ARCWISE CONNECTEDNESS OF THE COMPLEMENT IN A HYPERSPACE

By
Hiroshi HOSOKAWA

Abstract. The hyperspace $C(X)$ of a continuum $X$ is always arcwise connected. In [6], S. B. Nadler Jr. and J. Quinn show that if $C(X)-\{A_i\}$ is arcwise connected for each $i = 1, 2$, then $C(X)-\{A_1, A_2\}$ is also arcwise connected. Nadler raised questions in his book [5]: Is it still true with the two sets $A_1$ and $A_2$ replaced by $n$ sets, $n$ finite? What about countably many? What about a collection $\{A_\lambda : \lambda \in \Lambda\}$ which is a compact zero-dimensional subset of the hyperspace? In this paper we prove that if $\mathcal{A} \subset C(X)$ is a closed countable subset, $\mathcal{U}$ is an arc component of an open set of $C(X)$ and $C(X)-\{A\}$ is arcwise connected for each $A \in \mathcal{A}$, then $\mathcal{U}-\mathcal{A}$ is arcwise connected.

Key words and phrases: continuum, hyperspace, order arc, Whitney map, arcwise connectedness, indecomposable continuum, decomposable continuum.

AMS subject classifications (1980): 54B20, 54C05.

1. Notation and Preliminary Lemmas

A continuum is a nonempty compact connected metric space. The letter $X$ will always denote a nondegenerate continuum with a metric function $d$. Let $Y$ be a subcontinuum of $X$ and $\varepsilon$ a positive number. The set $N(Y; \varepsilon)$ denotes the $\varepsilon$-neighborhood of $Y$ in $X$, i.e., $N(Y; \varepsilon) = \{x \in X : d(x, y) < \varepsilon$ for some $y \in Y\}$ and $Y_\varepsilon$ denotes the component of the closure of $N(Y; \varepsilon)$ containing $Y$. The hyperspace $C(X)$ of $X$ is the space of all subcontinuum of $X$ with the Hausdorff metric $H_d$ defined by

$$H_d(A, B) = \inf\{\varepsilon > 0 : A \subset N(B; \varepsilon) \text{ and } B \subset N(A; \varepsilon)\}.$$ 

With this metric, $C(X)$ becomes a continuum. If $Y$ is a subcontinuum of $X$, then
we consider $C(Y)$ as a subspace of $C(X)$. For two subsets $\mathcal{A}$ and $\mathcal{B}$ of $C(X)$, let $H_d(\mathcal{A}, \mathcal{B}) = \inf \{ H_d(A, B) : A \in \mathcal{A} \text{ and } B \in \mathcal{B} \}$. A map is a continuous function. Any map $\mu : C(X) \to [0, 1]$ satisfying

1. if $A \subseteq B$ and $A \neq B$, then $\mu(A) < \mu(B)$,
2. $\mu(\{x\}) = 0$ for each $x \in X$ and $\mu(X) = 1$

is called a Whitney map for $C(X)$. Such a map always exists (see [7]). An order arc is a map $\sigma : [a, b] \to C(X)$ such that if $a \leq t_0 < t_1 \leq b$, then $\sigma(t_0) \subset \sigma(t_1)$ and $\sigma(t_0) \neq \sigma(t_1)$. It is also called an order arc from $\sigma(a)$ to $\sigma(b)$.

If $A, B$ are distinct elements of $C(X)$, then there is an order arc from $A$ to $B$ if and only if $A \subset B$ (see [1]).

We often use the following lemmas which are easy to prove hence we omit their proofs.

**Lemma 1.** Let $Y$ be a proper subcontinuum of $X$. If there is a subcontinuum $M$ of $X$ such that $M \cap Y \neq \emptyset \neq M - Y$, then for any $\varepsilon > 0$ and $y \in M \cap Y$, there is a subcontinuum $N$ of $M \cap Y$ such that $N \cap Y \neq \emptyset \neq N - Y$ and $y \in N$.

The diameter of a subset $A$ of $X$ is denoted by $\delta(A)$, i.e., $\delta(A) = \sup \{ d(x, y) : x, y \in A \}$.

**Remark.** If $\mathcal{A}$ is a connected subset of $C(X)$ such that $Y \in \mathcal{A}$ and $\delta(\mathcal{A}) \leq \varepsilon$, then $\mathcal{A} \subset C(Y_\varepsilon)$.

**Lemma 2.** If a subset $\{A, B, C, D\} \subset C(X)$ satisfies $A \subset B \cap C \subset B \cup C \subset D$, then $H_d(B, C) \leq H_d(A, D)$. In particular, if $\sigma$ is an order arc, then $\delta(\sigma([a, b])) = H_d(\sigma(a), \sigma(b))$.

Furthermore we need the following Krasinkiewiz-Nadler's Theorem (Theorem 3.1 of [2]).

**Proposition 3.** Let $\mu : C(X) \to [0, 1]$ be a Whitney map and $A_1, A_2 \in \mu^{-1}(t_0)$, where $t_0 \in [0, 1]$. Let $K$ be a subcontinuum of $A_1 \cap A_2$. Then there is a map $\alpha : [0, 1] \to \mu^{-1}(t_0) \cap C(A_1 \cup A_2)$ such that $\alpha(0) = A_1$, $\alpha(1) = A_2$ and $K \subset \alpha(t)$ for all $t \in [0, 1]$. If $A_1 \neq A_2$, then $\alpha$ can be taken to be an embedding.

In fact Theorem 3.1 of [2] is much more general, and from its proof we obtain the following lemma.

**Lemma 4.** Let $\mu : C(X) \to [0, 1]$ be a Whitney map and let $A, B, C$ be
subcontinua of $X$ such that $A \cap B \supset C$. Then there is a map $\alpha : [0, 1] \rightarrow \mu^{-1}(\mu(A) \cap C(A \cup B))$ such that $\alpha(0) = A, \alpha(t) \supset C$ for each $t \in [0, 1]$, and if $\mu(A) \leq \mu(B)$ then $\alpha(1) \subset B$, and if $\mu(A) > \mu(B)$ then $\alpha(1) \supset B$.

In the same paper they proved (Theorem 3.5 in [2]) that:

**Proposition 5.** Let $X$ be decomposable and $\mu : C(X) \rightarrow [0, 1]$ a Whitney map. Then there is $s_0 \in [0, 1]$ such that if $s \in [s_0, 1]$, then $\mu^{-1}(s)$ is arcwise connected.

The following proposition is Theorem 4.6 of [4].

**Proposition 6.** If $Y$ is a non-degenerate proper subcontinuum of $X$, then the following two statements are equivalent:

1. $C(X) - \{Y\}$ is not arcwise connected.
2. There is a dense subset $D$ of $Y$ such that if $M$ is a subcontinuum of $X$ satisfying $M \cap D \neq \emptyset \neq M - Y$, then $M \supset Y$.

**2. Bypass Lemma**

Let $K, L \in C(X)$ and $\mathcal{A} \subset C(X)$. An arc from $K$ to $L$ in $\mathcal{A}$ is a map $\alpha : [a, b] \rightarrow \mathcal{A}$ such that $\alpha(a) = K$ and $\alpha(b) = L$. If $\alpha$ is an embedding, then we call it an embedding arc. Following is a key lemma.

**Lemma 7.** Let $Y$ be a non-degenerate proper subcontinuum of a continuum $X$ such that $C(X) - \{Y\}$ is arcwise connected. Let $\alpha : [0, 1] \rightarrow C(X)$ be a map such that $\alpha(1) = Y$ and $\alpha(t) \in C(Y) - \{Y\}$ for each $t \in [0, 1]$. Then for a given $\varepsilon > 0$, there is a map $\beta : [0, 1] \rightarrow C(X) - \{Y\}$ such that $\alpha(0) = \beta(0)$, $H_\varepsilon(\alpha(t), \beta(t)) < \varepsilon$ for each $t \in [0, 1]$ and $\beta(1) - Y \neq \emptyset$.

**Proof.** First suppose that $Y$ is indecomposable. Put $\varepsilon_1 = \varepsilon / 3$. Since $\alpha$ is continuous, there is $t_0 \in [0, 1)$ such that $\delta(\alpha((t_0, 1])) < \varepsilon_1$. Let $\lambda$ be the composant of $Y$ such that $\alpha(t_0) \subset \lambda$. By Lemma 1 and Proposition 6, there is a subcontinuum $M$ of $Y_{t_0}$ such that $M - Y \neq \emptyset \neq Y - M$ and $M \cap \lambda \neq \emptyset$. We may assume that $M \cap \alpha(t_0) \neq \emptyset$. (Because let $\lambda'$ be a composant of $Y$ different from $\lambda$. Since $M$ is compact and $Y - M \neq \emptyset, \lambda' - M \neq \emptyset$. Thus we can replace $M$ by $M \cup N$, where $N$ is a continuum contained in $\lambda$ such that $M \cap N \neq \emptyset \neq N \cap \alpha(t_0)$.) Let $\sigma : [t_0, 1] \rightarrow C(X)$ be an order arc from $\alpha(t_0)$ to $M \cup \alpha(t_0)$. Then
\[ \delta(\sigma([t_0, 1])) = H_d(\alpha(t_0), M \cup \alpha(t_0)) \leq H_d(\alpha(t_0), Y) \]
\[ \leq H_d(\alpha(t_0), Y) + H_d(Y, Y) < 2\varepsilon. \]

Define an arc \( \beta \) in \( C(X) \) by
\[
\beta(t) = \begin{cases} 
\alpha(t) & \text{if } t \in [0, t_0], \\
\sigma(t) & \text{if } t \in (t_0, 1].
\end{cases}
\]

Clearly \( \beta \) is continuous and its image does not contain \( Y \). If \( t \in [0, t_0] \), then \( H_d(\alpha(t), \beta(t)) = 0 \). Suppose that \( t \in (t_0, 1] \). Then since \( \alpha(t_0) = \beta(t_0) \),
\[ H_d(\alpha(t), \beta(t)) \leq H_d(\alpha(t), \alpha(t_0)) + H_d(\beta(t_0), \beta(t)) \]
\[ \leq \delta(\alpha([t_0, 1])) + \delta(\sigma([t_0, 1])) < 3\varepsilon = \varepsilon. \]

For the second case, suppose that \( Y \) is decomposable. Put \( \varepsilon_1 = \varepsilon / 5 \) and let \( \mu \) be a Whitney map for \( C(X) \). By Proposition 5, there is \( s_0 < \mu(Y) \) such that if \( s \in [s_0, \mu(Y)] \), then \( \mu^{-1}(s) \cap C(Y) \) is arcwise connected. Moreover \( s_0 \) can be taken so that \( \delta(\mu^{-1}([s_0, 1]) \cap C(Y)) < \varepsilon_1 \). Since \( \alpha \) is continuous, there is \( t_0 \in [0, 1] \) such that \( \mu(\sigma([t_0, 1])) \subset [s_0, 1] \). For simplicity, put \( s_1 = \mu(\alpha(t_0)) \). By Proposition 6 and Lemma 1, there is a subcontinuum \( M \) of \( Y \), such that \( M \cap Y \neq \emptyset \) and \( M \cap Y \neq Y \). There are two cases.

(i) Suppose there is \( A \in \mu^{-1}(s_1) \cap C(Y) \) such that \( A \cap M \neq \emptyset \). Put \( t_1 = (t_0 + 1)/2 \) and let \( \sigma_1 : [t_0, t_1] \to \mu^{-1}(s_1) \cap C(Y) \) be an arc from \( \alpha(t_0) \) to \( A \) (such an arc exists since \( s_0 \leq s_1 < \mu(Y) \)) and \( \sigma_2 : [t_1, 1] \to C(X) \) an order arc from \( A \) to \( A \cup M \). Note that \( \delta(\sigma_2([t_1, 1])) < 2\varepsilon_1 \). Define an arc \( \beta \) in \( C(X) \) by
\[
\beta(t) = \begin{cases} 
\alpha(t) & \text{if } t \in [0, t_0], \\
\sigma_1(t) & \text{if } t \in (t_0, t_1], \\
\sigma_2(t) & \text{if } t \in (t_1, 1].
\end{cases}
\]

Clearly \( \beta \) is continuous and \( \beta(t) \neq Y \) for each \( t \in [0, 1] \). If \( t \in [0, t_0] \), then \( H_d(\alpha(t), \beta(t)) = 0 \). Suppose \( t \in [t_0, t_1] \). Then since \( \beta([t_0, t]) \subset \mu^{-1}(s_1) \cap C(Y) \),
\[ H_d(\alpha(t), \beta(t)) \leq H_d(\alpha(t), \alpha(t_0)) + H_d(\beta(t_0), \beta(t)) \]
\[ \leq \delta(\alpha([t_0, 1])) + \delta(\mu^{-1}(s_1) \cap C(Y)) < 2\varepsilon_1 < \varepsilon. \]

Finally suppose \( t \in [t_1, 1] \). Then
\[ H_d(\alpha(t), \beta(t)) \leq H_d(\alpha(t), \alpha(t_1)) + H_d(\alpha(t_1), \beta(t)) + H_d(\beta(t), \beta(t)) \]
\[ + \delta(\alpha([t_1, 1])) + \delta(\sigma_2([t_1, 1])) < \varepsilon_1 + 2\varepsilon_1 + 2\varepsilon_1 + 5\varepsilon_1 = 5\varepsilon_1 = \varepsilon. \]

(ii) Suppose that for each \( A \in \mu^{-1}(s_1) \cap C(Y) \), \( A \cap M \neq \emptyset \) implies \( Y \subset A \cup M \).
In this case, each element of \( \mu^{-1}(s_1) \cap C(Y) \) intersects \( M \). In particular,
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\( \alpha(t_0) \cap M \neq \phi \). Considering an order arc from \( M \) to \( M \cup Y \), we can enlarge \( M \) and hence we can assume \( \mu(M) > s_i \). By Lemma 4, there is a map \( \sigma_i : [t_0, t_1] \to \mu^{-1}(s_i) \cap C(Y \cup M) \) from \( \alpha(t_0) \) to \( \sigma_i(t_i) \subset M \), where \( t_1 = (t_0 + 1)/2 \).

Let \( \sigma_2 : [t_1, 1] \to C(X) \) be an order arc from \( \sigma_i(t_i) \) to \( M \). Define an arc \( \beta \) in \( C(X) \) by

\[
\beta(t) = \begin{cases} 
\alpha(t) & \text{if } t \in [0, t_0], \\
\sigma_i(t) & \text{if } t \in (t_0, t_1], \\
\sigma_2(t) & \text{if } t \in (t_1, 1].
\end{cases}
\]

As in case (i), \( \beta \) satisfies all the required conditions.

Now we prove the main lemma.

**Bypass Lemma 8.** Let \( Y \) be a subcontinuum of \( X \) such that \( C(X) - \{Y\} \) is arcwise connected and let \( \alpha : [0, 1] \to C(X) \) be an arc such that \( \alpha(t) = Y \) if and only if \( t = 1/2 \). Then for each \( \varepsilon > 0 \) and each \( a, b \), where \( 0 \leq a < 1/2 < b \leq 1 \), there is a map \( \beta : [0, 1] \to C(X) - \{Y\} \) such that \( \alpha(t) = \beta(t) \) for all \( t \in [a, b] \) and \( H_d(\alpha(t), \beta(t)) < \varepsilon \) for all \( t \in [0, 1] \).

**Proof.** If \( Y = X \), then \( X \) is decomposable (by Theorem 11.4 and Corollary 11.8 of [5]). Let \( \mu \) be a Whitney map for \( C(X) \). By Proposition 5, there is \( s_0 \in [0, 1) \) such that \( \mu^{-1}(s) \) is arcwise connected for each \( s \in [s_0, 1] \). Moreover \( s_0 \) can be chosen so that \( \delta(\mu^{-1}[s_0, 1]) < \varepsilon/2 \). Since \( \alpha \) is continuous, there exist two numbers \( t_0, t_1 \) such that \( a \leq t_0 < 1/2 < t_1 \leq b \), \( \mu(\alpha(t_0)) = \mu(\alpha(t_1)) \in [s_0, 1] \) and \( \delta(\alpha([t_0, t_1])) < \varepsilon/2 \). Put \( \mu(\alpha(t_0)) = s_i \). Then since \( s_i \in [s_0, 1] \), there is a map \( \sigma : [t_0, t_1] \to C(Y) \) from \( \alpha(t_0) \) to \( \alpha(t_1) \). Define an arc \( \beta \) in \( C(X) - \{Y\} \) by

\[
\beta(t) = \begin{cases} 
\alpha(t) & \text{if } t \in [0, t_0], \\
\sigma(t) & \text{if } t \in (t_0, t_1].
\end{cases}
\]

If \( t \in (t_0, t_1) \), then

\[
H_d(\alpha(t), \beta(t)) \leq H_d(\alpha(t), \alpha(t_0)) + H_d(\sigma(t_0), \sigma(t)) \\
\leq \delta(\alpha([t_0, t_1])) + \delta(\mu^{-1}(s_i)) < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]

Therefore \( \beta \) satisfies the required conditions.

Next suppose that \( Y \) is a proper subcontinuum of \( X \). Put \( \varepsilon_i = \varepsilon / 4 \). There exist two numbers \( t_0, t_1 \) such that \( a \leq t_0 < 1/2 < t_1 \leq b \) and \( \delta(\alpha([t_0, t_1])) < \varepsilon_i \). Note that \( \alpha([t_0, t_1]) \subset C(Y) \). If \( \alpha([t_0, 1/2]) \subset C(Y) \), then by Lemma 7, there is a map \( \sigma : [t_0, 1/2] \to C(Y) - \{Y\} \) such that \( \sigma(t_0) = \alpha(t_0), \sigma(1/2) \neq Y \neq \phi \) and \( H_d(\alpha(t), \sigma(t)) < \varepsilon_i \) for each \( t \in [t_0, 1/2] \). If \( \alpha([t_1, 1/2]) \subset C(Y) \neq \phi \), then put \( \sigma = \alpha[1/2, 1/2] \).

There is \( r \in (t_0, 1/2) \) such that \( \sigma(r) - Y \neq \phi \neq Y - \sigma(r) \). Let \( \tau : [r, 1/2] \to C(X) \) be
an order arc from $\sigma(r)$ to $Y_t$. Define $\beta_0 : [t_0, 1/2] \to C(X)$ by

$$\beta_0(t) = \begin{cases} 
\sigma(t) & \text{if } t \in [t_0, r], \\
\tau(t) & \text{if } t \in (r, 1/2].
\end{cases}$$

It is easy to see that $H_d(\alpha(t), \beta_0(t)) < 4\varepsilon_1$ for each $t \in [t_0, 1/2]$.

As in the same way, we can find a map $\beta : \left[1/2, t_1\right] \to C(X)$ such that $\beta_1(1/2) = Y_t$, $\beta_1(t_1) = \alpha(t)$ and $H_d(\alpha(t), \beta_1(t)) < 4\varepsilon_1$ for each $t \in [1/2, t_1]$. Then the arc $\beta$ defined by

$$\beta(t) = \begin{cases} 
\alpha(t) & \text{if } t \in [0, t_0] \cup [t_1, 1], \\
\beta_0(t) & \text{if } t \in [t_0, 1/2], \\
\beta_1(t) & \text{if } t \in [1/2, t_1]
\end{cases}$$

satisfies the required conditions.

3. Arcwise Connectedness of the Complement

Let $\mathcal{Y}$ be a closed subset of $C(X)$ such that $C(X) - \{Y\}$ is arcwise connected for each $Y \in \mathcal{Y}$. We will show that if $\mathcal{Y}$ is a finite set, then its complement is also arcwise connected. Using this, we show that the same is fold if $\mathcal{Y}$ is a closed countable set. If $\mathcal{A} \subset \mathcal{E}(X)$ and $\varepsilon > 0$, then we wright the $\varepsilon$-neighborhood of $\mathcal{A}$ in $C(X)$ by $N(\mathcal{A}; \varepsilon)$.

THEOREM 9. Let $\mathcal{Y}$ be a finite subset of $C(X)$ such that $C(X) - \{Y\}$ is arcwise connected for each $Y \in \mathcal{Y}$ and let $\alpha : [0, 1] \to C(X)$ be an arc from $K$ to $L$, where $K, L \in C(X) - \mathcal{Y}$. Then for each $\varepsilon > 0$, there is a map $\beta : [0, 1] \to C(X) - \mathcal{Y}$ from $K$ to $L$ such that $\beta([0, 1]) \subset N(\alpha([0, 1]); \varepsilon)$.

PROOF. If $K = L$, then we can take $\beta$ to be a constant map. Hence let us suppose $K \neq L$. There is an embedding $\alpha' : [0, 1] \to \alpha([0, 1])$ such that $\alpha(t) = \alpha'(t)$ for $t = 0, 1$. Therefore we can assume that $\alpha$ is an embedding arc and hence $\alpha^{-1}(\mathcal{Y})$ is a finite set. Let $\alpha^{-1}(\mathcal{Y}) = \{t_1, t_2, \ldots, t_n\}$, where $0 < t_i < t_{i+1} < 1$ for $i = 1, 2, \ldots, n - 1$.

(i) Suppose $n = 1$ and without loss of generality, assume $t_1 = 1/2$. Put $\alpha(1/2) = Y$. Then $\alpha([0, 1])$ and $\mathcal{Y}_1 = \mathcal{Y} - \{Y\}$ are closed and disjoint. Put $\delta = H_d(\alpha([0, 1]), \mathcal{Y}_1)$ and $\varepsilon_1 = \min(\varepsilon, \delta)$. Then $\varepsilon_1 > 0$. Applying Bypass Lemma, there is a map $\beta : [0, 1] \to C(X) - \{Y\}$ from $K$ to $L$ such that $H_d(\alpha(t), \beta(t)) < \varepsilon_1$. By the choice of $\varepsilon_1$, $\beta$ satisfies the required conditions.

(ii) Suppose $k \geq 2$ and the Theorem holds for $n = k - 1$. Let $\alpha^{-1}(\mathcal{Y}) = \{t_1, t_2, \ldots, t_k\}$ where $0 < t_i < t_{i+1} < 1$ for $i = 1, 2, \ldots, k - 1$. Put $\delta = H_d(\alpha([t_0, 1]), \mathcal{Y} - \{\alpha(t_k)\})$ and
\[ \varepsilon_i = \min \{ \varepsilon / 2, \delta \}, \quad \text{where} \quad t_0 = (t_{i-1} + t_i) / 2. \] Then partially applying Bypass Lemma, there is a map \( \beta : [0,1] \to C(X) \) such that \( \alpha|[0,t_0] = \beta|[0,t_0], \alpha(1) = \beta(1), \) \( H_\varepsilon(\alpha(t),\beta(t)) < \varepsilon, \) and \( \beta((0,1]) \) does not contain \( \alpha(t_1) \). Let \( \alpha_i \) be an embedding arc from \( K \) to \( L \) such that \( \alpha_i(0,1]) \subseteq \beta_i(0,1]) \). Then it is easy to see that the image of \( \alpha_i \) intersects at most \( n - 1 \) elements of \( \mathcal{Y} \). Therefore by the inductive hypothesis, there is an arc \( \beta \) from \( K \) to \( L \) in \( C(X) - \mathcal{Y} \) such that \( \beta((0,1]) \subseteq N(\alpha((0,1];\varepsilon/2)). \) Hence \( \beta \) is a required arc.

**Corollary 10.** Let \( \mathcal{F} \) be a closed subset of \( C(X) \) and let \( \mathcal{A} \) be an arc component of \( C(X) - \mathcal{F} \). If \( \mathcal{Y} \) is a finite subset of \( C(X) \) such that \( C(X) - \{ \mathcal{Y} \} \) is arcwise connected for each \( \mathcal{Y} \in \mathcal{Y} \), then \( \mathcal{A} - \mathcal{Y} \) is arcwise connected.

**Proof.** Let \( K, L \) be arbitrary elements of \( \mathcal{A} - \mathcal{Y} \). There is a map \( \alpha : [0,1] \to \mathcal{A} \) from \( K \) to \( L \). Put \( \varepsilon = (1/2)H_\varepsilon(\alpha([0,1]),\mathcal{F}). \) Then \( \varepsilon > 0 \) and hence by Theorem 9, there is a map \( \beta : [0,1] \to C(X) - \mathcal{Y} \) from \( K \) to \( L \) such that \( \beta([0,1]) \subseteq N(\alpha([0,1]);\varepsilon) \). By the definition of \( \varepsilon, N(\alpha([0,1]);\varepsilon) \cap \mathcal{F} = \emptyset. \) Therefore \( \beta \) is an arc in \( \mathcal{A} - \mathcal{Y} \) from \( K \) to \( L. \)

Let \( A' \) denote the derived set of the space \( A \). The derived set of \( A \) of order \( \lambda \) is defined by

\[ A^{(\lambda)} = A', \quad A^{(\nu+1)} = (A^{(\nu)})' \quad \text{and} \quad A^{(\lambda)} = \bigcap_{\nu < \lambda} A^{(\nu)} \]

if \( \lambda \) is a limit ordinal (see [3]).

We say that a triple \( \{ \mathcal{F}, \mathcal{A}, \mathcal{Y} \} \) is **admissible** if \( \mathcal{F} \) is a closed subset of \( C(X), \mathcal{A} \) is an arc component of \( C(X) - \mathcal{F}, \mathcal{Y} \) is a closed countable subset of \( C(X) \) such that \( C(X) - \{ \mathcal{Y} \} \) is arcwise connected for each \( \mathcal{Y} \in \mathcal{Y}. \)

**Theorem 11.** If \( \{ \mathcal{F}, \mathcal{A}, \mathcal{Y} \} \) is admissible, then \( \mathcal{A} - \mathcal{Y} \) is arcwise connected.

**Proof.** First observe that the least ordinal \( \nu \) such that \( \mathcal{Y}^{(\nu)} = \emptyset \) (such an ordinal \( \nu \) exists since \( \mathcal{Y} \) does not contain perfect sets) is not a limit ordinal. Therefore there is the least ordinal \( \lambda \) such that \( \mathcal{Y}^{(\lambda)} = \emptyset. \) Denote such the ordinal \( \lambda \) by \( \Lambda(\mathcal{Y}). \) To prove the Theorem, we shall proceed by transfinite induction on \( \Lambda(\mathcal{Y}). \)

If \( \Lambda(\mathcal{Y}) = 0, \) then \( \mathcal{Y} \) is a finite set. Hence Theorem follows from Corollary 10.

Suppose that the Theorem holds for any admissible triple \( \{ \mathcal{F}, \mathcal{A}, \mathcal{Y} \} \) such that \( \Lambda(\mathcal{Y}) < \lambda \). Let \( \{ \mathcal{F}, \mathcal{A}, \mathcal{Y} \} \) be an admissible triple such that \( \Lambda(\mathcal{Y}) = \lambda \) and let \( K, L \) be arbitrary elements of \( \mathcal{A} - \mathcal{Y}. \) It is sufficient to show that there is an arc from \( K \) to \( L \) in \( \mathcal{A} - \mathcal{Y}. \) Since \( \mathcal{Y}^{(\lambda+1)} = \emptyset, \mathcal{Y}^{(\lambda)} \) is a finite set. Therefore by Corollary 10,
there is a map \( \alpha : [0,1] \to \mathcal{A} - \mathcal{Y}^{(\lambda)} \) from \( K \) to \( L \). Put \( \varepsilon = (1/2)H_j(\alpha([0,1])), \mathcal{Y} = \mathcal{Y} - N(\mathcal{Y}^{(\lambda)}; \varepsilon), \mathcal{F}_1 = \mathcal{F} \cup N(\mathcal{Y}^{(\lambda)}; \varepsilon) \), where \( N(\mathcal{Y}^{(\lambda)}; \varepsilon) \) is the closure of \( N(\mathcal{Y}^{(\lambda)}; \varepsilon) \) in \( C(X) \), and let \( \mathcal{A}_1 \) be the arc component of \( C(X) - \mathcal{F}_1 \) containing \( K \) (and hence \( L \)). Note that \( \mathcal{A}_1 \subset \mathcal{A} \). The triple \( \{ \mathcal{F}_1, \mathcal{A}_1, \mathcal{Y}_1 \} \) is admissible and \( \Lambda(\mathcal{Y}_1) < \lambda \). Hence by inductive hypothesis, there is an arc from \( K \) to \( L \) in \( \mathcal{A}_1 - \mathcal{Y}_1 \). Since \( \mathcal{A}_1 - \mathcal{Y}_1 \subset \mathcal{A} - \mathcal{Y} \) and \( K, L \) are arbitrary elements of \( \mathcal{A} - \mathcal{Y} \), \( \mathcal{A} - \mathcal{Y} \) is arcwise connected.

**Corollary 12.** If \( \mathcal{Y} \) is a countable closed subset of \( C(X) \) such that \( C(X) - \{Y\} \) is arcwise connected for each \( Y \in \mathcal{Y} \), then \( C(X) - \mathcal{Y} \) is arcwise connected.

**References**


Department of Mathematics, Tokyo Gakugei Univ.
Nukuikitamachi, koganei-shi, Tokyo, 184, Japan