META-ABELIANIZATIONS OF $SL(2, Z[\frac{1}{p}])$ AND DENNIS-STEIN SYMBOLS

By
Jun Morita

Abstract. Using a Dennis-Stein symbol, we will study $K_2(2, Z[\frac{1}{p}])$ and the meta-abelianization of $SL(2, Z[\frac{1}{p}])$.

1. Introduction.

Let $Z$ be the ring of rational integers. For a given group $G$, we denote by $G'$ the commutator subgroup $[G, G]$ of $G$, and by $G''$ the second commutator subgroup $[G', G']$ of $G$. Then we put $G^{ab} = G / G'$, the abelianization of $G$, and $G^{mab} = G / G''$, the meta-abelianization of $G$. The cyclic group of order $m$ is denoted by $Z_m$, and the cyclic group of infinite order is denoted by $Z$ instead of $Z_\infty$. And, the semi-direct product $H = K \ltimes L$ of groups means $H = \langle K, L \rangle$, $K \cap L = 1$, and $H \triangleright L$. Then we will obtain the following results.

**THEOREM 1.** Let $p$ be a prime number. Then

$$SL(2, Z[\frac{1}{p}])^{mab} = \begin{cases} Z_3 \times (Z_2 \times Z_2) & p = 2; \\ Z_4 \times Z_3 & p = 3; \\ Z_{12} \times (Z_2 \times Z_6) & p \geq 5. \end{cases}$$

**COROLLARY.** Suppose $p \geq 5$. Then

$$SL(2, Z[\frac{1}{p}])^{mab} = SL(2, Z[\frac{1}{2}])^{mab} \times SL(2, Z[\frac{1}{3}])^{mab}.$$ 

**THEOREM 2.**

(1) Suppose $p = 2, 3$. Then $K_2(2, Z[\frac{1}{p}]) = Z \times Z_{p-1}$, and $K_2(2, Z[\frac{1}{p}])$ is central.

(2) Suppose $p \geq 5$. Then $K_2(2, Z[\frac{1}{p}]) \triangleright Z \times Z$, and $K_2(2, Z[\frac{1}{p}])$ is not central.

There is an algorithm to get a finite presentation of $SL(2, Z[\frac{1}{p}])$. Therefore, it might be possible to calculate the meta-abelianization of $SL(2, Z[\frac{1}{p}])$ when $p$ is

Received March 1, 1994
given. However, the main difficulty is that one cannot expect a uniform presentation of \( SL(2, \mathbb{Z}[\frac{1}{p}]) \) for all \( p \) (cf. [4]). Here we will find some element \( d(a,b) \), called a Dennis-Stein symbol, in \( K_2(2, \mathbb{Z}[\frac{1}{p}]) \) which leads to Theorem 1 as well as Theorem 2. Corollary can be also obtained from the result in [9].

This research was partially supported by SFB in Bielefeld 1992.

1. \( K_2(2, A) \) and symbols.

For a commutative ring, \( A \), with 1, we define the Steinberg group of rank one, called \( St(2, A) \), by generators: \( x_{12}(t) \), \( x_{21}(t) \) for \( t \in A \) and defining relations:

\[
x_{ij}(s)x_{ij}(t) = x_{ij}(s + t)
\]

and

\[
x_{ij}(u)x_{ij}(-u^{-1})x_{ij}(t)x_{ij}(u^{-1})x_{ij}(-u) = x_{ij}(-u^{-2}t)
\]

for \( s, t \in A \), \( u \in A^\times \) and \( {i, j} = \{1, 2\} \), where \( A^\times \) is the unit group of \( A \) (cf. [2], [5]). Then, there is a natural homomorphism, \( \pi \), of \( St(2, A) \) into \( SL(2, A) \) with

\[
\pi x_{12}(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \pi x_{21}(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.
\]

Put

\[
K_2(2, A) = \ker[\pi : St(2, A) \to SL(2, A)].
\]

Now we define several elements in \( St(2, A) \). Set

\[
w_{ij}(u) = x_{ij}(u)x_{ij}(-u^{-1})x_{ij}(u),
\]

\[
h_{ij}(u) = w_{ij}(u)w_{ij}(-1),
\]

\[
c(u, v) = h_{12}(uv)h_{12}(u)^{-1}h_{12}(v)^{-1},
\]

\[
d(a, b) = x_{21}(bu^{-1})x_{12}(-a)x_{21}(b)x_{12}(au^{-1})h_{12}(u)^{-1}
\]

for \( a, b \in A \), \( u, v \in A^\times \), \( {i, j} = \{1, 2\} \) with \( 1 - ab = u \). Then

\[
\pi w_{12}(u) = \begin{pmatrix} 0 & u \\ -u^{-1} & 0 \end{pmatrix}, \quad \pi w_{21}(u) = \begin{pmatrix} 0 & -u^{-1} \\ u & 0 \end{pmatrix},
\]

\[
\pi h_{12}(u) = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \quad \pi h_{21}(u) = \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix},
\]

and \( c(u, v), d(a, b) \in K_2(2, A) \). The symbols \( c(u, v) \) and \( d(a, b) \) are called Steinberg symbols and Dennis-Stein symbols respectively.
2. The case of $\mathbb{Z}_{1/2}$.

First, we shall recall that $\text{St}(2,\mathbb{Z})$ is isomorphic to the 3-braid group, $B_3 = \langle x, y | xyx = yxy \rangle$. Hence, we see $\text{St}(2,\mathbb{Z})^{\text{mab}} = \mathbb{Z} \ltimes (\mathbb{Z} \times \mathbb{Z})$. Since $K_2(2,\mathbb{Z}) = \langle c(-1,-1) \rangle = \mathbb{Z}$ and $c(-1,-1)$ is corresponding to $x^{12} \equiv 1 \mod B_3^{\nu}$ (cf. [5], [8]), we obtain $\text{SL}(2,\mathbb{Z})^{\text{mab}} = \mathbb{Z} \times (\mathbb{Z} \times \mathbb{Z})$.

Now we take a prime number $p$ and consider $A = \mathbb{Z}_{1/p}$. For each $p$, we define the group $G_p$ by the generators $x_1, x_2, y_1, y_2$ and the defining relations

$$\begin{align*}
(x_1, y_2, x_2)^2, y_1] &= [(x_2, y_2, x_2)^2, y_1] = 1.
\end{align*}$$

Then $\text{St}(2,\mathbb{Z}_{1/p}) = G_p$ (cf. [7]). In [8], we already confirmed that

$$\text{St}(2,\mathbb{Z}_{1/p})^{\text{mab}} = \begin{cases} 
\mathbb{Z}_3 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2) & p = 2; \\
\mathbb{Z}_5 \ltimes \mathbb{Z}_3 & p = 3; \\
\mathbb{Z}_{p^2-1} \ltimes (\mathbb{Z} \times \mathbb{Z}) & p \geq 5.
\end{cases}$$

To get this, we constructed $M_p$ as follows:

$$\begin{align*}
M_2 &= \langle \sigma, \tau_1, \tau_2 | \sigma^3 = \tau^2 = \tau_1^2 = 1, \sigma \tau_1, \sigma^{-1} = \tau_1, \tau_2, \sigma \tau_1 \sigma^{-1} = \tau_1 \rangle, \\
M_3 &= \langle \sigma, \tau | \sigma^8 = \tau^3 = 1, \sigma \tau \sigma^{-1} = \tau_1 \rangle, \\
M_p &= \langle \sigma, \tau_1, \tau_2 | \sigma^{p-1} = \tau_1, \tau_2 = 1, \sigma \tau_1, \sigma^{-1} = \tau_1, \tau_2^{-1}, \sigma \tau_1 \sigma^{-1} = \tau_1 \rangle
\end{align*}$$

with $p \geq 5$. Then we obtain $G_p^{\text{mab}} = M_p$ for every $p$, which gives the explicite group structure of $\text{St}(2,\mathbb{Z}_{1/p})^{\text{mab}}$ as above. In fact, we can easily see that there is a group homomorphism $\alpha_p$ of $G_p$ onto $M_p$ satisfying

$$\begin{align*}
\alpha_2(x_1) &= \sigma \tau_1, \alpha_2(y_1) = \sigma, \alpha_2(x_2) = \sigma^2, \alpha_2(y_2) = \sigma^2 \tau_1 \tau_2 \quad (p = 2), \\
\alpha_3(x_1) &= \sigma \tau, \alpha_3(y_1) = \sigma, \alpha_3(x_2) = \sigma^3, \alpha_3(y_2) = \sigma^3 \tau \quad (p = 3), \\
\alpha_p(x_1) &= \sigma \tau_1, \alpha_p(y_1) = \sigma, \alpha_p(x_2) = \sigma^p, \alpha_p(y_2) = \sigma^p \tau_1 \quad (p \equiv 1 \mod 6), \\
\alpha_p(x_1) &= \sigma \tau_1, \alpha_p(y_1) = \sigma, \alpha_p(x_2) = \sigma^p, \alpha_p(y_2) = \sigma^p \tau_1 \tau_2 \quad (p \equiv 5 \mod 6).
\end{align*}$$

This map $\alpha_p$ induces an isomorphism of $G_p^{\text{mab}}$ onto $M_p$.

If $p = 2$, then $K_2(2,\mathbb{Z}_{1/2})$ is generated by $c(-1,-1)$ (cf. [11]), and $c(-1,-1)$ is corresponding to $1 \in M_2$. Therefore, $\text{SL}(2,\mathbb{Z}_{1/2})^{\text{mab}} = \mathbb{Z} \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$. If $p = 3$, then $K_2(2,\mathbb{Z}_{1/3})$ is generated by $c(-1,-1)$ and $c(3,-1)$ (cf. [11]), which are corresponding to $1 \in M_3$ and $\sigma^2 \in M_3$ respectively. Therefore, $\text{SL}(2,\mathbb{Z}_{1/3})^{\text{mab}} = (\mathbb{Z}_2 \ltimes \mathbb{Z}_3)$. 
Next suppose $p \geq 5$. Then we will choose some Dennis-Stein symbols, and consider their images in $M_p$. Note that if $1 - ab = \pm p$, then $d(a, b) = w_{12}(1)x_{12}(\pm bp^{-1})w_{12}(-1)x_{12}(-a)x_{12}(\pm ap^{-1})w_{21}(\pm p)w_{12}(1) \in \text{St}(2, \mathbb{Z}[\frac{1}{p}])$, which is corresponding to
\[
e(a, b) = (x_1, y_1, x_1^2, y_1^2, x_1 x_2, y_1 y_2, x_1 y_1, x_1)
\in G_p.
\]
Since $y_j^{p-1} \equiv 1 \mod G''_p$ (cf. [8]), then $y_1 \equiv x_1^p \mod G''_p$ and $y_2 \equiv x_1^p \mod G''_p$. Hence, $e(a, b) \equiv x_2^{2b}x_1^{-b}x_2^{2a}(x_2y_2x_2)^{2l}(x_1y_1x_1) \mod G''_p$.

If $p = 6k + 1, k = 2'm, (2, m) = 1$, then
\[
\alpha_p e(-2^{l+1}, 3m) \\
= (\sigma^p)^{3m}(\sigma^l \tau_1)^{-2^{l+1}} \sigma^{-3m}(\sigma^l \tau_1)^{-2^{l+1}}(\sigma^p \sigma^l \tau_1 \sigma^p)^{-1}(\sigma \tau_1 \sigma \tau_1) \\
= \sigma^{-3m} \sigma^{-3m} \rho \sigma^{-3m} \rho^{-1} \sigma^{-2^{l+1}} \rho \sigma^{-3m} \rho^{-1} \\
= \sigma^{-3m} \rho \sigma^{-3m} \rho^{-1} \sigma^{-2^{l+1}} \rho \sigma^{-3m} \rho^{-1} \\
= \sigma^{-3m} \rho \sigma^{-3m} \rho^{-1} \sigma^{-2^{l+1}} \rho \sigma^{-3m} \rho^{-1}
\]
where $(\sigma \tau_1)^{2^{l+1}} = \sigma^{-2^{l+1}} \rho, \rho = \tau_1 \tau_2$ or $\tau_1^{-1} \tau_2 \rho$, and $\rho' \in \{\tau_1^{-2} \tau_2, \tau_1^{-2} \tau_2 \tau_1, \tau_1 \tau_2 \tau_1 \}$. In particular, the order of $d(-2^{l+1}, 3m)$ is infinite.

If $p = 6k - 1, k = 2'm, (2, m) = 1$, then
\[
\alpha_p e(2^{l+1}, 3m) \\
= (\sigma^p)^{3m}(\sigma^l \tau_1)^{-2^{l+1}} \sigma^{-3m}(\sigma^l \tau_1^{-1} \tau_1)^{-2^{l+1}}(\sigma^p \sigma^l \tau_1 \sigma^p)^{-1}(\sigma \tau_1 \sigma \tau_1) \\
= \sigma^{-3m} \sigma^{-2^{l+1}} \rho \sigma^{-3m} \rho^{-1} \sigma^{-2^{l+1}} \rho \sigma^{-3m} \rho^{-1} \\
= \sigma^{-3m} \rho \sigma^{-3m} \rho^{-1} \sigma^{-2^{l+1}} \rho \sigma^{-3m} \rho^{-1} \\
= \sigma^{-3m} \rho \sigma^{-3m} \rho^{-1} \sigma^{-2^{l+1}} \rho \sigma^{-3m} \rho^{-1}
\]
where $(\sigma \tau_1)^{2^{l+1}} = \sigma^{-2^{l+1}} \rho, \rho = \tau_1 \tau_2$ or $\tau_1 \tau_2 \rho$, and $\rho' \in \{\tau_1^{-2} \tau_2, \tau_1 \tau_2 \tau_1, \tau_1 \tau_1^{-2} \tau_2 \tau_1 \}$. In particular, also in this case, the order of $d(2^{l+1}, 3m)$ is infinite.

**Proof of Theorem 1.** For $p = 2, 3$, we already discussed completely. Suppose $p \geq 5$. Then the homomorphism $\pi$ of $\text{St}(2, \mathbb{Z}[\frac{1}{p}])$ onto $\text{SL}(2, \mathbb{Z}[\frac{1}{p}])$ induces the homomorphism, called $\bar{\pi}$, of $M_p$ onto $\text{SL}(2, \mathbb{Z}[\frac{1}{p}])^{\text{mob}}$. Since $\sigma^{12}, \tau_1^6, \tau_2^1 \tau_2^2 \in \ker \bar{\pi}$ as above, we obtain a homomorphism of $\mathbb{Z}_{12} \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_6)$ onto $\text{SL}(2, \mathbb{Z}[\frac{1}{p}])^{\text{mob}}$. On the other hand, we see that $\text{PSL}(2, \mathbb{Z}/3\mathbb{Z}) \simeq \mathbb{Z}_4 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_3)$ (cf. [3]) and $\text{SL}(2, \mathbb{Z}/14\mathbb{Z}) \simeq \mathbb{Z}_4 \ltimes \mathbb{Z}_4$ (cf. Section 3). Hence, $\text{SL}(2, \mathbb{Z}[\frac{1}{p}])^{\text{mob}} \simeq \mathbb{Z}_{12} \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_6)$. \(\square\)
Meta-Abelianization of $SL(2, \mathbb{Z}[1/p])$

Considering the action of $\sigma$ in $M_p$, one reaches Corollary easily. And, the result in Theorem 2(1) is already known (cf. [1], [5], [6], [7]).

**Proof of Theorem 2(2).** In $K_2(2, \mathbb{Z}[\frac{1}{p}])$, we have found three elements

\[ c_1 = c(-1,-1), \quad c_2 = c(p,-1), \quad d = d(\mp 2^{i+1}, 3m) \]

as before, where $p = 6k \pm 1$. Let $L$ be the subgroup of $K_2(2, \mathbb{Z}[\frac{1}{p}])$ generated by $c_1, c_2, d$. Then, $L$ is abelian, and $d$ is not central and of infinite order in $St(2, \mathbb{Z}[\frac{1}{p}])$ $St(2, \mathbb{Z}[\frac{1}{p}])$ by the structure of $M_p$. Therefore, $c_1^{n_1} c_2^{n_2} d^{n_3} = 1$ with $n_1, n_2, n_3 \in \mathbb{Z}$ implies $n_3 = 0$ and $c_1^{n_1} c_2^{n_2} = 1$. Then, since the image of $c_1$ (resp. $c_2$) in the stable $K_2$ over the field of real numbers is of infinite order (resp. trivial), $n_i$ must be 0 (cf. [5]). Hence, $L = \langle c_1, c_2, d \rangle = Z \times Z_n \times Z$, where $n$ is the order of $c_2$ and $\geq 2$. □

In particular, for every $p \geq 5$, we get $K_2(2, \mathbb{Z}[\frac{1}{p}]) \neq Z \times Z_{p-1}$ (cf. [8; Theorem 9]).

### 3. Some remarks around $SL(2, \mathbb{Z}/4\mathbb{Z})$

The group $SL(2, \mathbb{Z}/4\mathbb{Z})$ is generated by

\[ r_1 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad r_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad s = \begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix}, \]

and the subgroup generated by $r_1, r_2$ (resp. $s$) is isomorphic to $\mathfrak{A}_4$ (resp. $Z_4$). Hence, we see

(1) $SL(2, \mathbb{Z}/4\mathbb{Z}) \cong \mathfrak{A}_4 \times Z_4$.

In particular, $SL(2, \mathbb{Z}/4\mathbb{Z})^{mod} \cong Z_4 \times Z_3$. Furthermore, by some easy and routine calculation, we obtain the following as an appendix:

(2) $GL(2, \mathbb{Z}/4\mathbb{Z}) = GL(2, F_2[\xi]/(\xi^3)) \cong \mathfrak{S}_3 \times (Z_2)^4 \cong Z_3 \times (\mathfrak{S}_2 \times Z_2)$,

(3) $PGL(2, \mathbb{Z}/4\mathbb{Z}) = PGL(2, F_2[\xi]/(\xi^3)) \cong GL(2, F_2[\xi]/(\xi^3)) \cong \mathfrak{S}_4 \times Z_2$,

(4) $PSL(2, \mathbb{Z}/4\mathbb{Z}) = PSL(2, F_2[\xi]/(\xi^3)) \cong \mathfrak{S}_4$.

### References


Institute of Mathematics,
University of Tsukuba,
Tsukuba, Ibaraki, 305, Japan