A SUBMANIFOLD WHICH CONTAINS MANY EXTRINSIC CIRCLES

By

Koichi Ogiue and Ryoichi Takagi

Dedicated to Professor Kentaro Yano in honor of the Second Order of Merit

1. Introduction

There are many simple characterizations of a sphere in $E^3$, which are either elementary geometric or differential geometric. For example,

\((E_1)\) It "looks round" from an arbitrary point.

or

\((E_2)\) A section with an arbitrary plane is a circle.

gives an elementary geometric criterion for a surface to be a sphere. On the other hand,

or

\((D_2)\) Every geodesic is a plane curve.

gives a differential geometric criterion for a compact surface to be a sphere.

A condition such as \((D_2)\) is simple and logical but not practical, because it is not so easy for an observer in $E^3$ to know practically that a curve on a surface is a geodesic or not.

On the contrary, it is easy to know that a curve in $E^3$ is a circle or not and is contained in a surface or not.

Therefore we consider an elementary geometric condition such as

\((*)\) A circle in $E^3$ of (arbitrarily) given radius can be pressed entirely on an arbitrary position of a surface.

It is easy to see that \((*)\) is a condition for a compact surface to be a sphere. A condition such as \((*)\) is practical in the sense that it is available in verifying the sphericity of a given physical solid. We emphasize that such a condition is quite natural because an observer is an inhabitant of an ambient space. But, \((*)\) requires a very large quantity of information because of its condition "an arbitrary position".

Received September 2, 1983.
Therefore we are going to give in §4 practical criterion for a compact surface to be a sphere, which is better than (*). Moreover we will extend our situation to a general Riemannian submanifold and give characterizations of an extrinsic sphere.

2. Basic notions

Let $M$ be an $n$-dimensional submanifold immersed in an $m$-dimensional Riemannian manifold $\tilde{M}$. The Riemannian connections of $M$ and $\tilde{M}$ are denoted by $\nabla$ and $\tilde{\nabla}$, respectively, whereas the normal connection is denoted by $\nabla^\perp$.

The second fundamental form $\sigma$ is defined by

$$\sigma(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y,$$

where $X$ and $Y$ are vector fields tangent to $M$.

For a vector field $\xi$ normal to $M$, the tensor field $A_t$ of type $(1,1)$ on $M$ is defined by

$$\tilde{\nabla}_X \xi = -A_t X + \nabla^\perp_X \xi.$$

Then $\sigma$ and $A_t$ are related by

$$\langle \sigma(X, Y), \xi \rangle = \langle A_t X, Y \rangle,$$

where $\langle , \rangle$ denotes the inner product with respect to the respective Riemannian metrics.

The covariant derivative $\nabla'_X \sigma$ of $\sigma$ is defined by

$$(\nabla'_X \sigma)(Y, Z) = \nabla^\perp_X \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z).$$

The mean curvature vector field $\mathfrak{h}$ is defined by

$$\mathfrak{h} = \frac{1}{n} \mathop{\text{trace}} \sigma.$$

We say that $\mathfrak{h}$ is parallel if $\nabla^\perp_X \mathfrak{h} = 0$ for all $X$ tangent to $M$.

We say that $M$ is totally umbilic if

$$\sigma(X, Y) = \langle X, Y \rangle \mathfrak{h}$$

for all $X$ and $Y$. Equivalently, $M$ is totally umbilic if

$$A_t = \langle \xi, \mathfrak{h} \rangle I$$

for all $\xi$, where $I$ denotes the identity transformation. It is known that if $\tilde{M}$ is a space form (i.e., a Riemannian manifold of constant curvature), then a totally umbilic submanifold $M$ of $\tilde{M}$ has parallel mean curvature vector. A submanifold $M$ of an arbitrary Riemannian manifold $\tilde{M}$ is called an extrinsic sphere if it is totally umbilic and has non-zero parallel mean curvature vector.
A regular curve $\gamma = \gamma(s)$ on $\tilde{M}$ parametrized by arc length $s$ is called a circle if there exist a field $Y = Y(s)$ of unit vectors along $\gamma$ and a positive constant $k$ such that

$$\begin{align*}
\nabla_{\dot{\gamma}} Y &= k Y \\
\nabla_{\dot{\gamma}} \dot{Y} &= -k \dot{Y},
\end{align*}$$

where $\dot{\gamma}$ denotes the unit tangent vector of $\gamma$.

The number $k$ (resp. $1/k$) is called the curvature (resp. radius) of $\gamma$. It is easily seen that a circle $\gamma = \gamma(s)$ of curvature $k$ in $\tilde{M}$ satisfies

$$\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} \dot{\gamma} + k^2 \dot{\gamma} = 0.$$

3. A circle contained in a submanifold

Let $M$ be a submanifold of $\tilde{M}$. A curve $\gamma = \gamma(s)$ in $\tilde{M}$ is a circle of curvature $k$ if it satisfies

$$\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} \dot{\gamma} + k^2 \dot{\gamma} = 0.$$

Using the equation of Gauss $\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} \dot{\gamma} - \Gamma_{\dot{\gamma} \dot{\gamma}} \dot{\gamma} = \sigma(\dot{\gamma}, \dot{\gamma})$, we easily obtain the following result, which is useful throughout this paper.

**Lemma.** Let $M$ be a submanifold of $\tilde{M}$. Then a curve $\gamma = \gamma(s)$ in $M$ is a circle of curvature $k$ in $\tilde{M}$ if and only if it satisfies

$$\begin{align*}
\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} \dot{\gamma} + k^2 \dot{\gamma} &= 0 \\
\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} \sigma(\dot{\gamma}, \dot{\gamma}) + 3\sigma(\dot{\gamma}, \nabla_{\dot{\gamma}} \dot{\gamma}) &= 0.
\end{align*}$$

4. A practical characterization of an umbilical surface in $E^3$

We give an elementary geometric criterion for umbilicity, which is much better than (*) in §1. The following criterion requires the existence of two circles through each point, whereas (*) requires the existence of infinitely many circles through each point.

**Theorem 1.** Let $M$ be a surface in $E^3$. Suppose that, through each point $p \in M$, there exist two circles of $E^3$ such that

(i) they are contained in $M$ in a neighborhood of $p$,

(ii) they are tangent to each other at $p$.

Then $M$ is locally a plane or a sphere.

**Proof.** Let $p$ be an arbitrary point of $M$ and let $\gamma_1$ and $\gamma_2$ be two circles through $p$ which satisfy the conditions (i) and (ii).

Let $X_p$ be the common unit tangent vector of $\gamma_1$ and $\gamma_2$ at $p$. Then $X: p \rightarrow X_p$.
defines a vector field on \( M \), which may not be continuous. Let \( Y \) be a unit vector field on \( M \) which, together with \( X \), forms a right-handed orthonormal system. Then, since \( \dim M = 2 \), it follows from Lemma that there exist \( c_1 \neq c_2 \) such that

\[
\begin{align*}
\langle (F'_x \sigma)(X, X) + 3\sigma(X, c_1 Y) \rangle &= 0 \\
\langle (F'_x \sigma)(X, X) + 3\sigma(X, c_2 Y) \rangle &= 0
\end{align*}
\]

at each point. Therefore we have

\[
(4.2) \quad \sigma(X, Y) = 0,
\]

that is,

\[
\langle AX, Y \rangle = 0.
\]

Since \( \dim M = 2 \), we see that \( AX \) is parallel to \( X \). Thus \( X \) is (and hence \( Y \) is also) a principal vector at each point. Let \( \lambda \) and \( \mu \) be principal curvatures and \( \xi_1 \) and \( \xi_2 \) the corresponding principal unit vectors so that \( A\xi_1 = \lambda \xi_1 \) and \( A\xi_2 = \mu \xi_2 \). Put \( M_0 = \{ p \in M | \lambda(p) \neq \mu(p) \} \). If \( M_0 = \emptyset \), then \( M \) is totally umbilic. Therefore we suppose that \( M_0 \neq \emptyset \). Then \( \xi_1 \) and \( \xi_2 \) are \( C^\infty \) vector fields in some neighborhood of each point of \( M_0 \). Hence we may put

\[
\begin{align*}
F'_{\xi_1} \xi_1 &= \alpha \xi_2 \\
F'_{\xi_2} \xi_1 &= \beta \xi_2
\end{align*}
\]

so that

\[
\begin{align*}
F'_{\xi_1} \xi_2 &= -\alpha \xi_1 \\
F'_{\xi_2} \xi_2 &= -\beta \xi_1.
\end{align*}
\]

Put \( M_{0i} = \{ p \in M | X(p) = \xi_i(p) \} \) \((i = 1, 2)\). Then \( M_0 = M_{01} \cup M_{02} \), and it is easily seen that \( M_0 \subset M_{01} \cup \text{Int} M_{02} \) and \( M_0 \subset M_{02} \cup \text{Int} M_{01} \) and hence that \( M_{01} \) or \( M_{02} \) has interior points, or \( M_{01} \) or \( M_{02} \) is dense in \( M_0 \). We may assume without loss of generality that \( M_{01} \) has interior points or \( M_{01} \) is dense in \( M_0 \). Hence it is sufficient to consider the case where \( p \in M_{01} \).

Using \( (4.2) \) we obtain

\[
\begin{align*}
(F'_{\xi_1} \sigma)(\xi_1, \xi_1) &= F'_{\xi_1} \cdot \sigma(\xi_1, \xi_1) - 2\sigma(\xi_1, \xi_2) \\
&= F'_{\xi_1} (\lambda e_3) - \sigma(\xi_1, a \xi_2) \\
&= (F'_{\xi_1} \lambda) e_3,
\end{align*}
\]

where \( e_3 \) is a local field of unit normals of \( M \).

On the other hand, it follows from \( (4.1) \) and \( (4.2) \) that

\[
(4.3) \quad (F'_{X} \sigma)(X, X) = 0.
\]

Therefore we have \( F'_{X} \lambda = 0 \) on \( M_{01} \). If \( p \) is an interior point of \( M_{01} \), then

\[
(4.4) \quad F'_{\xi_1} \lambda = 0
\]

holds in some neighborhood of \( p \). If \( M_{01} \) is dense in \( M_0 \), then, by continuity, \( (4.4) \) holds on \( M_0 \).
We choose an orthonormal frame field $e_1, e_2$ in a sufficiently small tubular neighborhood of $\gamma_1$ in such a way that

$$e_1 = \dot{\gamma}_1, \text{ along } \gamma_1,$$

and put

$$\mathcal{R}_{e_1} e_1 = ae_2 \text{ and } \mathcal{R}_{e_2} e_1 = be_2$$

so that $\mathcal{R}_{e_1} e_2 = -ae_1$ and $\mathcal{R}_{e_2} e_2 = -be_2$.

Let $(h_{ij})$ be the matrix of $A$ with respect to $e_1$ and $e_2$. Then it follows from Lemma that, along $\gamma_1$,

(4.5) \hspace{1cm} k^2 = a^2 + h_{11}^2

(4.6) \hspace{1cm} \mathcal{R}_{e_1} a = h_{11}/h_{12}

(4.7) \hspace{1cm} ah_{12} + \mathcal{R}_{e_1} h_{11} = 0,$$

where $k$ is the curvature of $\gamma_1$ as a circle in $E^3$.

Letting $\theta$ be the angle between $\xi_1$ and $e_1$ so that

(4.8) \hspace{1cm} \begin{cases} e_1 = \xi_1 \cos \theta + \xi_2 \sin \theta \\ e_2 = -\xi_2 \sin \theta + \xi_2 \cos \theta \end{cases}

we get

(4.9) \hspace{1cm} \begin{cases} h_{11} = \lambda \cos^2 \theta + \mu \sin^2 \theta \\ h_{12} = -\lambda \cos \theta \sin \theta \\ h_{22} = \lambda \sin^2 \theta + \mu \cos^2 \theta \end{cases}

By differentiating (4.8), we obtain

(4.10) \hspace{1cm} a = a \cos \theta + \beta \sin \theta + \mathcal{R}_{e_1} \theta,

which is nothing but the transformation lar for Christoffel's symbols.

Moreover, by the equation of Codazzi \(\langle F_{\xi_1} A \rangle \xi_2 - \langle F_{\xi_2} A \rangle \xi_1 = 0\), we get

(4.11) \hspace{1cm} \begin{cases} \alpha(\lambda - \mu) = F_{\xi_2} \lambda \\ \beta(\lambda - \mu) = F_{\xi_2} \mu \end{cases}

Therefore, by (4.4), (4.7), (4.8), (4.9), (4.10) and (4.11) we obtain

(4.12) \hspace{1cm} [3(\lambda - \mu) \cos \theta \cdot \mathcal{R}_{e_1} \theta - \sin^2 \theta \cdot F_{\xi_2} \mu] \sin \theta = 0.

Note that $\theta = 0$ at $p$. The point $p$ under consideration has one of the following properties:

(A) There exists no sequence \(\{p_n \in \gamma_1 | \theta(p_n) \neq 0\}\) with $p = \lim p_n$

(B) There exists a sequence \(\{p_n \in \gamma_1 | \theta(p_n) = 0\}\) with $p = \lim p_n$.

If $p$ is a point of kind (A), then it is clear that the integral curve of $\xi_1$ through
\( p \) coincides with \( y \), on the connected component of \( \{ q \in \gamma_j | \dot{\theta}(q) = 0 \} \) containing \( p \).

If \( p \) is a point of kind (B), it follows from (4.12) that
\[
3(\lambda - \mu) \cos \theta \cdot R_{e_1} \theta - \sin \theta \cdot R_{\xi_2} \mu = 0
\]
holds on \( \{ p_n \in \gamma_j | \theta(p_n) \neq 0 \} \).

Taking the limit of (4.13), we obtain
\[
(4.14) \quad R_{e_1} \theta = 0 \quad \text{at} \quad p,
\]
since \( \lambda \neq \mu \).

After applying \( R_{e_1} \) to (4.12) and then multiplying by \( \sin \theta \), we evaluate it at \( p \) to obtain
\[
(4.15) \quad R_{e_1} R_{e_1} \theta = 0 \quad \text{at} \quad p.
\]
It is clear that (4.14) and (4.15) hold even if \( p \) is a point of kind (A).

It follows from (4.10) and (4.14) that
\[
a(p) = a(p).
\]
This, together with (4.5) and (4.9), yields
\[
(4.11) \quad k^2 = (\alpha(p))^2 + (\lambda(p))^2,
\]
which implies that the curvature of \( \gamma \) is \( \sqrt{(\alpha(p))^2 + (\lambda(p))^2} \).

Applying \( R_{e_1} \) to (4.10) and evaluating at \( p \), we obtain
\[
R_{\xi_1} a = R_{\xi_1} a \quad \text{at} \quad p
\]
because of (4.14) and (4.15).

On the other hand, from (4.6) we get
\[
R_{\xi_1} a = R_{e_1} a = h_1, h_1 = 0 \quad \text{at} \quad p
\]
Therefore we have
\[
R_{\xi_1} a = 0 \quad \text{at} \quad p.
\]
Since \( p \) is arbitrary, we get \( R_{\xi_1} a = 0 \) on \( M_0 \).

If \( p \) is an interior point of \( M_0 \), then
\[
(4.17) \quad R_{\xi_1} a = 0
\]
holds in some neighborhood of \( p \). If \( M_0 \) is dense in \( M_0 \), then, by continuity, (4.17) holds on \( M_0 \).

Eliminating \( \xi_2 \) from \( R_{\xi_1} a = \alpha \xi_2 \) and \( R_{\xi_1} \xi_2 = -\alpha \xi_1 \), we obtain
\[
R_{\xi_1} R_{\xi_1} \xi_1 + \alpha^2 \xi_1 = 0.
\]
Moreover, since \( \xi_1 \) is a principal vector, we get
Therefore we have
\begin{equation}
F_{\xi_1} \cdot F_{\xi_1} \cdot \xi_1 + (\alpha^2 + \lambda^2) \xi_1 - A_{\xi_1, \xi_1} \xi_1 = 0. 
\end{equation}
Furthermore, since $F_{\xi_1} \cdot \xi_1 = \alpha \xi_2$, it follows from (4.2) and (4.3) that $\langle F_{\xi_1} \cdot \xi_1, \xi_1 \rangle \rangle + 3\sigma(\xi_1, F_{\xi_1} \cdot \xi_1) = 0$ on $M_0$.
\begin{equation}
\langle F_{\xi_1} \cdot \xi_1, \xi_1 \rangle \rangle + 3\sigma(\xi_1, F_{\xi_1} \cdot \xi_1) = 0 
\end{equation}
holds in some neighborhood of $p$. If $M_0$ is dense in $M$, then, by continuity, (4.19) holds on $M_0$.
By (4.18), (4.19) and Lemma, we see that the integral curve of $\xi_1$ through $p$ is a circle of curvature $\sqrt{\langle \alpha(p) \rangle^2 + \langle \lambda(p) \rangle^2}$ in $E^3$.

Since we can apply the same argument to $\gamma_2$, letting $\theta_i (i=1,2)$ be the angle between $\xi_i$ and $\gamma_i$, we consider the following cases:

(A)$_i$: There exists no sequence $\{p_n \in \gamma_i : \theta_i(p_n) = 0\}$ with $p = \lim p_n$
(B)$_i$: There exists a sequence $\{p_n \in \gamma_i : \theta_i(p_n) = 0\}$ with $p = \lim p_n$

It is clear that the case (A)$_1$ and (A)$_2$ does not occur. If (A)$_1$ and (B)$_2$, then $\gamma_1$ and $\gamma_2$ have the same curvature and the integral curve of $\xi_1$ through $p$ coincides with $\gamma_1$. If (B)$_1$ and (A)$_2$, then $\gamma_1$ and $\gamma_2$ have the same curvature and the integral curve of $\xi_1$ through $p$ coincides with $\gamma_2$. If (B)$_1$ and (B)$_2$, then $\gamma_1$, $\gamma_2$ and the integral curve of $\xi_1$ through $p$ have the same curvature and hence, the integral curve of $\xi_1$ through $p$ coincides with $\gamma_1$ or $\gamma_2$, since $\dim M = 2$. This is a contradiction so that the last case does not occur.

Since we can apply the same argument to the possible two cases, we suppose that the (B)$_1$ and (A)$_2$ is the case, that is, the integral curve of $\xi_1$ through $p$ coincides with $\gamma_2$. Then, from
\begin{equation}
\langle F_{\xi_1} \cdot \xi_1, F_{\xi_1} \cdot e_1 \rangle = \langle F_{\xi_1} \cdot \xi_1 + \sigma(\xi_1, \xi_1), F_{e_1} \cdot e_1 + 3\sigma(\xi_1, \xi_1) \rangle \rangle = \alpha \xi_2 + \lambda e_2 + h_1 e_3,
\end{equation}
we get
\begin{equation}
\langle F_{\xi_1} \cdot \xi_1, F_{\xi_1} \cdot e_1 \rangle = \lambda^2 + \alpha^2 \text{ at } p,
\end{equation}
since $\alpha = \alpha$, $\lambda = h_1$, and $\xi_2 = e_2$ at $p$.

Let $\gamma_1 \gamma_2$ denote the angle between $\gamma_1$, plane and $\gamma_2$-plane. Then we have
\begin{equation}
\cos \gamma_1 \gamma_2 = \frac{\langle F_{\xi_1} \cdot \xi_1, F_{\xi_1} \cdot e_1 \rangle}{||F_{\xi_1} \cdot \xi_1|| ||F_{\xi_1} \cdot e_1||} = 1.
\end{equation}
Thus $\gamma_1$-plane and $\gamma_2$-plane coincide along $\gamma_2$. This implies that the set of $\gamma_i$'s along $\gamma_2$ near $p$ forms a part of a plane, which contradicts the non-umbilicity of $M_0$. 

Therefore $M_6 = \emptyset$ so that $M$ is totally umbilic.

(Q.E.D.)

Remark. The assumption (i) in Theorem 1 is not sufficient for a surface to be totally umbilic, because an ordinary torus and a suitable part of an ellipsoid or an elliptic paraboloid satisfy the assumption (i). Moreover, it is not difficult to see that a torus actually contains four circles of $E^3$ through each point ([1]). On the contrary, it is easily seen that a surface is a plane if it contains three straight lines through each point (cf. §7). It seems to be natural to conjecture that a simply connected complete surface in $E^3$ which contains two circles through each point must be totally umbilic. This conjecture is hairbreadth, because an ellipsoid and an elliptic paraboloid actually satisfy the assumption except at only four and two points respectively.

5. An extrinsic sphere in a general Riemannian manifold

Let $M$ be an $n$-dimensional submanifold in $\tilde{M}$. If $n = 2$, then $(\partial_{i_1}^i \hat{t}_i)_p$ and $(\partial_{i_2}^i \hat{t}_i)_p$ are linearly dependent for two circles $\hat{t}_1$ and $\hat{t}_2$ in $\tilde{M}$ which are tangent to each other at $p \in M$ and contained in $M$. On the contrary, the situation is not so simple if $n > 2$. Therefore we introduce the following notion.

Circles $\hat{t}_1, \ldots, \hat{t}_k$ in $\tilde{M}$ which are tangent to one another at $p \in M$ and contained in $M$ in a neighborhood of $p$ are said to be $r$-independent if $\dim(\langle \partial_{i_1}^i \hat{t}_i \rangle_p, \ldots, \langle \partial_{i_r}^i \hat{t}_i \rangle_p) = r$ and there exist $c_1, \ldots, c_k$ with $\sum c_i \neq 0$ such that $\sum c_i \langle \partial_{i_1}^i \hat{t}_i \rangle_p = 0$, where $\langle \cdot \rangle$ denotes the linear space spanned by $\cdot$.

We shall give an elementary geometric characterization of an extrinsic sphere, that is, we shall prove the following.

Theorem 2. Let $M$ be an $n$-dimensional submanifold immersed in a Riemannian manifold $\tilde{M}$. Then $M$ is either a totally geodesic submanifold or an extrinsic sphere if, through each point $p \in M$, there exist $n^2$ circles of $\tilde{M}$ such that

(i) they are contained in $M$ in a neighborhood of $p$,
(ii) they are mutually tangent $n$ by $n$ and respective $n$ circles are $(n-1)$-independent,
(iii) none of them are orthogonal to each other at $p$,
(iv) the set of all tangent vectors to them at $p$ spans $T_p(M)$.

Proof. Let $\gamma_{ij}(1 \leq i, j \leq n)$ be $n^2$ circles satisfying the assumption and suppose that $\gamma_{i_1}, \ldots, \gamma_{i_n}$ are tangent to one another at $p$. We denote by $X_i$ the common unit tangent vector of $\gamma_{i_1}, \ldots, \gamma_{i_n}$ at $p$. Then by (i) and Lemma we have

\[
\begin{cases}
(P_{X_i^i} \sigma)(X_i, X_i) + 3\sigma(X_i, (P_{\hat{t}_{i_1}^1} \hat{t}_{i_1})) = 0 \\
\cdots \\
(P_{X_i^i} \sigma)(X_i, X_i) + 3\sigma(X_i, (P_{\hat{t}_{i_1}^1} \hat{t}_{i_1})) = 0
\end{cases}
\]

(5.1)
A Submanifold which Contains many

for $1 \leq i \leq n$.

On the other hand, since $\tilde{\gamma}_i, \ldots, \tilde{\gamma}_n$ are $(n-1)$-independent by (ii), we see that $\{X_i\} = \{(F_{\tilde{\gamma}_i})_p, \ldots, (F_{\tilde{\gamma}_n})_p\}$. Thus, from (5.1) we obtain

(5.2) \[(F_{\tilde{\gamma}_i})_p(X_i, X_i) = 0\]

and

(5.3) \[\sigma(X_i, \{X_i\}) = 0\].

It follows from (5.3) that

\[<A_{\xi}X_i, \{X_i\} > = 0\]

for an arbitrary normal vector $\xi$ at $p$.

This implies that $X_i$ is a principal vector with respect to $\xi$.

Therefore we see from (iii) and (iv) that $A_\xi$ is proportional to the identity transformation for an arbitrary normal vector $\xi$ at $p$, which means that $p$ is an umbilic point.

Since $p$ is arbitrary, $M$ is totally umbilic.

Moreover, (5.2), together with the umbilicity of $M$, allows the identity

\[F_{\tilde{\gamma}_i} = (F_{\tilde{\gamma}_i})_p = (F_{\tilde{\gamma}_i})_p + 2\sigma(\tilde{\gamma}_i, F_{\tilde{\gamma}_i})_{\tilde{\gamma}_i}\]

to boil down to

\[F_{\tilde{\gamma}_i} = 0\].

Thus we get $F_{\hat{\gamma}} = 0$ at $p$ by (iv), and hence we have

\[F_{\hat{\gamma}} = 0 \text{ on } M\],

since $p$ is arbitrary.

Thus $M$ is totally umbilic with parallel mean curvature vector. More precisely, $M$ is a totally geodesic submanifold or an extrinsic sphere according as $\hat{h} = 0$ or $\hat{h} \neq 0$.

(Q.E.D.)

The following result gives another characterization of an extrinsic sphere.

**Theorem 3.** Let $M$ be an $n$-dimensional submanifold immersed in a Riemannian manifold $\tilde{M}$. Then $M$ is either a totally geodesic submanifold or an extrinsic sphere if, at each point $p$ of $M$, there exist an orthonormal basis $e_i, \ldots, e_n$ of $T_p(M)$ and real numbers $a_1, \ldots, a_n (0 < a_j < \pi/2)$ with the following properties: For each pair $(X, Y) = (e_i, e_j)$ and $(X, Y) = (e_i, \cos a_j + e_j \sin a_j, e_i \sin a_j - e_j \cos a_j), 1 \leq i < j \leq n$, there exist two circles $\gamma_1$ and $\gamma_2$ of $\tilde{M}$ such that

(i) $\gamma_1(0) = \gamma_2(0) = p$
(ii) \( \dot{r}_1(0) = \dot{r}_3(0) = X \)

(iii) \( (F_{F_1} r_{3})_p = c_1 Y \) and \( (F_{F_2} r_{3})_p = c_2 Y \) for some \( c_1 \neq c_2 \) (i.e., \( \dot{r}_1 \) and \( \dot{r}_3 \) are 1-independent)

(iv) \( \dot{r}_1 \) and \( \dot{r}_3 \) are contained in \( M \) in a neighborhood of \( p \).

**Proof** By (iv) and Lemma, we have

\[
\begin{align*}
(F'_k a)(X, X) + 3a(X, c_1 Y) &= 0 \\
(F'_k a)(X, X) + 3a(X, c_2 Y) &= 0.
\end{align*}
\]

Therefore

\[
\sigma(X, Y) = 0
\]

holds for all pairs \( (X, Y) = (e_i, e_j) \) and \( (e_i \cos \alpha_j + e_j \sin \alpha_j, e_i \sin \alpha_j - e_j \cos \alpha_j), 1 \leq i < j \leq n \).

Thus we obtain

\[
\sigma(e_i, e_j) = 0 \quad (1 \leq i < j \leq n)
\]

\[
\sigma(e_i, e_1) = \cdots = \sigma(e_n, e_n),
\]

which implies that \( p \) is an umbilic point.

Since \( p \) is arbitrary, \( M \) is totally umbilic.

Moreover, by (5.4) and (5.5) we get

\[
(F'_k a)(X, X) = 0
\]

for \( X = e_1, \cdots, e_{n-1} \) and \( e_1 \cos \alpha_n + e_n \sin \alpha_n \).

Applying the same argument as in the proof of Theorem 2, we obtain

\[
F: \mathbb{H} = 0
\]

so that we can complete the proof.

**Corollary 1.** Let \( M \) be an \( n \)-dimensional submanifold of \( E^m \). If \( M \) satisfies the assumption of Theorem 2 or Theorem 3, then \( M \) is locally \( E^n \) or \( S^n \).

**Corollary 2.** Let \( M \) be a surface in \( E^m \). Suppose that, through each point of \( M \), there exist four circles of \( E^m \) such that

(a) they are contained in \( M \) in a neighborhood of \( p \),

(b) they are tangent two by two at \( p \),

(c) none of them are orthogonal to each other at \( p \).

Then \( M \) is locally a plane or a sphere.

6. A submanifold with parallel second fundamental form

We shall give an elementary geometric characterization for a submanifold with parallel second fundamental form. We first prove the following general
A Submanifold which Contains many

result.

Proposition. Let \( M \) be a submanifold of \( \tilde{M} \). Suppose, through each point of \( M \) and in each direction tangent to \( M \), there exist \( r \) (which may depend on the point and the direction) circles of \( \tilde{M} \) such that
(a) they are contained in \( M \) in a neighborhood of the point,
(b) they are \((r-1)\)-independent.
Then \( (P_\sigma)(X,X)=0 \) holds for all \( X \) tangent to \( M \).

Proof Let \( p \) be an arbitrary point of \( M \) and \( X \in T_p(M) \) be an arbitrary unit vector. Then, by assumption, there exist \( r \) circles \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_r \) of \( M \) which satisfy \( \tilde{\gamma}_i(0)=p, \tilde{\gamma}_i(0)=X \) and (a) and (b). Thus, by Lemma, we have

\[
\begin{align*}
(P_\sigma(X,X) + 3a(X,(P_\tilde{\gamma}_1)_p=0 \\
\quad \cdots \cdots \\
(P_\sigma(X,X) + 3a(X,(P_\tilde{\gamma}_r)_p=0.
\end{align*}
\]

Therefore, by (b) we get \( (P_\sigma)(X,X)=0 \). (Q.E.D.)

It is easily seen that if the equation of Codazzi reduces to \( (P_\sigma)(Y,Z) - (P_\tilde{\gamma})(X,Z)=0 \), then we can deduce \( P_\sigma=0 \) from \( (P_\sigma)(X,X)=0 \). For example, we have the following.

Corollary. Let \( M \) be a submanifold of a space form. If \( M \) satisfies the assumption of Proposition, then the second fundamental form of \( M \) is parallel.

It is clear that the same result holds for a Kaehler or totally real submanifold of a complex space form (i.e., a Kaehler manifold of constant holomorphic curvature).

7. A submanifold which contains many extrinsic geodesics

An extrinsic circle is not necessarily an intrinsic circle, whereas an extrinsic geodesic is an intrinsic geodesic. Therefore it is natural to expect that a submanifold with extrinsic geodesics is simpler than a submanifold with extrinsic circles. The following result gives a sufficient condition for a submanifold to be totally geodesic.

Theorem 4. Let \( M \) be an \( n \)-dimensional submanifold immersed in a Riemannian manifold \( \tilde{M} \). Then a point \( p \in M \) is a geodesic point if there exist \( n(n+1)/2 \) geodesics \( \gamma_{ij}(1 \leq i \leq j \leq n) \) of \( \tilde{M} \) such that

\( i \) \( \gamma_{ij}(0)=p \)
(ii) \( T_p(M) = \{ (\dot{r}_{1i}(0), \ldots, \dot{r}_{ni}(0)) \} \)

(iii) \( \dot{r}_{ij}(0) \in [(\dot{r}_{iu}(0), \dot{r}_{jj}(0))] \)

(iv) \( \dot{r}_{ij} \) are contained in \( M \) in a neighborhood of \( p \)

**Proof.** By (iv) and the equation of Gauss, we have

\[
\sigma(\dot{r}_{ij}, \dot{r}_{ij}) = \tau_{ij}^2 - \tau_{ij} = 0 \quad (1 \leq i \leq j \leq n)
\]

in a neighborhood of \( p \).

Therefore it follows from (iii) that

\[
\sigma(\dot{r}_{iu}(0), \dot{r}_{iu}(0)) = 0 \quad (1 \leq i \leq n)
\]

\[
\sigma(a_i \dot{r}_{iu}(0) + b_j \dot{r}_{jj}(0), a_i \dot{r}_{iu}(0) + b_j \dot{r}_{jj}(0)) = 0 \quad (1 \leq i < j \leq n)
\]

for some \( a_i \) and \( b_j \).

Hence we have

\[
\sigma(\dot{r}_{iu}(0), \dot{r}_{iu}(0)) = 0 \quad (1 \leq i \leq j \leq n),
\]

which, together with (ii), implies \( \sigma = 0 \) at \( p \).

**Corollary.** Let \( M \) be a surface in a Riemannian manifold \( \tilde{M} \). If, through each point \( p \) of \( M \), there exist three geodesics of \( \tilde{M} \) which are contained in \( M \) in a neighborhood of then \( M \) is totally geodesic.

**Remark.** The condition (iii) is automatically satisfied when \( \dim M = 2 \), from which Corollary follows.

On the contrary, in the case \( \dim M > 2 \), \( p \) is not necessarily a geodesic point even if there exist infinitely many geodesics of \( \tilde{M} \) which satisfy (i), (ii) and (iv).

**References**


Department of Mathematics
Tokyo Metropolitan University
and
Department of Mathematics
University of Tsukuba