a \land c \in P_1$. Accordingly, $a \land c \in P_1 \cap P_2$, and because $a$ is the least element in this convex, $a = a \land c \geq c$. Similarly we see that $b \leq c$. Because there is an upper bound $c$ for $a$ and $b$, the element $a \lor b$ exists, and as well known, an $a \lor b$ geodesic goes over $a \lor b$ in the Hasse diagram graph of a finite distributive lattice. Thus $a \lor b \in P_1$. Because $c, b \in P_3$ and $c \geq a \lor b$, the element $a \lor b$ belongs to $P_3$, and analogously we see that $a \lor b \in P_2$. Now, $a \lor b \in P_1 \cap P_2 \cap P_3$, which intersection should be empty, and hence the assumption $0 \in \cap \{ \bar{P} | P \in \mathcal{X} \}$ must be false. This proves the property (ii).

Assume conversely that $G$ is a graph satisfying the properties (i) and (ii) of the theorem. We choose an arbitrary point from the set $\cap \{ \bar{P} | P \in \mathcal{X} \}$ and denote it by $h$. Let $a$ and $b$ be two arbitrary points in $V$ and let us consider the intersection $\langle h, a \rangle \cap \langle h, b \rangle \cap \langle a, b \rangle$. Because the convexes $\langle h, a \rangle$, $\langle h, b \rangle$ and $\langle a, b \rangle$ are the intersections of corresponding prime convexes, we can substitute the intersection $\langle h, a \rangle \cap \langle h, b \rangle \cap \langle a, b \rangle$ by the expression

$$(\cap \{ P_i \} \cap \langle h, a \rangle \subset P_1) \cap (\cap \{ U_j \} \cap \langle h, b \rangle \subset U_j) \cap (\cap \{ W_k \} \cap \langle a, b \rangle \subset W_k).$$

Now, $P_i \cap W_k, P_i \cap U_j, U_j \cap W_k \neq \phi$, and if $\langle h, a \rangle \cap \langle h, b \rangle \cap \langle a, b \rangle = \phi$, then $h \in \cap \{ \bar{P} | P \in \mathcal{X} \}$, which is a contradiction. Thus $\langle h, a \rangle \cap \langle h, b \rangle \cap \langle a, b \rangle \neq \phi$. Moreover, this intersection contains exactly one element. This can be seen as follows: Every prime convex $P$ of $G$ (or its complement $\bar{P}$) contains at least two of the points $a, b, h$. If the intersection $\langle h, a \rangle \cap \langle h, a \rangle \cap \langle a, b \rangle$ contains two disjoint points $x$ and $y$, then every $P$ (or $\bar{P}$) contains both $x$ and $y$, and the convex $x$ cannot be separated from the point $y$, which contradicts (i) by Lemma 1. Thus $\langle h, a \rangle \cap \langle h, b \rangle \cap \langle a, b \rangle = [d]$. According to Lemma 2, a convex $\langle x, y \rangle$ consists of points on $x-z$ geodesics. Thus the relation $[d] = \langle h, a \rangle \cap \langle h, b \rangle \cap \langle a, b \rangle$ shows that every triple $h, a, b$, where $a$ and $b$ are arbitrary points of $G$, has a unique median.

We order now the points of $V$ as follows:
Graphs and finite distributive partial
dimensional 397

\[ a \leq b \iff a \text{ is on a } b-h \text{ geodesic } \iff a \in \langle h, b \rangle. \]

This definition suggests us to define the meet \( a \wedge b \) as the unique median \( d \) of the points \( a, b \) and \( h \). Assume that \( c \) is a point such that \( c \in \langle h, a \rangle \cap \langle h, a \rangle \) and \( c \in \langle h, d \rangle \). The intersection \( \langle h, d \rangle \cap \langle c, b \rangle \) is empty, because if \( x \) belongs to this intersection, then the \( d-x-c-h \) path is a \( d-h \) geodesic and \( c \in \langle d, h \rangle \), which is a contradiction. There is a prime convex \( P \) separating the convexes \( \langle h, d \rangle \) and \( \langle c, b \rangle : \langle h, d \rangle \subset P \) and \( \langle c, b \rangle \subset P \). Indeed, as seen above, the points \( h, d \) and \( c \) have a median \( u \) which is on a \( d-h \) geodesic and thus belongs to the convex \( \langle d, h \rangle \). By the prime convex intersection property of \( G \) and Lemma 1, there is a prime convex \( P \) separating \( \langle c, b \rangle \) and \( u \) \( \langle \langle c, b \rangle \subset P \) and \( u \in \overline{P} \). If now \( h \) or \( d \) belongs to \( P \), then also \( u \) belongs to \( P \) because \( u \) is on a \( c-h \) geodesic as well as on a \( c-d \) geodesic. Thus \( h, d \in P \), whence also \( \langle h, d \rangle \subset P \). If \( a \in P \), then \( c \in P \) because it is on an \( a-h \) geodesic, and thus \( a \) must belong to \( P \). Because \( d \) is on an \( a-b \) geodesic, the relation \( a, b \in P \) implies a contradiction, and hence \( c \in \langle h, d \rangle \). This proves that \( d \) is a maximum lower bound of \( a \) and \( b \), and thus the order defined on \( V \) is a meetsemilattice order. Accordingly, \( V \) is a meetsemilattice with \( h \) as the least element. Because \( V \) is finite, it is a partial lattice. The Hasse diagram graph of \( V \) is isomorphic to \( G \): When a line belongs to an \( x-h \) geodesic, there is nothing to prove, and hence we assume that the line \( \langle a, b \rangle \) of \( G \) does not belong to any \( x-h \) geodesic. This is possible only if \( d(a, h) = d(b, h) \). But then \( a, b \) and \( h \) have no median, which is absurd, and the isomorphism follows.

It remains to show that every set \( \langle k \rangle = \{ v \mid v \in V \text{ and } v \leq k \} \) is a distributive lattice. By the order definition above, \( \langle h, k \rangle = \langle k \rangle \). Every convex \( A \) of a prime convex intersection graph induces a prime convex intersection graph. By Mulder and Schrijver [1, Theorem 2.2], a prime convex intersection graph \( \langle h, k \rangle \) is a median graph (and then the Hasse diagram graph of a distributive lattice with \( h \) as the least element and \( k \) as the greatest element by [1, Theorem 3.1]) if its prime convexes needed to separate its convexes satisfy the Helly property. The prime convexes needed to separate the convexes of \( \langle h, k \rangle \) are obtained from the prime convexes of \( \mathcal{X} \) by intersecting them with \( \langle h, k \rangle \). Let now \( P_1, P_2 \ldots, P_m \) be prime convexes of \( \mathcal{X} \) such that \( P_i \cap P_j \cap \langle h, k \rangle = \emptyset \). We denote the corresponding prime convexes of \( \langle h, k \rangle \) by \( P_i^h \). By Lemma 2, the convex \( \langle h, k \rangle \) consists of points on \( h-k \) geodesics in \( G \). If \( h, k \in P_i^h \), then \( P_i^h \) is not prime because its every point is on some \( h-k \) geodesic. Hence either \( h \) or \( k \) belongs to \( P_i^h \). The relation \( h \in P_i^h \) contradicts the property \( h \in \cap \{ P \mid P \in \mathcal{X} \} \), and thus \( k \in P_i^h \), and this relation holds for every \( i, i = 1, \ldots, m \). Then \( k \in P_i^h \cap P_j^h \cap \cdots \cap P_m^h \), and the Helly property of the prime convexes needed to separate the convexes of \( \langle h, k \rangle \) holds.


$k)$ follows. This proves the distributivity of $\langle h, k \rangle = \langle k \rangle$, and thus $G$ is the Hasse diagram graph of a finite distributive partial lattice.

The author likes to express his sincere thanks to the referee for his valuable suggestions and comments.

References