HOMOGENEOUS LORENTZ MANIFOLDS WITH ISOTROPY SUBGROUP $U(2)$ OR $SO(2)$

Dedicated to Professor Tsunero Takahasi on his 60th birthday

By

Hiroo Matsuda

1. Introduction.

Let $(M, \langle , \rangle)$ be an $n$-dimensional connected Lorentz manifold with metric $\langle , \rangle$ of signature $(-, +, \cdots, +)$. In this note, we assume that an isometry group has compact isotropy subgroup at every point in $M$.

In [8], we showed that, if $n \geq 6$, there is no $r$-dimensional isometry group for $(n-1)(n-2)/2 + 3 \leq r \leq n(n-1)/2 - 1$, and we determined simply connected $n$-dimensional Lorentz manifolds admitting an isometry of dimension $(n-1)(n-2)/2 + 2$ for $n \geq 6$. However, there exists a 5-dimensional Lorentz manifold admitting a 9 $(=(5-1)(5-2)/2 + 3 = 5(5-1)/2 - 1)$-dimensional isometry group (see Remark 1.3 in [8]). In §3, we will determine simply connected 5-dimensional Lorentz manifolds admitting an isometry group of dimension 9. That is, we have the following Theorem A.

**THEOREM A.** Let $(M, \langle , \rangle)$ be a simply connected 5-dimensional Lorentz manifold admitting a connected 9-dimensional isometry group $G$ with compact isotropy subgroup at every point in $M$. Then $(M, \langle , \rangle)$ is one of the following:

1. $(M, \langle , \rangle)$ is isometric to $(\mathbb{R} \times M_2, -dt^2 + ds^2)$ where $(M_2, ds^2)$ is a 2-dimensional simply connected complex space form;

2. $(M, \langle , \rangle)$ is isometric to a simply connected 5-dimensional Lie group with a left-invariant Lorentz metric $\langle , \rangle$ and $G$ is isomorphic to a semi-direct product $U(2) \times M$;

3. $(M, \langle , \rangle)$ is a principal fibre bundle, with a 1-dimensional structure group, over a 2-dimensional simply connected complex space form.

In [6], [7], we determined $n$-dimensional Lorentz manifolds admitting an isometry group of dimension $n(n-1)/2 + 1$ (for $n \geq 4$). In §4, we will determine simply connected $n$-dimensional Lorentz manifolds admitting a connected isometry group of dimension $n(n-1)/2 + 1$ for $n=3$. This is, we have the follow-
Theorem B. Let \((M, <, >)\) be a simply connected 3-dimensional Lorentz manifold admitting a connected 4-dimensional isometry group \(G\) with compact isotropy subgroup at every point in \(M\). Then \((M, <, >)\) is one of the following:

1. \((M, <, >)\) is isometric to \((\mathbb{R} \times M, -dt^2 + ds^2)\) where \((M, ds^2)\) is a simply connected 2-dimensional Riemannian space form;

2. \((M, <, >)\) is isometric to a simply connected 3-dimensional Lie group with a left-invariant Lorentz metric and \(G\) is isomorphic to a semi-direct product \(SO(2) \ltimes M\);

3. \((M, <, >)\) is a principal fibre bundle, with a 1-dimensional structure group, over a simply connected 2-dimensional Riemannian space form.

In [8], we determined simply connected \(n\)-dimensional Lorentz manifolds \(M\) admitting an isometry group of dimension \((n-1)(n-2)/2+2\) for \(n \geq 6\). In § 4, we will determine simply connected \(n\)-dimensional Lorentz manifolds \(M\) admitting an isometry group of dimension \((n-1)(n-2)/2+2\) for \(n = 4\). That is, we will show the following Theorem C.

Theorem C. Let \((M, <, >)\) be a simply connected 4-dimensional Lorentz manifold admitting a connected 5-dimensional isometry group \(G\) with compact isotropy subgroup at every point in \(M\). Then \((M, <, >)\) is one of the following:

1. \((M, <, >)\) is isometric to \((M_1 \times M_2, ds_1^2 + ds_2^2)\) where \((M_1, ds_1^2)\) is a simply connected 2-dimensional Lie group with a left-invariant Lorentz metric \(ds_1^2\) and \((M_2, ds_2^2)\) is a simply connected 2-dimensional Riemannian space form;

2. \((M, <, >)\) is isometric to a simply connected 4-dimensional Lie group with a left-invariant Lorentz metric and \(G\) is a semi-direct product \(SO(2) \ltimes M\);

3. \((M, <, >)\) is a principal fibre bundle, with a 2-dimensional abelian structure group, over a simply connected 2-dimensional Riemannian space form.

The author would like to express his thanks to the referee for his kind advice.

2. Preliminaries.

Let \((M, <, >)\) be a connected Lorentz manifold with metric \(<, >\) of signature \((- , + , \cdots , + )\) and let \(G\) be a connected isometry group acting on \(M\) such that the isotropy subgroup \(H\) at \(p \in M\) is compact. Then the linear isotropy subgroup \(\tilde{H} = \{ dh; h \in H \}\) is a closed subgroup of \(O(1, n-1) = \{ A \in GL(n, \mathbb{R}); ^tASA = S \}\) where \(S\) is the matrix
Homogeneous Lorentz manifolds

\[ \left( \begin{array}{cc} -1 & 0 \\ 0 & I_{n-1} \end{array} \right) \]

\((I_{n-1} \text{ is the unit matrix of degree } n-1)\). So \(\tilde{H}\) is conjugate to a closed subgroup of \(O(1) \times O(n-1)\).

**Proposition 2.1.** Let \((M, \langle , \rangle)\) be a simply connected 5-dimensional Lorentz manifold admitting a 9-dimensional isometry group \(G\) with compact isotropy subgroup at every point in \(M\). Then \(G\) acts on \(M\) transitively and the linear isotropy subgroup is conjugate to \(1 \times U(2)\).

**Proof.** Suppose that \(G\) does not act on \(M\) transitively. Then \(\dim G(o) \leq 4(o \in M)\). Hence the dimension of the isotropy subgroup \(H\) at \(o\) is not less than 5. On the other hand, since \(H\) is compact, the linear isotropy subgroup is isomorphic to a subgroup of \(O(1) \times O(4)\), so \(\dim H \leq 4(4-1)/2 = 6\). Thus \(5 \leq \dim H \leq 6\). Then we have \(\dim H = 6\) (c.f., [2], [9]), so that we have \(\dim G(o) = 3\), which contradicts Lemma 1.2 in [8]. Therefore \(G\) is transitive on \(M\).

Since \(M\) is simply connected, \(H\) is connected and the linear isotropy subgroup \(\tilde{H}\) is isomorphic to a subgroup of \(1 \times SO(4)\). Since \(\dim H = \dim G - \dim M = 4\), \(\tilde{H}\) is conjugate to \(1 \times U(2)\) (c.f., [9]).

By the same way as the proof of Proposition 2.1, we have

**Proposition 2.2.** Let \((M, \langle , \rangle)\) be a simply connected 4(resp. 3)-dimensional Lorentz manifold admitting a 5(resp. 4)-dimensional isometry group with compact isotropy subgroup at every point in \(M\). Then \(G\) acts on \(M\) transitively and the linear isotropy subgroup is conjugate to \(I_3 \times SO(2)\) (resp. \(1 \times SO(2)\)).

In view of Propositions 2.1 and 2.2, we consider homogeneous Lorentz manifolds \(G/H = M\) (\(H\) is the isotropy subgroup of \(G\) at some point \(o \in M\)). We denote Lie algebras of \(G\) and \(H\) by \(\mathfrak{g}\) and \(\mathfrak{h}\) respectively. Since \(H\) is compact, there exists a subspace \(m\) of \(\mathfrak{g}\) such that

\[ \mathfrak{g} = \mathfrak{h} \oplus m, \quad [\mathfrak{h}, m] \subseteq m. \]

Let \(\pi: G \to G/H = M\) be the natural projection. We identify the tangent space \(T_oM\) and \(m\) by \(d\pi\). The Lorentz inner product on \(T_oM\) induces the Lorentz inner product \(\langle , \rangle_m\) on \(m\) so that \(d\pi: m \to T_oM\) is a linear isometry.
3. Proof of Theorem A.

Let $(M, \langle , \rangle)$ be a simply connected 5-dimensional Lorentz manifold admitting a connected isometry group $G$ of dimension 9. By the Proposition 2.1, $M$ is a simply connected homogeneous Lorentz manifold $G/H$ and the linear isotropy subgroup is conjugate to $1 \times U(2)$. Then $Ad(H)$ acts on $m$ as $1 \times U(2)$, so there exists a 1-dimensional subspace $m_1$ and a 4-dimensional subspace $m_2$ of $m$ such that

$$m = m_1 \oplus m_2$$

and $Ad(H) = id.$ on $m_1$ (so, $[\mathfrak{h}, m_1] = \{0\}$), $Ad(H) = U(2)$ on $m_2$. Since $U(2)$ acts on $m_2$ irreducibly and contains $-I_2$, we have Lemma 3.1 by using Schur's Lemma.

**Lemma 3.1.** $m_2$ is spacelike and $m_1$ is perpendicular to $m_2$ (so, $m_1$ is timelike).

Let $p_0$, $p_1$ and $p_2$ be orthogonal projections from $\mathfrak{g}$ to $\mathfrak{h}$, $m_1$ and $m_2$ respectively. Since $\mathfrak{h}$, $m_1$ and $m_2$ are $Ad(H)$-invariant, we see
\begin{equation}
    p_i Ad(h) = Ad(h) p_i \quad (i=0, 1, 2)
\end{equation}
for any $h \in H$. Since there exists $E \in \mathfrak{h}$ such that

$$Ad(\exp tE) = \begin{pmatrix}
    \cos tl_2 & -\sin tl_2 \\
    \sin tl_2 & \cos tl_2
\end{pmatrix}$$

on $m_2$, we have
\begin{equation}
    [E, X] = JX
\end{equation}
for any $X \in m_2$, where $J$ is an almost complex structure on $m_2$.

**Remark 3.2.** $E$ belongs to the center of $\mathfrak{h}$.

**Lemma 3.3.** $[m_1, m_2] \subseteq m_2$. More precisely, there exist linear maps $L_1, L_2 : m_1 \to \mathbb{R}$ such that
\begin{equation}
    [A, X] = L_1(A)X + L_2(A)JX
\end{equation}
for any $A \in m_1$ and any $X \in m_2$.

**Proof.** For any fixed $A \in m_1$, we define a linear map $f_A : m_2 \to \mathfrak{g}$ by $f_A(X) = [A, X](X \in m_2)$. Since $Ad(H) = id.$ on $m_1$, we have
\begin{equation}
    f_A Ad(h) = Ad(h) f_A
\end{equation}
for any $h \in H$. By (3.1) and (3.3), we have

\[(p_i f_A) Ad(h) = Ad(h) (p_i f_A) \quad (i = 0, 1, 2)\]

for any $h \in H$.

**Step 1.** We claim $p_i [m_1, m_2] = \{0\}$. Since $\ker (p_i f_A)$ is $Ad(H)$-invariant by (3.4) and $Ad(H)$ acts on $m_2$ irreducibly, we have $\ker (p_i f_A) = \{0\}$ or $m_2$. On the other hand, there exist a non-zero $X \in m_2$ and $h \in H$ such that $Ad(h) X - X \neq 0$. We have $p_i f_A (Ad(h) X - X) = 0$, which implies that $Ad(h) X - X \in \ker (p_i f_A)$. Therefore we have $\ker (p_i f_A) = m_2$, that is, $p_i [A, m_2] = \{0\}$. Since $A$ is arbitrary, we have $p_i [m_1, m_2] = \{0\}$.

**Step 2.** We claim $p_0 [m_1, m_2] = \{0\}$. By the same procedure as step 1, we have $\ker (p_0 f_A) = \{0\}$ or $m_2$. Assume $\ker (p_0 f_A) = \{0\}$. Since $p_0 f_A : m_2 \to \frak{h}$ is injective, we have $\dim (p_0 f_A (m_2)) = 4 = \dim \frak{h}$, so we have $[A, m_2] = \frak{h}$. On the other hand, $[A, m_2]$ is spanned by $[A, X]$'s ($X \in m_2$) and we have

\[
[A, X] = [A, -J^2 X] = -[A, [E, JX]]
\]

because $E$ belongs to the center of $\frak{h}$. Thus we have $[A, m_2] = \{0\}$, which is a contradiction. Therefore, we have $p_0 [A, m_2] = \{0\}$. Since $A$ is arbitrary, we have $p_0 [m_1, m_2] = \{0\}$.

**Step 3.** $f_A$ is a linear map from $m_2$ to $m_2$ by step 1 and step 2, and $f_A$ commutes with $Ad(h)$ for any $h \in H$, so by Schur's Lemma, there exist linear maps $L_1, L_2 : m_1 \to \mathbb{R}$ such that

\[
f_A (X) = L_1 (A) X + L_2 (A) JX \quad (X \in m_2).
\]

For a non-zero $A_1 \in m_1$, set $m'_1 = \mathbb{R} \{ A_1 - L_2 (A_1) E \}$. Since $E$ belongs to the center of $\frak{h}$, we have $[m'_1, m_2] = \{0\}$. It is trivial that

\[
[A, X] = L_1 (p_0 (A)) X \quad (A \in m'_1, X \in m_2).
\]

Thus we have a new decomposition of $\mathfrak{g}$:

\[
\mathfrak{g} = m' \oplus \mathfrak{h},
\]

(\text{where } m' = m_1 \oplus m_2), according to which we define a Lorentz inner product on $m'$ as in §2. Then $m_2$ is spacelike and perpendicular to $m'_1$, and we have Lemma 3.3'.

**Lemma 3.3'.** $[m'_1, m_2] \subseteq m_2$. More precisely, there exists a linear map $L'_1 : m'_1 \to \mathbb{R}$ such that

\[
[A, X] = L'_1 (A) X \quad (A \in m'_1, X \in m_2).
\]
We use notations $m$, $m_1$ and $L$ instead of $m', m'_1$ and $L'_1$ respectively.

**Lemma 3.4.** (1) If $L \neq 0$, then $[m_2, m_2] = 0$.
(2) If $L = 0$, then $[m_2, m_2] \subseteq \mathfrak{h} \oplus m_1$.

**Proof.** For any $Z, W \in m_2$, we have


by the equality

$$[E, [Z, W]] = [[E, Z], W] + [Z, [E, W]].$$

Therefore, for a basis $X, JX, Y, JY$ of $m_2$, we have

$$p_t[X, JX] = p_t[Y, JY] = 0$$

$$p_t[X, Y] + p_t[JX, JY] = 0$$

$$p_t[X, JY] + p_t[Y, JX] = 0,$$

so $\dim p_t[m_2, m_2] \leq 2$. On the other hand, $p_t[m_2, m_2]$ is $Ad(H)$-invariant subspace of $m_2$ by (3.1). Thus we have $p_t[m_2, m_2] = \{0\}$, that is, $[m_2, m_2] \subseteq \mathfrak{h} \oplus m_1$. Thus, for any $X, Y \in m_2$, we can set $[X, Y] = U + A(U \subseteq \mathfrak{h}, A \in m_1)$. Then, for $B \subseteq m_1$, we have $2L(B)[X, Y] = [B, A] = 0$. If $L \neq 0$, then $[X, Y] = 0$. $\blacksquare$

**Lemma 3.5.** If $L = 0$, then $m_1 = \mathfrak{z}(g)$ where $\mathfrak{z}(g)$ is a center of $g$.

**Proof.** Since $[m_1, \mathfrak{h}] = \{0\} = [m_1, m_2]$, it is trivial that $m_1 \subseteq \mathfrak{z}(g)$.

Let $Z$ be any vector in $\mathfrak{z}(g)$. For any $X \in m_2$, we have $[p_t(Z), X] + [p_t(Z), X] = 0$. Since $[p_t(Z), X] \in m_2$ and $[p_t(Z), X] \in \mathfrak{h} \oplus m_1$, we have $[p_t(Z), X] = 0$, which implies $p_t(Z) = 0$. We have $p_t(Z) = 0$ by equalities $0 = [E, Z] = J p_t(Z)$. Therefore we have $Z \in \mathfrak{z}(m_1)$. $\blacksquare$

By the above argument, we have following possibilities;

(i) $[m_1, m_2] = m_2$, $[m_2, m_2] = \{0\}$;

(ii) $[m_2, m_2] = \{0\}$, $[m_2, m_2] \subseteq \mathfrak{h}$;

(iii) $[m_1, m_2] = \{0\}$, $[m_2, m_2] \subseteq \mathfrak{g} = m_1$;

(iv) $[m_1, m_2] = \{0\}$, $p_t[m_2, m_2] \neq \{0\}$, $p_t[m_1, m_2] \neq \{0\}$.

**Case (ii).** By the same way as in the proof of the Theorem B in [7], we have the space (1).

**Case (i) and (iii).** $m_1 \oplus m_2$ is an ideal in $g$. Let $K$ be a connected Lie subgroup of $G$ whose Lie algebra is $m$. Then $K$ is a closed normal subgroup of
Since the dimension of the isotropy subgroup of $K$ at $o \in M$ is equal to $\dim (K \cap H) = \dim (\mathfrak{m} \cap \mathfrak{h}) = 0$, we have $\dim K(o) = \dim M$. Therefore $K(o)$ is open in $M$. Since $K$ is a normal subgroup of $G$, each $K$-orbit is open in $M$. By the connectedness of $M$, we have $K(o) = M$. Thus $M$ is isometric to the Lie group $K$ with a left invariant Lorentz metric. Since the sequence

$$1 \rightarrow H \rightarrow G \rightarrow G/H = K(o) \rightarrow 1$$

is exact and there exists a cross section $s : K(o) \rightarrow G$ such that $s = \text{id}$. $G$ is a semi-direct product of $H = \text{U}(2)$ and $M = K(o)$. Thus we have space (2).

**Case (iv).** Let $C$ be a Lie subgroup of $G$ whose Lie algebra is $\mathfrak{g} = \mathfrak{m}_1$. Then $C$ is a closed, commutative and normal subgroup of $G$, and acts on $M$ freely (because $C \cap H = \{1\}$). Therefore, each $C$-orbit is a 1-dimensional closed submanifold and timelike (because $\mathfrak{m}_1$ is timelike).

**Lemma 3.6.** The orbit space $M/C$ has a differentiable manifold structure.

**Proof.** Since $H$ is compact and $C$ is closed, $C$ acts on $M$ properly (c.f., [5], [11]). Then $M/C$ is a Hausdorff space and satisfies the second countable axiom (c.f., [3]). Since each $C$-orbit $C(x)$ of $x \in M$ is timelike, there exists an open set $V$ in $\mathbb{R}^4$ such that a normal exponential map $\exp^\sharp : V \rightarrow S = \exp^\sharp(V)$ is a diffeomorphism and $\langle T_xS, T_xC(x) \rangle = 0$. Then $M/C$ has a differentiable manifold structure (c.f., [3]).

By the same procedure as in the proof of Theorem 30.2 in [3], we have

**Lemma 3.7.** $C \rightarrow M \rightarrow M/C$ is a principal fibre bundle with a structure group $C$.

We introduce a Riemannian metric $h$ on $M/C$ so that $\rho : M \rightarrow M/C$ is a semi-Riemannian submersion as follows: Let $S(y)$ be a neighborhood of $y = \rho(y)$ in $M/C$ and $\chi_{S(y)}$ be a local cross section from $S(y)$ to $M$. We define a Riemannian metric $h_{S(y)}$ on $S(y)$ by

$$h_{S(y)}(X, Y) = \langle d\chi_{S(y)}(X), d\chi_{S(y)}(Y) \rangle (\chi_{S(y)})$$

for any vector fields $X$ and $Y$ on $M/C$. Since $\chi_{S(y)}(x)$ and $\chi_{S(o)}(x)$ belong to the same $C$-orbit for $x \in S(y) \cap S(x)$, there exists $c \in C$ such that $c\chi_{S(y)}(x) = \chi_{S(o)}(x)$. Therefore we have

$$h_{S(o)}(X, Y)(x) = h_{S(y)}(X, Y)(x).$$

Thus $\{h_{S(y)}\}$ defines a Riemannian metric on $M/C$. 

Homogeneous Lorentz manifolds
G/C is an isometry group acting on M/C effectively and transitively, and
the isotropy subgroup is H/C=H=U(2). So M/C is a simply connected 2-
dimensional complex space form (c.f., [4]).

Thus M is a principal fibre bundle with an abelian structure group C of
dimension 1, over a simply connected 2-dimensional complex space form. We
complete the proof of the Theorem A.

REMARK 3.8. When L≠0, the space (2) in the Theorem A is isometric to
the Lie group G₄ in [10] and G is a semi-direct product U(2)×G₄.

REMARK 3.9. By the similar way as the proof of the Theorem A, we have
the following. Let (M, ⟨, ⟩) be a simply connected 6-dimensional Lorentz mani-
fold on which a connected isometry group G acts transitively. If the linear
isotropy subgroup H at o∈M acts on TₒM as Iₜ×U(2), then (M, ⟨, ⟩) is one
of the following:

(1) (M, ⟨, ⟩) is isometric to (N₁×M₄, dt²+ds₄) where (M₁, dt²) is a simply
connected 2-dimensional Lie group with a left-invariant Lorentz metric dt² and
(M₄, ds₄) is a 2-dimensional simply connected complex space form;

(2) (M, ⟨, ⟩) is isometric to a simply connected 6-dimensional Lie group
with a left-invariant Lorentz metric and G is isomorphic to a semi-direct pro-
duct U(2)×M;

(3) (M, ⟨, ⟩) is a principal fibre bundle, with a 2-dimensional abelian struc-
ture group, over a 2-dimensional simply connected complex space form.

4. Proofs of the Theorem B and Theorem C.

Let (M, ⟨, ⟩) be a simply connected n-dimensional Lorentz manifold admit-
ting a connected isometry group of dimension n(n−1)/2+1 (resp. (n−1)(n−2)/
2+2) for n=3 (resp. n=4). By the Proposition 2.2, M is a simply connected
homogeneous Lorentz manifold G/H and the linear isotropy subgroup is con-
jugate to Iₙ₋₂×SO(2). Then Ad(H) acts on m as Iₙ₋₂×SO(2), so there exist
an (n−2)-dimensional subspace m₁ and a 2-dimensional subspace m₂ of m such
that m=m₁⊕m₂, Ad(H)=Iₙ₋₂ on m₁ and Ad(H)=SO(2) on m₂. By the same
way as the proof of Lemma 3.1, we have

LEMMA 4.1. m₂ is spacelike and perpendicular to m₁ (therefore, m₁ is time-
like).

Let pₒ, p₁ and p₂ be orthogonal projection from g to h, m₁ and m₂ respec-
tively. Then by the same reason as in §3, we have
Homogeneous Lorentz manifolds

\[ p_i Ad(h) = Ad(h)p_i \quad (i=0, 1, 2) \]

for any \( h \in H \). Since there exists \( E \in \mathfrak{h} \) such that

\[ Ad(\exp tE) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \]

on \( \mathfrak{m}_2 \). We have

\[ [E, X] = JX \quad (X \in \mathfrak{m}_2) \]

where \( J \) is an almost complex structure on \( \mathfrak{m}_2 \).

**Lemma 4.2.** There exists linear maps \( L_1, L_2: \mathfrak{m}_1 \rightarrow \mathbb{R} \) such that

\[ [A, X] = L_1(A)X + L_2(A)JX \]

for any \( A \in \mathfrak{m}_1 \) and any \( X \in \mathfrak{m}_2 \).

**Proof.** For any fixed \( A \in \mathfrak{m}_1 \), we define a linear map \( f_A: \mathfrak{m}_2 \rightarrow \mathfrak{g} \) by \( f_A(X) = [A, X](X \in \mathfrak{m}_2) \). By the same procedure as in the proof of Lemma 3.3, we have \( \ker (p_0 f_A) = \{0\} \) or \( \mathfrak{m}_2 \). Suppose that \( \ker (p_0 f_A) = \{0\} \). Then \( p_0 f_A: \mathfrak{m}_2 \rightarrow \mathfrak{h} \) is injective, so \( \dim \mathfrak{h} \geq 2 \) which contradicts the fact that \( \dim \mathfrak{h} = 1 \). Since \( A \) is arbitrary, we have \( p_1 [\mathfrak{m}_1, \mathfrak{m}_2] = \{0\} \). We can show \( p_1 [\mathfrak{m}_1, \mathfrak{m}_2] = \{0\} \) by the same way as in the proof of Lemma 3.3. Therefore we have Lemma 4.2 by Schur's Lemma.

Let \( A_1, \ldots, A_{n-2} \) be a basis of \( \mathfrak{m}_1 \) such that \( L_2(A_j) = 0 \) \((j \neq 1)\). Set \( \mathfrak{m}_1' = R \{ A_1, \ldots, A_{n-2} \} \). Then we have a new decomposition \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}' \) (where \( \mathfrak{m}' = \mathfrak{m}_1' \oplus \mathfrak{m}_2 \)) of \( \mathfrak{g} \) and we have

\[ [A'X] = L_1(p_1(A')X \quad (A' \in \mathfrak{m}_1', X \in \mathfrak{m}_2). \]

By the same procedure as in §3, \( \mathfrak{m}_1 \) is spacelike and perpendicular to \( \mathfrak{m}_1' \) and we have

**Lemma 4.2'.** There exists a linear map \( L_1': \mathfrak{m}_1' \rightarrow \mathbb{R} \) such that

\[ [A', X] = L_1'(A')X \quad (A' \in \mathfrak{m}_1', X \in \mathfrak{m}_2). \]

We use the notation \( \mathfrak{m}_1, \mathfrak{m} \) and \( L \) instead of \( \mathfrak{m}_1', \mathfrak{m}' \) and \( L_1' \) respectively.

**Lemma 4.3.** \( [\mathfrak{m}_1, \mathfrak{m}_1] \subseteq \ker (L) \) for \( n=4 \).

**Proof.** For any \( A, B \in \mathfrak{m}_1 \), we have
Hiroo Matsuda

\[ \text{Ad}(h)p_*[A, B] = p_*[\text{Ad}(h)A, \text{Ad}(h)B] = p_*[A, B], \]

for any \( h \in H \). Since \( \text{Ad}(H) \) acts on \( m_2 \) irreducibly, we have \( p_*[A, B] = 0 \). Thus we can set \( [A, B] = aE + C \) (for some \( a \in \mathbb{R} \) and for some \( C \in m_1 \)). For any \( X \in m_2 \), we have

\[ [X, [A, B]] = [[[X, A], B] + [A, [X, B]]] = L(A)L(B)X - L(A)L(B)X = 0 \]

Thus we have

\[ 0 = [X, [A, B]] = [X, aE + C] = -aJX - L(C)X. \]

Since \( X \) and \( JX \) are linearly independent, we have \( a = 0 \) and \( L(C) = 0 \), so we have \( [A, B] = C \) where \( C \in \ker(L) \).

**Lemma 4.4.**

(1) If \( L \neq 0 \), then \( [m_2, m_2] = \{0\} \) (resp. \( [m_3, m_3] \subseteq \ker(L) \)) for \( n = 3 \) (resp. \( n = 4 \)).

(2) If \( L = 0 \), then \( [m_2, m_2] \subseteq \mathfrak{h} \oplus \mathfrak{h}([m_1]) \), where \( \mathfrak{h}([m_1]) \) is a center of \( m_1 \) in \( m_2 \).

**Proof.** For any \( X, Y \in m_2 \), we can set \( [X, Y] = U + A(U \in \mathfrak{h}, A \in m_1) \) by the same way as the proof of Lemma 3.4. Then for \( B \in m_1 \), we have \( 2L(B)[X, Y] = [A, B] \). If \( L \neq 0 \), then \( [X, Y] = [B, A]/2L(B) \) for a nonzero \( B \), so we have (1). If \( L = 0 \), then \( [B, A] = 0 \) for any \( B \in m_1 \), so \( A \in \mathfrak{h}([m_1]) \).

By the same way as the proof of Lemma 3.5, we have

**Lemma 4.5.** If \( L = 0 \), then \( \mathfrak{h}([m_1]) = \mathfrak{g} \) where \( \mathfrak{g} \) is a center of \( \mathfrak{g} \).

**Remark 4.6.** If \( n = 4 \), then \( \dim m_1 = 2 \), so \( \mathfrak{g}([m_1]) = \{0\} \) or \( m_1 \).

By the same way as in § 3, we have Theorem B and Theorem C.

**References**


195-219.


Department of Mathematics
Kanazawa Medical University
Uchinada-machi,
Ishikawa, 920-02
Japan