ON \textit{q}-PSEUDOCONVEX OPEN SETS IN A COMPLEX SPACE

By

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In a series of (perhaps not widely known) papers T. Kiyosawa ([1], [2], [3], [4], [5]) introduced and developed the notion of Levi \textit{q}-convexity. Here we show how to use this notion to improve one of his results ([2] Th. 2) (for a different extension, see [7]). To state and prove our results, we recall few definitions.

Let \( M \) be a complex manifold of dimension \( n \); a real \( C^2 \) function \( u \) on \( M \) is said to be \( q \)-convex at a point \( P \) of \( M \) if the hermitian from \( L(u)(P)=\sum_{i,j}(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j})_P \times (P)a_i a_j, z_1, \ldots, z_n \) local coordinates around \( P \), has at least \( n-q+1 \) strictly positive eigenvalues; we say that \( u \) is Levi \( q \)-convex at \( P \) if either \( (du)_P=0 \) and \( u \) is \( q \)-convex at \( P \) or \( (du)_P\neq0 \) and the restriction of \( L(u)(P) \) to the hyperplane \( \left\{ \sum_{i} \left( \frac{\partial u}{\partial z_i} \right)(P) a_i = 0 \right\} \) has at least \( n-q \) strictly positive eigenvalues. Let \( X \) be a complex space, \( A \in X \), and \( f: X \rightarrow \mathbb{R} \) a \( C^2 \) function; we say that \( f \) is \( q \)-convex (or Levi \( q \)-convex) at \( A \) if there is a neighborhood \( V \) of \( A \) in \( X \), a closed embedding \( \varphi: V \rightarrow U \) with \( U \) open subset of an euclidean space, a \( C^2 \) function \( u \) on \( U \) such that \( f| V = u \circ \varphi \) and \( u \) is \( q \)-convex (or respectively Levi \( q \)-convex) at \( P = \varphi(A) \). It is well-known that a \( q \) convex function is Levi \( q \) convex and that both notions do not depend upon the choice of charts and local coordinates; for any fixed choice of charts and local coordinates we will call \( L(u)(P) \) the Levi form of \( u \) at \( P \) and of \( f \) at \( A \).

An open subset \( D \) of a complex space \( X \) is said to have regular Levi \( q \)-convex boundary if we can take a covering \( \{V_i\} \) of a neighbourhood of the boundary \( bD \) of \( D \) with closed embeddings \( \varphi_i: V_i \rightarrow U_i \) and \( U_i \) open in an euclidean space and \( C^2 \) functions \( f_i \) on \( U_i \), with \( V_i \cap D = \{ x \in V_i : f_i \circ \varphi_i(x) < 0 \} \) and such that if \( x \in V_i \cap V_j \), there is a neighborhood \( A \) of \( x \) in \( V_i \cap V_j \) such that on \( A = f_i(f_j \circ \varphi_i) \circ A \) with \( f_{ij} > 0, f_{ij} \in \mathbb{C}^2 \) on \( A \). The last condition is always satisfied for a domain \( D \) defined locally by Levi \( q \)-convex functions \( s_i \) if the set of points of \( bD \) at which either \( dS \) vanishes or \( X \) is singular is discrete.

A complex space \( X \) is called \( q \)-complete if it has a \( C^2 \) \( q \)-convex exhausting function \( f \); if \( f \) is both \( q \)-convex and weakly plurisubharmonic, \( X \) is called very

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strongly $q$-convex (in the sense of T. Ohsawa [6]).

Now we can state our results.

**Theorem.** Let $D$ be a regular Levi $q$-convex open subset of a complex space $X$. Then there exist a neighbourhood $V$ of the boundary $bD$ and a $q$-convex real function $t$ such that $D \cap V = \{x \in V : t(x) < 0\}$.

**Corollary.** Let $X$ be a very strongly $q$-convex space and $D$ an open subset of $X$ with regular Levi $q$-convex boundary. Then $D$ is $q$-complete.

Compare the corollary with the main result in [7].

**Proof** of the theorem. Note that the proof of [2] Theorem 2 goes on verbatim even if $D$ is not relatively compact in $X$. The quoted result gives a neighbourhood $W$ of $bD$ and a Levi $q$-convex function $g$ in $W$ such that $D \cap W = \{x \in W : g(x) < 0\}$. Consider a strictly positive real function $v$ on $W$. Set $t = ge^{vq}$. Since $g$ vanishes on $bD$, the Levi form of $t$ at a point $y$ in $bD$ is proportional to the Levi form at $y$ of $e^{vq}$, with $c = g(y)$. Hence if $g(y)$ is sufficiently high, $t$ is $q$-convex at $y \in bD$ ([3] Prop. 2 or [5] Lemma 2); how big must be $g(y)$ depend only from the eigenvalues of the Levi form of $g$ at $y$; hence the same constant works also in a neighbourhood of $y$. Let $\{V_n\}$, $\{U_n\}$ be locally finite coverings of $W$ with $V_n$ relatively compact in $U_n$, $\{U_n\}$ fine enough (in particular with local charts on which $g$ may be find constants $c_n > 0$ such that if $u < c_n$ on $V_n$, $t = ge^{vq}$ is $q$-convex at every point of $bD$, hence in a neighbourhood $V$ of $bD$. Q.E.D.

**Proof** of the corollary. By the theorem we may find an open neighbourhood $V$ of $bD$ and a real $C^2$ $q$-convex function $f$ on $V$ such that $V \cap D = \{x \in V : f(x) < 0\}$. Let $W$ be an open neighbourhood of $bD$ with closure contained in $V$. Note that the function $s := -f^{-1}$ is $q$-convex on $V \cap D$ and goes to infinity near $bD$. Let $u$ be a real non-negative $C^2$ function on $U$ with support containd in $V \cap D$, $u = 1$ in $W \cap D$. We may consider $us$ as a function on $D$ setting $(us)(x) = 0$ if $x \notin V$. Take an exhaustive, positive, $q$-convex function $h$ on $X$. Take an increasing sequence $\{K_n\}$ or compact subset of $X$, with union $X$ and a sequence $\{c_n\}$ of strictly positive real numbers. Take a $C^2$ function $b : R \to R$ with $b(t) = 0$ for $t \leq -1$, $b(t) \geq c_j$ for $j \leq t \leq j + 1$ and $b'(t) > 0$ for $t > -1$. Set $g(t) = \int_{-\infty}^{t} b(x) dx$

and set $F = g \circ h$. For every $P \in X$ and any choice of local coordinates, we have $L(F)(P) \geq b(h(P))L(g)(P)$. Hence we may choose the constants $c_j$ with $c \geq j$ and such that $F + s$ is $q$-convex on $(D \setminus W) \cap K_j$ for every $j$. Since $F$ is plurisubharmonic, $F + s$ is $q$-convex on $D$. If $\{x_n\}$ is a sequence in $D$ without accumula-
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On points in $X$, then $(F(x_n))$ and $\{F(x_n)+s(x_n)\}$ are unbounded on $\{x_n\}$. The function $s$ is unbounded on every sequence of points in $D$ converging to a point in $bD$, hence $F+s$ is an exhaustion function on $D$. Q. E. D.

References


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