ON SOME STARLIKENESS CONDITIONS FOR ANALYTIC FUNCTIONS

By
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Let $A(p)$ denote the class of functions $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ which are analytic in the open disk $E = \{ z : |z| < 1 \}$.

A function $f(z) \in A(p)$ is called $p$-valently starlike with respect to the origin iff

$$\text{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in} \ E.$$

We denote by $S^*(p)$ the subclass of $A(p)$ consisting of functions which are $p$-valently starlike in $E$.

Mocanu [3, Theorem 1] proved that if $f(z) \in A(1)$ and

$$|\arg f'(z)| < \frac{\pi}{2} \alpha_0 = 0.968 \ldots, \quad z \in E,$$

where $\alpha_0 = 0.6165 \ldots$ is the unique root of the equation

$$2 \tan^{-1}(1-\alpha) + \pi(1-2\alpha) = 0,$$

then $f(z) \in S^*(1)$.

In [5], Nunokawa proved the following theorem.

THEOREM A. Let $p \geq 2$. If $f(z) \in A(p)$ satisfies

$$|\arg f^{(p)}(z)| < \frac{3}{4} \pi \quad \text{in} \ E,$$

then $f(z)$ is $p$-valent in $E$.

DEFINITION 1. Let $F(z)$ be analytic and univalent in $E$, and suppose that $F(E) = D$. If $f(z)$ is analytic in $E$, $f(0) = F(0)$, and $f(E) \subset D$, then we say that $f(z)$ is subordinate to $F(z)$ in $E$, and we write

$$f(z) \prec F(z).$$

DEFINITION 2. If the function $f(z)$ is analytic in $E$ and if for every non-
real \( z \) in \( E \)

\[
\text{sign}(Im f(z)) = \text{sign}(Im z),
\]

then \( f(z) \) is said to be typically-real in \( E \). We owe this definition to [1, p. 184].

We shall use the following lemmas to prove our results.

**Lemma 1.** Let \( \beta^* = 1.218 \ldots \) be the solution of

\[
\pi \beta = \frac{3\pi}{2} - \tan^{-1} \beta
\]

and let

\[
\alpha = \alpha(\beta) = \beta + \frac{2}{\pi} \tan^{-1} \beta
\]

for \( 0 < \beta \leq \beta^* \).

If \( p(z) \) is analytic in \( E \), with \( p(0) = 1 \), then

\[
p(z) + z p'(z) < \left(\frac{1+z}{1-z}\right)^{\alpha} \implies p(z) < \left(\frac{1+z}{1-z}\right)^{\beta}
\]

We owe this lemma to [2, Theorem 5].

**Lemma 2.** Let \( f(z) \in A(p) \) and suppose

\[
\Re \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} > 0 \text{ in } E.
\]

Then we have

\[
\Re \frac{zf^{(k)}(z)}{f^{(k-1)}(z)} > 0 \text{ in } E,
\]

or

\[
f^{(p-k)}(z) \in \mathbb{S}^*(k)
\]

for \( k = 1, 2, 3, \ldots, p \).

We owe this lemma to [4, Theorem 5].

**Theorem 1.** Let \( p \geq 2 \). If \( f(z) \in A(p) \) satisfies

\[
|\arg f^{(p)}(z)| < \frac{3}{4} \pi \text{ in } E
\]

and \( f^{(p-1)}(z)/z \) is typically-real in \( E \), then \( f(z) \in \mathbb{S}^*(p) \).

**Proof.** Let us put

\[
p(z) = \frac{f^{(p-1)}(z)}{p! z}.
\]
From the assumption (1), Lemma 1 and applying the same method as the proof of [5, Main theorem], we have

\[ p(z) + z p'(z) = \frac{f^{(p)}(z)}{p!} \left( \frac{1+z}{1-z} \right)^{\frac{1}{2}} \text{ in } E, \]

\[ p(0) = 1 \text{ and therefore we have} \]

\[ \frac{f^{(p-1)}(z)}{p! z} \left( \frac{1+z}{1-z} \right) \text{ in } E. \]

This shows that

\( (2) \quad \text{Re} \frac{f^{(p-1)}(z)}{z} > 0 \text{ in } E. \)

By the same calculation as [6, p. 276], we have

\( (3) \quad \frac{f^{(p-2)}(z)}{z f^{(p-1)}(z)} = \int_0^1 \frac{f^{(p-1)}(t z)}{f^{(p-1)}(z)} \, dt \)

\[ = \int_0^1 z \cdot \frac{t z}{z} \cdot \frac{f^{(p-1)}(t z)}{t z} \, dt \]

On the other hand, we easily have

\( (4) \quad \left| \arg \left( \frac{z}{f^{(p-1)}(z)} \cdot \frac{t z}{f^{(p-1)}(t z)} \right) \right| = \left| \arg \frac{f^{(p-1)}(t z)}{t z} - \arg \frac{f^{(p-1)}(z)}{z} \right| < \frac{\pi}{2}. \)

Since \( f^{(p-1)}(z)/z \) is typically-real in \( E \) and satisfies the condition (2). From (3) and (4), we easily have

\[ \text{Re} \frac{f^{(p-2)}(z)}{z f^{(p-1)}(z)} > 0 \text{ in } E. \]

This shows that

\( (5) \quad \text{Re} \frac{z f^{(p-1)}(z)}{f^{(p-1)}(z)} > 0 \text{ in } E. \)

From Lemma 2 and (5), we easily have

\[ \text{Re} \frac{zf''(z)}{f(z)} > 0 \text{ in } E. \]

This completes our proof.

**Theorem 2.** Let \( p \geq 2 \). If \( f(z) \in A(p) \) satisfies

\( (6) \quad |\arg f^{(p)}(z)| < \frac{\pi}{2} \cdot \alpha_1 \text{ in } E. \)

where \( \alpha_1 = 1/2 + (2/\pi) \tan^{-1}(1/2) = 0.79516 \cdots \), then \( f(z) \in S^*(p) \).
PROOF. Let us put
\[ p(z) = \frac{f_{p-1}(z)}{p!z}. \]
From the assumption (6), Lemma 1 and by the same calculation as in the proof of Theorem 1, we have
\[ p(z) + z p'(z) = \frac{f_{p}(z)}{p!} < \left( \frac{1+z}{1-z} \right)^{\alpha_1} \text{ in } E, \]
and therefore, we have
\[ \frac{f_{p-1}(z)}{p!z} < \left( \frac{1+z}{1-z} \right)^{1/2} \text{ in } E. \]
This shows that
\[ (7) \]
\[ | \arg \frac{f_{p-1}(z)}{z} | < \frac{\pi}{4} \text{ in } E. \]
By the same calculation as the proof of Theorem 1, we have
\[ \frac{f_{p-2}(z)}{z f_{p-1}(z)} = \int \frac{z}{t z} \cdot \frac{f_{p-1}(t z)}{tz} \, dt. \]
From (7), we easily have
\[ \left| \arg \left( \frac{z}{t z} \cdot \frac{f_{p-1}(t z)}{tz} \right) \right| \leq \left| \arg \frac{f_{p-1}(z)}{z} \right| + \left| \arg \frac{f_{p-1}(t z)}{tz} \right| < \frac{\pi}{2} \text{ in } E. \]
Therefore, we have
\[ \text{Re} \left( \frac{f_{p-2}(z)}{z f_{p-1}(z)} \right) > 0 \text{ in } E. \]
This shows that
\[ \text{Re} \left( \frac{z f_{p-2}(z)}{f_{p-1}(z)} \right) > 0 \text{ in } E. \]
or \( f(z) \) is \( p \)-valently starlike in \( E \). This completes our proof.

From Theorem 2, we easily have the following corollary.

**Corollary 1.** Let \( f(z) \in A(2) \) satisfies
\[ | \arg f^*(z) | < \frac{\pi}{2} \alpha_1 \text{ in } E. \]
then \( f(z) \) is 2-valently starlike in \( E \).

**Remark.** \( \alpha_0 = 0.6165 \ldots < \alpha_1 = 0.79516 \ldots. \)
On some starlikeness conditions for analytic functions

References


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