A NOTE ON SPAN UNDER REFINABLE MAPS

By

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1. Introduction.

All spaces considered in this note are metric, and all maps are continuous functions. A compactum is a compact metric space. A continuum is a connected compactum. In [1], Ford and Rogers defined a map \( r: X \to Y \) from a compactum \( X \) onto a compactum \( Y \) to be refinable if for each \( \varepsilon > 0 \), there is an \( \varepsilon \)-map \( f: X \to Y \) from \( X \) onto \( Y \) whose distance from \( r \) is less than \( \varepsilon \). Refinable maps are useful in continuum theory, and many properties in continuum theory are preserved by refinable maps. For example, decomposability [1], aposyndesis [2], property \([k] \), irreducibility, hereditary indecomposability and being the pseudo-arc [6] (see for other properties [4] and [5]).

Lelek [8] defined the surjective span of a continuum \( X \), \( \sigma^+(X) \) (resp. the surjective semi-span, \( \sigma^+(X) \)) to be the least upper bound of all real numbers \( \alpha \) with the following property; there exist a continuum \( C \) and maps \( f_1, f_2: C \to X \) such that \( f_1(C) = X = f_2(C) \) (resp. \( f_1(C) = X \)) and \( \text{dist}(f_1(c), f_2(c)) \leq \alpha \) for every \( c \in C \). The span \( \sigma(X) \) and the semi-span \( \sigma_\alpha(X) \) of \( X \) are defined by the formulas;

\[
\sigma(X) = \sup\{\sigma^+(A) | A \text{ is a subcontinuum of } X\},
\]

\[
\sigma_\alpha(X) = \sup\{\sigma^+(A) | A \text{ is a subcontinuum of } X\}.
\]

Recently, many authors have been investigating span theory and finding interesting properties. Concerning span and special classes of maps, the following problems are raised in the University of Houston Problem Book;

Problem 86. Do confluent maps of continua preserve span zero?

Problem 92. If \( M \) is a continuum with positive span such that each of its proper subcontinua has span zero, does every nondegenerate monotone continuous image of \( M \) have positive span?

Ingram, [3, Theorem 2], showed that monotone maps of continua preserve span zero.

In this note we will show that refinable maps of continua preserve surjective
(semi-) span zero, and refinable preimages of continua with surjective (semi-) span zero have surjective (semi-) span zero. We note that for a refinable map \( r: X \to Y \), if \( Y \) has property \([k]\), then \( r \) is confluent ([6, Theorem (2.3)]), and moreover if \( Y \) is locally connected, \( r \) is monotone ([1, Corollary 1.2]).

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2. Results.

In this note an ANR means an absolute neighborhood retract for the class of metric spaces. For a metric space \( X \) and points \( x, x' \) of \( X \), \( d(x, x') \) is the distance from \( x \) to \( x' \) under a metric of \( X \).

Theorem. Let \( r: X \to Y \) be a refinable map of continua. Then \( \tau(X) = 0 \) if and only if \( \tau(Y) = 0 \), where \( \tau = \sigma*, \sigma^0, \sigma \) or \( \sigma_0 \).

For the proof we need the following lemmas:

Lemma 1 ([10, Lemma 1]). Let \( f: X \to P \) be a map from a compactum \( X \) to a compact ANR \( P \). Then for every \( \varepsilon > 0 \), there is a positive number \( \delta > 0 \) such that if \( g: X \to Y \) is a \( \delta \)-map from \( X \) onto a compactum \( Y \), then there is a map \( h: Y \to P \) such that \( f \) and \( hg \) are \( \varepsilon \)-near.

By a slight modification of the proof of [9, 3.1], we have the following.

Lemma 2. Let \( X \) be a non-empty continuum contained in a compactum \( Z \). If \( \beta \) is a real number and for \( n = 1, 2, 3, \ldots \), there exists a continuum \( Z_n \) in \( Z \) such that \( \beta \leq \tau(Z_n) \) and \( \text{Lim} Z_n = X \), then \( \beta \leq \tau(X) \), where \( \tau = \sigma*, \sigma^0, \sigma \) or \( \sigma_0 \).

Proof of Theorem. Since surjective span zero is a topological invariant in the class of continua, we may assume that both \( X \) and \( Y \) are subsets of the Hilbert cube \( Q \).

Suppose that \( \sigma*(X) = 0 \). Let \( C \) be a continuum, and let \( f_1, f_2: C \to Y \) be maps such that \( f_1(C) = f_2(C) = Y \). Let take a compact ANR neighborhood \( U \) of \( X \) in \( Q \) and a continuous extension \( g: U \to Q \) of \( r \). For each integer \( n \geq 1 \), there is a positive number \( \varepsilon_n > 0 \) such that

\[
\text{(2) if } d(x, x') < \varepsilon_n, \text{ then } d(g(x), g(x')) < \frac{1}{n}.
\]

Since \( r \) is a refinable map, there exists a sequence \( \{r_i\} \) of maps \( r_i: X \to Y \) such that for each \( i \geq 1 \),

\[
\text{(2) } r_i(X) = Y,
\]
A note on span under refinable maps

(3) $d(r(x), r_i(x)) < \frac{1}{i}$ for all $x \in X$, and

(4) $\operatorname{diam} r^{-1}(y) < \frac{1}{i}$ for all $y \in Y$.

Then by (2), (4) and Lemma 1, we have integers $n \leq i(1) < i(2) < \cdots$ and maps $h_j : Y \to U$, $j = 1, 2, \cdots$, such that

(5) $d(h_j r_{i(j)}(x), x) < \frac{1}{j}$ for all $x \in X$, $j = 1, 2, \cdots$.

By (2) and (5), we easily have that $\lim_j h_j(Y) = \lim_j h_j r_{i(j)}(X) = X$. Since $\sigma*(X) = 0$, by Lemma 2, there exists an integer $j_0 \geq 1$ such that

(6) $\sigma*(h_j(Y)) < \epsilon_n$ for all $j \geq j_0$.

Now take an integer $j \geq j_0$ with $1/j < \epsilon_n$, and put the maps

$r' = r_{i(j)} : X \to Y$ and $h = h_j : Y \to h_j(Y) = h(Y)$.

Considering two maps $h f_1$, $h f_2 : C \to h(Y)$, by (6), there exists a point $c_n \in C$ such that

(7) $d(h f_1(c_n), h f_2(c_n)) < \epsilon_n$.

Then by (1),

(8) $d(g h f_1(c_n), g h f_2(c_n)) < \frac{1}{n}$.

By (2), take points $x_1$, $x_2 \in X$ such that $r'(x_i) = f_i(c_n)$ for $i = 1, 2$. Then by (3), (5) and (1), we have that for $i = 1, 2$,

(9) $d(f_i(c_n), g h f_1(c_n)) = d(r'(x_i), g h r'(x_i))$
\[< d(r'(x_i), r(x_i)) + d(r(x_i), g h r'(x_i))\]
\[< \frac{1}{i(j)} + d(g(x_i), g h r'(x_i))\]
\[< \frac{1}{n} + \frac{1}{n} = \frac{2}{n}\]

Hence by (8) and (9),

(10) $d(f_1(c_n), f_2(c_n)) < \frac{5}{n}$.

Let $c_0 \in C$ be an accumulation point of the sequence $\{c_n\}$. Then by (10), $d(f_1(c_0), f_2(c_0)) = 0$. It follows that $\sigma*(Y) = 0$.

Conversely, we suppose that $\sigma*(Y) = 0$. Let $C$ be a continuum, and let $f_1, f_2 : C \to X$ be maps such that $f_1(C) = X = f_2(C)$. For each $n \geq 1$, there is an $1/n$-map $r_n : X \to Y$ from $X$ onto $Y$, since $r$ is a refinable map. Since $r_n f_1(C) = Y =$
240

Akira Koyama

$r_n f_2(C)$ and $\sigma^*(Y) = 0$, there exists a point $c_n \in C$ such that $r_n f_1(c_n) = r_n f_2(c_n)$. Then $d(f_1(c_n), f_2(c_n)) < 1/n$. Hence, as in the first part of the proof, we have the point $c_\theta \in C$ such that $f_1(c_\theta) = f_2(c_\theta)$. Therefore $\sigma^*(X) = 0$.

The above proof may be used to prove similar theorems for $\sigma_0^*$, $\sigma$ and $\sigma_\alpha$.

**Corollary 1.** Let $r : X \rightarrow Y$ be a refinable map of continua. Then $\tau(X) > 0$ if and only if $\tau(Y) > 0$, where $\tau = \sigma^*$, $\sigma_0^*$, $\sigma$ or $\sigma_\alpha$.

Therefore refinable maps of compacta preserve positive span.

In the latter part of the proof of the Theorem we needed only the fact that there exists an $1/n$-map from $X$ onto $Y$ for each $n \geq 1$. Hence the following is obtained. The case $\tau = \sigma$ is included in [11, Lemma 21].

**Corollary 2.** Let $X$ and $Y$ be continua. If $X$ is $Y$-like and $\tau(Y) = 0$, then $\tau(X) = 0$, where $\tau = \sigma^*$, $\sigma_0^*$, $\sigma$ or $\sigma_\alpha$.

By [4, Corollary 3.4], every hereditarily decomposable circle-like continuum admits a refinable map onto a circle. Therefore we have

**Corollary 3.** Every hereditarily decomposable circle-like continuum has positive surjective (semi-) span.

**References**


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