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CERTAIN MINIMAL OR HOMOLOGICALLY VOLUME MINIMIZING SUBMANIFOLDS IN COMPACT SYMMETRIC SPACES

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1. Introduction.

In this paper we shall study minimal submanifolds in compact symmetric spaces and homologically volume minimizing submanifolds in compact simple Lie groups and quanternionic Kähler manifolds.

The first subject is studied by computing the second fundamental forms of submanifolds. In Section 2 using the structure theorem of the first conjugate loci of compact symmetric spaces (Takeuchi [5]) we compute the second fundamental form of a certain submanifold which is open and dense in the first conjugate locus of a compact symmetric space and prove the minimality of it. Moreover we show that the submanifold has no geodesic point.

The second subject is studied by using the notion "calibration" introduced by Harvey and Lawson [2]. This notion is used in Sections 3 and 4. The fundamental 2-form of a Kähler manifold is one of important examples of calibrations. It satisfies Wirtinger's inequality, which can be stated as follows. Let M be a Kähler manifold with fundamental 2-form ω. Then

$$\frac{1}{k!}\omega^k|_\xi \leqq \mathrm{vol}_\xi$$

for $1 \leqq k \leqq \dim_C M$ and any oriented tangent $2k$-plane ξ on M. The equality holds if and only if ξ is a complex plane with a suitable orientation. From the above inequality it follows that a compact Kähler submanifold of a Kähler manifold is homologically volume minimizing. Only closedness and the above inequality are needed to prove this assertion.

Here we explain the notion of calibration. Let M be a Riemannian manifold with a closed p-form ϕ on M which satisfies the following inequality:

$$\phi|_\xi \leqq \mathrm{vol}_\xi$$

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for any oriented tangent p-plane ξ on M. Such a form ϕ is called a *calibration.* Then any compact oriented p-dimensional submanifold N in M with the property:

$$\phi|_N = \mathrm{vol}_N$$

is homologically volume minimizing in M, that is,

$$\mathrm{vol}\,(N) \leqq \mathrm{vol}\,(N')$$

for any compact oriented p-dimensional submanifold N' such that $[N]=[N']$ in the homology group $H_p(M;\boldsymbol{R})$. In fact

$$\mathrm{vol}\,(N) = \int_N \mathrm{vol}_N = \int_N \phi = \int_{N'} \phi \leqq \int_{N'} \mathrm{vol}_{N'} = \mathrm{vol}\,(N')\,.$$

The equality holds if and only if $\phi|_{N'} = \mathrm{vol}_{N'}$. Even if N is noncompact, a similar argument shows that N is minimal and stable under variations of compact supports.

The purpose of Section 3 is to construct a calibration of degree 3 on a compact simple Lie group and to show that a 3-dimensional compact simple Lie subgroup associated with the highest root is homologically volume minimizing and that a certain submanifold which is open and dense in the first conjugate locus is stable under variations of compact supports. Note that the codimension of the first conjugate locus of a compact simple Lie group is equal to 3.

In Section 4 we shall prove a quaternionic version of Wirtinger's inequality, that is, on a quaternionic Kähler manifold M with the fundamental 4-form Ω

$$\frac{1}{k!}\Omega^k|_\xi \leqq \mathrm{vol}_\xi$$

for any oriented tangent $4k$-plane ξ on M. The equality holds if and only if ξ is a quaternionic plane with a suitable orientation. It follows from the result that a compact quaternionic Kähler submanifold of a quaternionic Kähler manifold is homologically volume minimizing. In particular, we obtain a stronger fact for the quaternionic projective space, which is stated as Theorem 11.

2. The first conjugate loci of compact symmetric spaces.

For a complete Rimannian manifold M and a point p in M, we denote by $F_p(M)$ the first conjugate locus of M with respect to p. This section is devoted to constructing submanifolds $F_p^0(M)$ and $F_p^1(M)$ of a compact symmetric space M which are open and dense in $F_p(M)$ and to verifying that $F_p^0(M)$ is a minimal submanifold in M and that $F_p^1(M)$ has no geodesic point if the rank of M is greater than 1.

Let G be a compact connected Lie group and θ an involutive automorphism of G. Put

$$G_\theta=\{g\in G\,;\,\theta(g)=g\}\,.$$

For a closed subgroup K of G which lies between G_θ and the identity component of G_θ, (G, K) is a symmetric pair. A bi-invariant Riemannian metric $\langle\ ,\ \rangle$ on G naturally induces a G-invariant Riemannian metric on the homogeneous space $M=G/K$, which is also denoted by $\langle\ ,\ \rangle$. Then M is a compact symmetric space with respect to $\langle\ ,\ \rangle$. It is known that any compact symmetric space is constructed in this way. From now on we assume that M is irreducible.

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively. The involutive automorphism θ of G induces an involutive automorphism of $\mathfrak{g}$, which is also denoted by θ. Since K lies between G_θ and the identity component of G_θ,

$$\mathfrak{k}=\{X\in\mathfrak{g}\,;\,\theta(X)=X\}\,.$$

Put

$$\mathfrak{m}=\{X\in\mathfrak{g}\,;\,\theta(X)=-X\}\,.$$

Since θ is involutive, we have a direct sum decomposition of $\mathfrak{g}$:

$$\mathfrak{g}=\mathfrak{k}+\mathfrak{m}\,. \tag{1}$$

Take a maximal Abelian subspace $\mathfrak{a}$ in $\mathfrak{m}$ and a maximal Abelian subalgebra $\mathfrak{t}$ in $\mathfrak{g}$ containing $\mathfrak{a}$, then the complexification $\mathfrak{t}^C$ of $\mathfrak{t}$ is a Cartan subalgebra of the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$. The bi-invariant Riemannian metric $\langle\ ,\ \rangle$ on G induces an $\mathrm{Ad}(G)$-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$. For an element $\alpha\in\mathfrak{t}$, put

$$\mathfrak{g}_\alpha=\{X\in\mathfrak{g}^C\,;\,[H,X]=\sqrt{-1}\langle\alpha, H\rangle X \text{ for each } H\in\mathfrak{t}\}\,.$$

An element $\alpha\in\mathfrak{t}-\{0\}$ is called a *root* if $\mathfrak{g}_\alpha\neq\{0\}$. Let Δ denote the set of all roots, then we obtain a direct sum decomposition of $\mathfrak{g}^C$:

$$\mathfrak{g}^C=\mathfrak{t}^C+\sum_{\alpha\in\Delta}\mathfrak{g}_\alpha\,.$$

For an element $\gamma\in\mathfrak{a}$, we define a subspace $\tilde{\mathfrak{g}}_\gamma$ of $\mathfrak{g}^C$ by

$$\tilde{\mathfrak{g}}_\gamma=\{X\in\mathfrak{g}^C\,;\,[H,X]=\sqrt{-1}\langle\gamma, H\rangle X \text{ for each } H\in\mathfrak{a}\}$$

and put

$$\Sigma=\{\gamma\in\mathfrak{a}-\{0\}\,;\,\tilde{\mathfrak{g}}_\gamma\neq\{0\}\}\,.$$

Let $H\longmapsto\bar{H}$ denote the orthogonal projection from $\mathfrak{t}$ to $\mathfrak{a}$, then

$$\Sigma=\{\bar{\alpha}\,;\,\alpha\in\Delta \text{ and } \bar{\alpha}\neq 0\}\,.$$

Choose lexicographic orderings $>$ on $\mathfrak{t}$ and $\mathfrak{a}$ such that

$$\alpha\in\Delta,\ \bar{\alpha}>0 \Longrightarrow \alpha>0 .$$

We denote by Δ_+ and Σ_+ the sets of all positive roots in Δ and Σ respectively. Put

$$\mathfrak{k}_\gamma=\mathfrak{k}\cap(\tilde{\mathfrak{g}}_\gamma+\tilde{\mathfrak{g}}_{-\gamma}), \qquad \mathfrak{m}_\gamma=\mathfrak{m}\cap(\tilde{\mathfrak{g}}_\gamma+\tilde{\mathfrak{g}}_{-\gamma})$$

for $\gamma\in\Sigma_+$ and

$$\mathfrak{k}_0=\{X\in\mathfrak{k}\,;[H,X]=0 \text{ for each } H\in\mathfrak{a}\}.$$

We have the following lemma. For proof of this lemma, see Section 3 of Chapter VI in Helgason [3] or Lemma 1.1 in Takeuchi [5].

LEMMA 1. *The direct sum decomposition*

$$\mathfrak{k}=\mathfrak{k}_0+\sum_{\gamma\in\Sigma_+}\mathfrak{k}_\gamma, \qquad \mathfrak{m}=\mathfrak{a}+\sum_{\gamma\in\Sigma_+}\mathfrak{m}_\gamma .$$

are orthogonal. We can choose $S_\alpha\in\mathfrak{k}$ and $T_\alpha\in\mathfrak{m}$ for each $\alpha\in\Delta_+$ with $\bar{\alpha}\neq 0$ in such a way that:

i) *For each $\gamma\in\Sigma_+$, $\{S_\alpha\,;\alpha\in\Delta_+,\bar{\alpha}=\gamma\}$ and $\{T_\alpha\,;\alpha\in\Delta_+,\bar{\alpha}=\gamma\}$ are orthonormal bases of $\mathfrak{k}_\gamma$ and $\mathfrak{m}_\gamma$ respectively;*

ii) *For each $\alpha\in\Delta_+$ with $\bar{\alpha}=\gamma\in\Sigma_+$ and each $H\in\mathfrak{a}$, we have*

$$\begin{aligned}&[H,S_\alpha]=\langle\gamma,H\rangle T_\alpha, \qquad [H,T_\alpha]=-\langle\gamma,H\rangle S_\alpha,\\ &\mathrm{Ad}(\exp H)S_\alpha=\cos\langle\gamma,H\rangle S_\alpha+\sin\langle\gamma,H\rangle T_\alpha,\\ &\mathrm{Ad}(\exp H)T_\alpha=-\sin\langle\gamma,H\rangle S_\alpha+\cos\langle\gamma,H\rangle T_\alpha .\end{aligned}$$

As M is irreducible, the root system Σ is irreducible and there exists a unique highest root δ in Σ. Let r be the rank of M and $\{\gamma_1,\cdots,\gamma_r\}$ be the fundamental root system of Σ. Put

$$\begin{aligned}&S=\{H\in\mathfrak{a}\,;\langle\delta,H\rangle=\pi,\ \langle\gamma_i,H\rangle\geqq 0 \text{ for } 1\leqq i\leqq r\},\\ &S^0=\{H\in\mathfrak{a}\,;\langle\delta,H\rangle=\pi,\ \langle\gamma_i,H\rangle>0 \text{ for } 1\leqq i\leqq r\},\\ &S^1=\left\{H\in S^0\,;\langle\gamma,H\rangle\notin\frac{\pi}{2}\boldsymbol{Z} \text{ for some } \gamma\in\Sigma_+ \text{ with } \langle\gamma,\delta\rangle\neq 0\right\}.\end{aligned}$$

LEMMA 2. *The first conjugate locus of M with respect to the origin o of M is described as follows:*

$$F_o(M)=\bigcup_{k\in K}k\,\mathrm{Exp}(S).$$

The sets $\bigcup_{k\in K}k\,\mathrm{Exp}(S^0)$ and $\bigcup_{k\in K}k\,\mathrm{Exp}(S^1)$ are submanifolds of M and open and dense in $F_o(M)$.

For the proof of this lemma, see Section 3 of Chapter VII in Helgason [3] or Corollary 3 in Takeuchi [5]. Let $F_o^0(M)$ denote the submanifold $\bigcup_{k\in K} k\,\mathrm{Exp}\,(S^0)$ of M. For any point p in M there is an element g in G such that $go=p$, so $F_p(M)$ coincides with $gF_o(M)$. Therefore $gF_o^0(M)$ is open and dense in $F_p(M)$. We denote by $F_p^0(M)$ the submanifold $gF_o^0(M)$ in M. By the definition of $F_o^0(M)$, $gF_o^0(M)$ is independent of the choice of g. When M is reducible, decomposing the orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$ into a product of irreducible orthogonal symmetric Lie algebras, we can define a submanifold $F_p^0(M)$ of M which is open and dense in $F_p(M)$. Through the decomposition of $(\mathfrak{g}, \theta)$ we can reduce a proof of the following theorem to one in case M is irreducible. For details about the construction of $F_p^0(M)$ in general case, see Section 1 in Takeuchi [5]. The definition of $F_p^1(M)$ is similar to $F_p^0(M)$.

THEOREM 3. *Let M be a compact symmetric space and p be a point in M. Then $F_p^0(M)$ is a minimal submanifold in M. If the rank of M is greater than* 1, *then $F_p^1(M)$ has no geodesic point.*

REMARK. It is well known that, if the rank of M is 1, $F_p(M)$ is a totally geodesic submanifold in M.

PROOF. As mentioned above, it may be assumed that M is irreducible. At first we shall compute the second fundamental form of the homogeneous submanifold $K\mathrm{Exp}\,(H)$ for each H in $\mathfrak{a}$. For each X in $\mathfrak{g}$, we define a vector field X^* on M by

$$X_x^* = \left.\frac{d}{dt}\right|_{t=0} \exp tX\cdot x \tag{2}$$

for each x in M. Simple calculations imply the following equations:

$$g_*X^* = (\mathrm{Ad}\,(g)X)^*\,, \tag{3}$$

$$g_*(\nabla_{X^*}Y^*) = \nabla_{(\mathrm{Ad}(g)X)^*}(\mathrm{Ad}\,(g)Y)^* \tag{4}$$

for $g\in G$ and $X, Y\in\mathfrak{g}$, where ∇ is the covariant derivative of M. The equation (4) follows from (3). For each $X\in\mathfrak{g}$, let $X_{\mathfrak{m}}$ denote the component of X in $\mathfrak{m}$ with respect to the orthogonal direct sum decomposition (1). Identifying $\mathfrak{m}$ with the tangent space $T_o(M)$ of M at the origin o, we obtain

$$(\nabla_{X^*}Y^*)_o = -[X, Y]_{\mathfrak{m}} \tag{5}$$

for $X\in\mathfrak{m}$ and $Y\in\mathfrak{g}$. Because

$$
\begin{aligned}
(\nabla_{X^*}Y^*)_o &= \lim_{t\to 0}\frac{(\exp(-tX))_* Y^*_{\exp tX\cdot o} - Y^*_o}{t} \\
&= [X^*, Y^*]_o \\
&= -[X, Y]_{\mathfrak{m}} .
\end{aligned}
$$

Note that $(\exp tX)_*$ is the parallel translation along the geodesic $\exp tX\cdot o$. The tangent space of $K\mathrm{Exp}(H)$ at $\mathrm{Exp}(H)$ is given as follows:

$$T_{\mathrm{Exp}(H)}(K\mathrm{Exp}(H)) = \left\{ \frac{d}{dt}\bigg|_{t=0} \exp tX\cdot \mathrm{Exp}(H)\,;\, X\in\mathfrak{k} \right\} .$$

If $X\in\mathfrak{k}_0$, then

$$\frac{d}{dt}\bigg|_{t=0} \exp tX\cdot \mathrm{Exp}(H) = 0 .$$

By Lemma 1,

$$\frac{d}{dt}\bigg|_{t=0} \exp tS_\alpha\cdot \mathrm{Exp}(H) = -(\exp H)_*(\sin\langle\alpha, H\rangle T_\alpha) , \tag{6}$$

so

$$
\begin{aligned}
T_{\mathrm{Exp}(H)}(K\mathrm{Exp}(H)) &= \{(\exp H)_* T_\alpha\,;\, \alpha\in\Delta_+,\ \langle\alpha, H\rangle\notin\pi\boldsymbol{Z}\}_{\boldsymbol{R}} \\
&= (\exp H)_* \sum_{\gamma\in\Sigma_+\ \langle\gamma, H\rangle\notin\pi\boldsymbol{Z}} \mathfrak{m}_\gamma .
\end{aligned}
$$

For $\alpha, \beta\in\Delta_+$ with $\langle\alpha, H\rangle, \langle\beta, H\rangle\notin\pi\boldsymbol{Z}$,

$$
\begin{aligned}
&[(\exp H)_*^{-1}(\nabla_{S_\alpha^*}S_\beta^*)]_o \\
&= [\nabla_{(\mathrm{Ad}(\exp H)^{-1}S_\alpha)^*}(\mathrm{Ad}(\exp H)^{-1}S_\beta)^*]_o \\
&= [\nabla_{(\cos\langle\alpha, H\rangle S_\alpha - \sin\langle\alpha, H\rangle T_\alpha)^*}(\cos\langle\beta, H\rangle S_\beta - \sin\langle\beta, H\rangle T_\beta)^*]_o \\
&= \sin\langle\alpha, H\rangle\cos\langle\beta, H\rangle [T_\alpha, S_\beta]
\end{aligned} \tag{7}
$$

by Lemma 1, the above equations (4), (5), and $\nabla_{X^*}Y^*=0$ at o for any $X\in\mathfrak{k}$ and any $Y\in\mathfrak{g}$. Identify the tangent space $T_{\mathrm{Exp}(H)}(M)$ with $\mathfrak{m}$ under the differential map $(\exp H)_*$ and let h_H be the second fundamental form of $K\mathrm{Exp}(H)$ at $\mathrm{Exp}(H)$. The equations (2), (6), and (7) yield

$$h_H(T_\alpha, T_\beta) = \cot\langle\beta, H\rangle [T_\alpha, S_\beta]^\perp$$

for $\alpha, \beta\in\Delta_+$ with $\langle\alpha, H\rangle, \langle\beta, H\rangle\notin\pi\boldsymbol{Z}$, where $\cdot^\perp$ is the component of $\cdot$ in $\left(\sum_{\gamma\in\Sigma_+\ \langle\gamma, H\rangle\notin\pi\boldsymbol{Z}} \mathfrak{m}_\gamma\right)^\perp$ with respect to the orthogonal direct sum decomposition of $\mathfrak{m}$ in Lemma 1. Furthermore

$$h_H(T_\alpha, T_\alpha) = -\cot\langle\alpha, H\rangle\bar{\alpha} .$$

Hence the mean curvature vector $\mathfrak{m}_H$ of $K\mathrm{Exp}(H)$ at $\mathrm{Exp}(H)$ is given as follows:

$$\mathfrak{m}_H=-\sum_{\alpha\in\Delta_+\ \langle\alpha,H\rangle\notin\pi\boldsymbol{Z}}\cot\langle\alpha,H\rangle\bar{\alpha}\ .$$

In particular the mean curvature vector $\mathfrak{m}_H$ is tangent to the submanifold $\mathrm{Exp}(\mathfrak{a})$.

Now we shall prove that the mean curvature vector of $K\mathrm{Exp}(S^0)$ vanishes. Let H be any element of S^0. The isometry of M induced by the reflection of $\mathfrak{a}$ in the hyperplane $\{X\in\mathfrak{a}\,;\langle\delta,X\rangle=\pi\}$ of $\mathfrak{a}$ fixes the points o and $\mathrm{Exp}(H)$, and leaves $K\mathrm{Exp}(S^0)$ invariant. Since S^0 is an open subset of $\{X\in\mathfrak{a}\,;\langle\delta,X\rangle=\pi\}$, $\mathfrak{m}_H$ is tangent to S^0. $\mathrm{Exp}(\mathfrak{a})$ is a totally geodesic submanifold in M, so it follows from the definition of the mean curvature vector that the mean curvature vector of $K\mathrm{Exp}(S^0)$ at $\mathrm{Exp}(H)$ vanishes for any H in S^0. Therefore the mean curvature vector of $K\mathrm{Exp}(S^0)$ at any point in it vanishes, that is, $K\mathrm{Exp}(S^0)$ is a minimal submanifold in M.

Next we shall show that $K\mathrm{Exp}(S^1)$ has no geodesic point, if the rank of M is greater than 1. Let H be any element of S^1. Take α in Δ_+ so that $\langle\alpha,H\rangle\notin\frac{\pi}{2}\boldsymbol{Z}$ and $\langle\bar{\alpha},\delta\rangle\neq0$.

$$[(\exp H)_*^{-1}(\nabla_{S_\alpha^*}S_\alpha^*)]_o=\sin\langle\alpha,H\rangle\cos\langle\alpha,H\rangle[T_\alpha,S_\alpha]$$

by (7). The choice of α impies that $\sin\langle\alpha,H\rangle\cos\langle\alpha,H\rangle\neq0$. $[T_\alpha,S_\alpha]\in\boldsymbol{R}\bar{\alpha}+\mathfrak{m}_{2\bar{\alpha}}$ and the component of it in $\boldsymbol{R}\bar{\alpha}$ does not vanish. Since the tangent space $T_{\mathrm{Exp}(H)}(K\mathrm{Exp}(S^1))$ is $(\exp H)_*\left(\{X\in\mathfrak{a}:\langle\delta,X\rangle=0\}+\sum_{\gamma\in\Sigma_+\ \langle\gamma,H\rangle\notin\pi\boldsymbol{Z}}\mathfrak{m}_\gamma\right)$, the second fundamental form of $K\mathrm{Exp}(S^1)$ at $\mathrm{Exp}(H)$ evaluated by $((S_\alpha^*)_{\mathrm{Exp}(H)},(S_\alpha^*)_{\mathrm{Exp}(H)})$ does not vanish. Therefore $\mathrm{Exp}(H)$ is not a geodesic point of $K\mathrm{Exp}(S^1)$ for any $H\in S^1$ and $K\mathrm{Exp}(S^1)$ has no geodesic point.

3. Compact simple Lie groups.

Let G be a connected compact simple Lie group with a bi-invariant Riemannian metric $\langle\ ,\ \rangle$. In this section we shall use calibrations to show that a certain 3-dimensional compact simple Lie subgroup G_1 of G is homologically volume minimizing in G and that the submanifold $F_p^0(G)$ is stable under variations of compact supports.

On G a calibration ϕ of degree 3 will be constructed. Let $\mathfrak{g}$ be the Lie algebra of G and take a maximal Abelian subalgebra $\mathfrak{k}$ of $\mathfrak{g}$. Define the root system Δ of $\mathfrak{g}$ with respect to $\mathfrak{k}$ like as in Section 2. Let δ be the highest root in Δ with respect to some ordering on $\mathfrak{k}$ and put

$$\phi(X,Y,Z)=\frac{1}{|\delta|}\langle[X,Y],Z\rangle$$

for X, Y, Z in $\mathfrak{g}$. By regarding an element of $\mathfrak{g}$ as a left-invariant vector field on

G, ϕ is a bi-invariant 3-form on G. In particular, ϕ is a closed form. Later on one will find that ϕ is a calibration on G. In a way similar to a proof of Lemma 1 in Section 2, we can prove the following lemma.

LEMMA 4. *There exist unit vectors E_α, F_α in $\mathfrak{g}$ for each $\alpha \in \Delta_+$ in such a way that:*

i)
$$\mathfrak{g}=\mathfrak{k}+\sum_{\alpha\in\Delta_+}\boldsymbol{R}E_\alpha+\sum_{\alpha\in\Delta_+}\boldsymbol{R}F_\alpha$$

is an orthogonal direct sum decomposition of $\mathfrak{g}$;

ii)
$$[H, E_\alpha]=\langle\alpha, H\rangle F_\alpha\,,\qquad [H, F_\alpha]=-\langle\alpha, H\rangle E_\alpha\,,\qquad [E_\alpha, F_\alpha]=\alpha$$

for $\alpha\in\Delta_+$ and $H\in\mathfrak{k}$.

Set
$$\mathfrak{g}_1=\boldsymbol{R}\delta+\boldsymbol{R}E_\delta+\boldsymbol{R}F_\delta\,,$$

then $\mathfrak{g}_1$ is a compact 3-dimensional simple Lie subalgebra of $\mathfrak{g}$. Let G_1 be the analytic subgroup of G corresponding to $\mathfrak{g}_1$. It is known that G_1 is simply connected and isomorphic to SU(2). See the proof of Theorem 5.4 in Wolf [6]. We introduce an orientation on $\mathfrak{g}_1$ such that $\{\delta, E_\delta, F_\delta\}$ is a positive basis of $\mathfrak{g}_1$.

THEOREM 5. *For each 3-dimensional oriented subspace ξ in $\mathfrak{g}$, the inequality*
$$\phi|_\xi \leqq \mathrm{vol}_\xi$$
holds. The equality holds if and only if there is an element g in G such that
$$\xi=\mathrm{Ad}\,(g)\mathfrak{g}_1$$
and that $\mathrm{Ad}\,(g):\mathfrak{g}_1\to\xi$ is orientation preserving. In particular, ϕ is a calibration on G.

PROOF. Since ϕ is $\mathrm{Ad}\,(G)$-invariant, we may assume that $\xi\cap\mathfrak{k}\neq\{0\}$. Take a positive basis $\{T, X, Y\}$ of ξ with $T\in\xi\cap\mathfrak{k}$. Put
$$X=T_0+\sum_{\alpha\in\Delta_+}s_\alpha E_\alpha+\sum_{\alpha\in\Delta_+}t_\alpha F_\alpha\,,$$
where $T_0\in\mathfrak{k}$ and $s_\alpha, t_\alpha\in\boldsymbol{R}$. By i) of Lemma 4,
$$|X|^2=|T_0|^2+\sum_{\alpha\in\Delta_+}|s_\alpha|^2+\sum_{\alpha\in\Delta_+}|t_\alpha|^2\,.$$
Owing to the formulas in ii) of Lemma 4,
$$|[T, X]|^2=\Big|\sum_{\alpha\in\Delta_+}s_\alpha\langle\alpha, T\rangle F_\alpha-\sum_{\alpha\in\Delta_+}t_\alpha\langle\alpha, T\rangle E_\alpha\Big|^2$$

$$\begin{aligned}&=\sum_{\alpha\in\Delta_+}|s_\alpha|^2|\langle\alpha, T\rangle|^2+\sum_{\alpha\in\Delta_+}|t_\alpha|^2|\langle\alpha, T\rangle|^2\\ &\leqq\sum_{\alpha\in\Delta_+}(|s_\alpha|^2+|t_\alpha|^2)|\alpha|^2|T|^2\\ &\leqq|\delta|^2\sum_{\alpha\in\Delta_+}(|s_\alpha|^2+|t_\alpha|^2)|T|^2\\ &\leqq|\delta|^2|T|^2|X|^2.\end{aligned}$$

Hence

$$|[T, X]|\leqq|\delta||T||X|. \tag{8}$$

The equality holds if and only if there is an $\alpha\in\Delta_+$ with the properties that:

$$|\alpha|=|\delta|, \qquad T\in\boldsymbol{R}\alpha, \qquad X\in\boldsymbol{R}E_\alpha+\boldsymbol{R}F_\alpha.$$

By the inequality (8),

$$\begin{aligned}|\phi(T, X, Y)|&=\frac{1}{|\delta|}|\langle[T, X], Y\rangle|\\ &\leqq|T||X||Y|.\end{aligned}$$

Therefore

$$\phi|_\xi\leqq\mathrm{vol}_\xi.$$

If the equality holds, then there is a root $\alpha\in\Delta_+$ so that

$$|\alpha|=|\delta|, \qquad T\in\boldsymbol{R}\alpha, \qquad X\in\boldsymbol{R}E_\alpha+\boldsymbol{R}F_\alpha$$

and $[T, X]//Y$, $\phi(T, X, Y)>0$. Since $X=s_\alpha E_\alpha+t_\alpha F_\alpha$,

$$[T, X]=\langle\alpha, T\rangle(s_\alpha F_\alpha-t_\alpha E_\alpha).$$

Accordingly $Y\in\boldsymbol{R}E_\alpha+\boldsymbol{R}F_\alpha$ and

$$\begin{aligned}&\xi=\boldsymbol{R}\alpha+\boldsymbol{R}E_\alpha+\boldsymbol{R}F_\alpha,\\ &\phi(\alpha, E_\alpha, F_\alpha)=|\alpha|>0,\end{aligned}$$

so $\{\alpha, E_\alpha, F_\alpha\}$ is a positive basis of ξ.

Noting that δ is the highest root and that $|\alpha|=|\delta|$, we have

$$\left|\frac{2\langle\beta, \alpha\rangle}{\langle\alpha, \alpha\rangle}\right|<2$$

for any root $\beta\in\Delta-\{\alpha, -\alpha\}$ and consequently

$$\frac{2\langle\beta, \alpha\rangle}{\langle\alpha, \alpha\rangle}=-1, 0, \text{ or } 1$$

for $\beta\in\Delta-\{\alpha, -\alpha\}$. According to Theorem 4.2 in Wolf [7], α is the highest root for some lexicographic ordering. Take an element w of the Weyl group of G with respect to $\mathfrak{k}$ which transports the fundamental Weyl chamber with respect to the old ordering to the one with respect to the new ordering and an element n of the normalizer of $\mathfrak{t}$ in G such that $\mathrm{Ad}(n)|_{\mathfrak{t}}=w$. As $\mathrm{Ad}(n)\delta=\alpha$,

$$\mathrm{Ad}(n)\mathfrak{g}_1=\xi .$$

$\mathrm{Ad}(n):\mathfrak{g}_1\to\xi$ is orientation preserving, because this map is a Lie algebra isomorphism.

THEOREM 6. *G_1 is a homologically volume minimizing submanifold in G.*

PROOF. Take ϕ as a calibration on G. By Theorem 5,

$$\phi|_{G_1}=\mathrm{vol}_{G_1} ,$$

so G_1 is homologically volume minimizing in G.

REMARK. The author does not know whether some isometry of G transports M to G_1 for any compact oriented 3-dimensional submanifold M with $[M]=[G_1]$ in $H_3(M;\boldsymbol{R})$ and $\mathrm{vol}(M)=\mathrm{vol}(G_1)$. In the quaternionic projective space $P^n(\boldsymbol{H})$, the problem is affermatively solved for $P^k(\boldsymbol{H})$ $(1\leqq k\leqq n)$ in Section 4. See Theorem 11.

THEOREM 7. *For any point p in G, $F_p^0(G)$ is minimal and stable under variations of compact supports.*

PROOF. We may assume that $p=e$. Choose an orientation of G and fix it. As G is an oriented Riemannian manifold, we can consider the Hodge star operator $*$. Let $\dim G=n$. By the definition of the Hodge star operator, $*\phi$ is also a calibration on G and for any 3-dimensional oriented plane ξ of $\mathfrak{g}$

$$\phi|_{\xi}=\mathrm{vol}_{\xi} \Longleftrightarrow *\phi|_{\xi^{\perp}}=\mathrm{vol}_{\xi^{\perp}} ,$$

where an orientation of $\xi^{\perp}$ is defined in such a way that: if $\{v_1, v_2, v_3\}$ and $\{v_4, \cdots, v_n\}$ are positive bases of ξ and $\xi^{\perp}$ respectively, then $\{v_1, \cdots, v_n\}$ is a positive basis of $\mathfrak{g}$.

Let r be the rank of G and $\{\alpha_1, \cdots, \alpha_r\}$ be the fundamental root system of Δ. Then

$$S^0=\{H\in\mathfrak{t}\,;\langle\delta, H\rangle=2\pi,\ \langle\alpha_i, H\rangle>0 \text{ for } 1\leqq i\leqq r\}$$

and

$$F_e^0(G)=\exp \mathrm{Ad}(G)S^0 .$$

As $0<\langle\alpha, H\rangle<2\pi$ for $\alpha\in\Delta_+-\{\delta\}$ and $H\in S^0$, the tangent space $T_{\exp H}(F_e^0(G))$ of $F_e^0(G)$

at $\exp H$ is given as follows:

$$T_{\exp H}(F_e^0(G))=(\exp H)_*\Big(\{X\in\mathfrak{k};\langle\delta, X\rangle=0\}+\sum_{\alpha\in\Delta_+-\{\delta\}}(\boldsymbol{R}E_\alpha+\boldsymbol{R}F_\alpha)\Big).$$

Therefore

$$*\phi|_{F_e^0(G)}=\mathrm{vol}_{F_e^0(G)}$$

for a suitable orientation on $F_e^0(G)$ and so $F_e^0(G)$ is minimal and stable under variations of compact supports, because $*\phi$ is a calibration on G.

4. Quaternionic Kähler submanifold.

Kraines has introduced a closed 4-form Ω on a quaternionic Kähler manifold in [4], which is analogous to the fundamental 2-form on a Kähler manifold. This section is devoted to showing that $\frac{1}{k!}\Omega^k$ is a calibration on a quaternionic Kähler manifold for each k, which is applied to quaternionic Kähler submanifolds.

First of all we give definitions of quaternionic Kähler manifolds and submanifolds. Let $\boldsymbol{H}$ be the quaternionic division algebra. The action of $\mathrm{Sp}(n)\times\mathrm{Sp}(1)$ to $\boldsymbol{H}^n$ is defined by

$$(A, z)x=Axz^{-1}$$

for $(A, z)\in\mathrm{Sp}(n)\times\mathrm{Sp}(1)$ and $x\in\boldsymbol{H}^n$. The action is isometric with respect to the standard inner product $\langle\ ,\ \rangle$ on $\boldsymbol{H}^n$. The image of the homomorphism from $\mathrm{Sp}(n)\times\mathrm{Sp}(1)$ to $\mathrm{SO}(\boldsymbol{H}^n)$ is denoted by $\mathrm{Sp}(n)\mathrm{Sp}(1)$. A $4n$-dimensional connected Riemannian manifold M is called a *quaternionic Kähler manifold*, if M has the following property: There is a point x in M such that, through an identification of $T_x(M)$ with $\boldsymbol{H}^n$, the linear holonomy group of M at x is contained in $\mathrm{Sp}(n)\mathrm{Sp}(1)$. Under the situation, take a piecewise smooth curve τ from x to y for any point y in M and put

$$S_y=P_\tau\,\mathrm{Sp}(1)P_\tau^{-1},$$

where P_τ is the parallel translation along the curve τ. Since $\mathrm{Sp}(1)$ is a normal subgroup of $\mathrm{Sp}(n)\mathrm{Sp}(1)$, the definition of S_y is independent of the choice of τ. We call $S=\{S_y\}_{y\in M}$ a *quaternionic structure on M.* A connected submanifold N of M is called a *quaternionic Kähler submanifold of M,* if $T_y(N)$ is invariant under the action of S_y for each y in N. Since a quaternionic Kähler submanifold N of M is totally geodesic (Alekseevskii [1]), N is also a quaternionic Kähler manifold with respect to the induced Riemannian metric.

Next we construct a closed 4-form on a quaternionic Kähler manifold due to Kraines [4]. The 2-forms Ω_I, Ω_J, and Ω_K on $\boldsymbol{H}^n$ are defined by

$$\Omega_I(X, Y)=\langle Xi, Y\rangle, \qquad \Omega_J(X, Y)=\langle Xj, Y\rangle$$

and

$$\Omega_K(X, Y)=\langle Xk, Y\rangle.$$

Kraines has proved that the 4-form

$$\Omega=\frac{1}{6}(\Omega_I\wedge\Omega_I+\Omega_J\wedge\Omega_J+\Omega_K\wedge\Omega_K)$$

on $\boldsymbol{H}^n$ is invariant under the action of $\mathrm{Sp}(n)\mathrm{Sp}(1)$. So we can extend Ω to a parallel 4-form on a quaternionic Kähler manifold, which is also denoted by Ω. The 4-form Ω is a closed form. We call Ω the *fundamental* 4-*form* on a quaternionic Kähler manifold. The reason of the multiplication by 1/6 in the definition of Ω is as follows. According to Wirtinger's inequality,

$$\frac{1}{2}\Omega_I^2|_\xi\leqq\mathrm{vol}_\xi, \qquad \frac{1}{2}\Omega_J^2|_\xi\leqq\mathrm{vol}_\xi, \qquad \frac{1}{2}\Omega_K^2|_\xi\leqq\mathrm{vol}_\xi$$

for any oriented 4-plane ξ in $\boldsymbol{H}^n$. Hence

$$\Omega|_\xi\leqq\mathrm{vol}_\xi.$$

The equality holds if and only if ξ is a $\mathrm{Sp}(1)$-invariant plane and has an orientation such that $\{v, vi, vk, vj\}$ is a positive basis of ξ for nonzero $v\in\xi$.

Now we shall show an inequality of Ω^m similar to the above inequality, which is analogous to Wirtinger's inequality on a Kähler manifold.

THEOREM 8. *Let M be a $4n$-dimensional quaternionic Kähler manifold with quaternionic structure S and fundamental 4-form Ω, then,*

$$\frac{1}{m!}\Omega^m|_\xi\leqq\mathrm{vol}_\xi$$

for each oriented tangent $4m$-plane ξ on M and $1\leqq m\leqq n$. The equality holds if and only if ξ is an S-invariant plane with such an orientation that $\{v_1, v_1i, v_1k, v_1j, \cdots, v_m, v_mi, v_mk, v_mj\}$ is a positive basis of ξ for some $v_1, \cdots, v_m$ in ξ. In particular, $\frac{1}{m!}\Omega^m$ is a calibration on M.

PROOF. It is sufficient to prove the inequality of Ω^m on $\boldsymbol{H}^n$. Define the action of $\mathrm{Sp}(1)$ on the space $\Lambda^{4m}(\boldsymbol{H}^n)$ of real $4m$-forms on $\boldsymbol{H}^n$ by

$$(z^*\Phi)(X_1, \cdots, X_{4m})=\Phi(X_1z, \cdots, X_{4m}z)$$

for $z\in\mathrm{Sp}(1)$, $\Phi\in\Lambda^{4m}(\boldsymbol{H}^n)$, and $X_1, \cdots, X_{4m}\in\boldsymbol{H}^n$. Let $\int_{\mathrm{Sp}(1)}$ be the invariant measure on Sp(1) with $\int_{\mathrm{Sp}(1)} 1=1$ and consider the form

$$\Psi=\int_{z\in \mathrm{Sp}(1)} z^*\Omega_I^{2m}.$$

The form Ω_I is invariant under the action of Sp(n) and the action of Sp(1) commutes with the one of Sp(n), so Ψ is Sp(n)-invariant. By the definition of Ψ, Ψ is Sp(1)-invariant. Therefore Ψ is Sp(n) Sp(1)-invariant. Since the space of Sp(n) Sp(1)-invariant $4m$-forms on $\boldsymbol{H}^n$ is generated by Ω^m, there is a real number c such that

$$\Psi=c\Omega^m.$$

We estimate the form Ψ. For $X_1, \cdots, X_{4m}\in \boldsymbol{H}^n$,

$$\Psi(X_1, \cdots, X_{4m})=\int_{z\in \mathrm{Sp}(1)} \Omega_I^{2m}(X_1 z, \cdots, X_{4m} z),$$

so by Wirtinger's inequality

$$\begin{aligned}|\Psi(X_1, \cdots, X_{4m})| &\leqq \int_{z\in \mathrm{Sp}(1)} |\Omega_I^{2m}(X_1 z, \cdots, X_{4m} z)| \\ &\leqq (2m)!\int_{z\in \mathrm{Sp}(1)} |X_1 z|\cdots|X_{4m} z| \\ &\leqq (2m)!|X_1|\cdots|X_{4m}|.\end{aligned}$$

Hence for any oriented $4m$-plane ξ in $\boldsymbol{H}^n$

$$\Psi|_\xi \leqq (2\mathrm{m})!\,\mathrm{vol}_\xi$$

and the equality holds if and only if ξz is invariant under the right multiplication by i and has a suitable orientation for any $z\in$ Sp(1). In order to simplify the condition we prepare the following lemma.

LEMMA 9. *Let V be a $4m$-dimensional real vector subspace of $\boldsymbol{H}^n$. Then the following conditions are equivalent.*

i) *V is* Sp(1)*-invariant.*

ii) *Vz is $U(1)$-invariant for each $z\in$* Sp(1), *where* $U(1)=\{x\in \boldsymbol{R}+\boldsymbol{R}i\,;\,|x|=1\}$.

PROOF. It is obvious that i) implies ii). So assume $Vz_0\neq V$ for some $z_0\in$ Sp(1). Since $U(1)$ is a maximal torus of Sp(1), there exists $z_1\in$ Sp(1) such that $z_2=z_1^{-1}z_0 z_1\in U(1)$. Then $Vz_1 z_2\neq Vz_1$. Thus the lemma is proved.

According to the lemma,

$$\Psi|_\xi=(2m)!\,\mathrm{vol}_\xi$$

if and only if ξ is Sp(1)-invariant and has a positive basis $\{v_1, v_1 i, v_1 k, v_1 j, \cdots, v_m, v_m i, v_m k, v_m j\}$ for some $v_1, \cdots, v_m$ in ξ. When the above equality holds for ξ,

$$\begin{aligned}(2m)!\,\mathrm{vol}_\xi &= c\Omega^m|_\xi \\ &= cm!\,\mathrm{vol}_\xi .\end{aligned}$$

Therefore

$$c=\frac{(2m)!}{m!}, \qquad \Omega^m=\frac{m!}{(2m)!}\int_{z\in \mathrm{Sp}(1)} z^*\Omega_I^{2m},$$

and the theorem has been proved.

THEOREM 10. *Let M be a quaternionic Kähler manifold and N be a $4m$-dimensional quaternionic Kähler submanifold of M. If N is compact, then*

$$\mathrm{vol}(N)\leqq \mathrm{vol}(N')$$

for any compact oriented $4m$-dimensional submanifold N' such that $[N]=[N']$ in the homology group $H_{4m}(M;\boldsymbol{R})$. The equality holds if and only if N' is also a quaternionic Kähler submanifold of M. If N is noncompact, N is stable under variations of compact supports.

PROOF. Take $\frac{1}{m!}\Omega^m$ as a calibration on M. The theorem follows from the explanation of calibrations in Introduction and Theorem 8.

Now we show a theorem on $P^n(\boldsymbol{H})$ stronger than Theorem 10.

THEOREM 11. *Let N be a $4m$-dimensional oriented compact submanifold of $P^n(\boldsymbol{H})$ such that $[N]=[P^m(\boldsymbol{H})]$ in $H_{4m}(M;\boldsymbol{R})$. Then*

$$\mathrm{vol}(P^m(\boldsymbol{H}))\leqq \mathrm{vol}(N)$$

and the equality holds if and only if N is congruent with $P^m(\boldsymbol{H})$ in $P^n(\boldsymbol{H})$.

PROOF. If

$$\mathrm{vol}(P^m(\boldsymbol{H}))=\mathrm{vol}(N),$$

then N is a quaternionic Kähler submanifold. As mentioned above N is a totally geodesic submanifold. Without loss of generality we may assume that $N\cap P^m(\boldsymbol{H})\neq\phi$. Take $x\in N\cap P^m(\boldsymbol{H})$. Since $T_x(N)$ and $T_x(P^m(\boldsymbol{H}))$ are S_x-invariant, there is and isometry g of $P^n(\boldsymbol{H})$ such that $g_*T_x(N)=T_x(P^m(\boldsymbol{H}))$. Both of N and $P^m(\boldsymbol{H})$ are totally geodesic, so $gN=P^m(\boldsymbol{H})$.

REMARK. $P^{n-1}(\boldsymbol{H})$ is the first conjugate locus of $P^n(\boldsymbol{H})$. So the first conjugate locus of $P^n(\boldsymbol{H})$ is homologically volume minimizing. But similar facts for other quaternionic symmetric spaces, which are classified by Wolf [6], does not hold. In fact, the codimension of the first conjugate locus of another quaternionic symmetric space is not equal to 4.

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Added in proof. The problem for a compact simple Lie group remarked below Theorem 6 has been affermatively solved by Ohnita and the author in the forthcoming paper entitled "Uniqueness of certain 3-dimensional homologically volume minimizing submanifolds in compact simple Lie groups."