R-SPACES ASSOCIATED WITH A HERMITIAN
SYMMETRIC PAIR

By
Hiroyuki Tasaki and Osami Yasukura

1. Introduction.

The linear isotropy representation of a Riemannian symmetric pair \((G, K)\) is defined as the differential of the left action of \(K\) on \(G/K\) at the origin. Every orbit of the linear isotropy representation of \((G, K)\) is called an \(R\)-space associated with \((G, K)\), which is an important example of equivariant homogeneous Riemannian submanifolds in a Euclidean sphere (See Takagi-Takahashi [2] and Takeuchi-Kobayashi [3]).

This paper is concerned with the linear isotropy representation of a Hermitian symmetric pair \((G, K)\). Its restriction to the center of \(K\) defines an \(S^1\)-action on the associated \(R\)-spaces. We determine all \(R\)-spaces associated with Hermitian symmetric pairs \((G, K)\) on which the semisimple part of \(K\) acts transitively. In particular, we know all irreducible Hermitian symmetric pairs such that each of the associated \(R\)-spaces has such a property. This result is utilizable for the classification of orthogonal transformation groups by their cohomogeneity (See the forthcoming paper [4] concerned with this problem in low cohomogeneity).

The authors are profoundly grateful to Professor Ryoichi Takagi for his helpful suggestion and critical reading of a primary manuscript.

2. Statement of the result.

Let \((G, K)\) be an irreducible Hermitian symmetric pair of compact type and \(\mathfrak{g}\) [resp. \(\mathfrak{k}\)] the Lie algebra of \(G\) [resp. \(K\)]. Then \(\mathfrak{g}\) has the canonical direct sum decomposition:

\[
\mathfrak{g} = \mathfrak{k} + \mathfrak{m},
\]

where \(\mathfrak{m}\) is the subspace of \(\mathfrak{g}\) satisfying

\[
[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m} \quad \text{and} \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}.
\]

The tangent space of \(G/K\) at the origin can be naturally identified with \(\mathfrak{m}\). Then
the linear isotropy representation of \((G,K)\) is nothing but the adjoint action \(\text{Ad}\) of \(K\) on \(\mathfrak{m}\).

Let \(K_s\) be the analytic subgroup of \(K\) corresponding to the semisimple part \(\mathfrak{k}_s = [\mathfrak{k}, \mathfrak{k}]\) of \(\mathfrak{k}\) and \(\mathfrak{z}\) be the 1-dimensional center of \(\mathfrak{k}\). We can take an element \(H_o\) in \(\mathfrak{z}\) such that

\[
(\text{ad} H_o|_\mathfrak{m})^2 = -i\mathbf{id}_\mathfrak{m},
\]

because \((G,K)\) is a Hermitian symmetric pair.

Take a maximal Abelian subalgebra \(\mathfrak{h}\) in \(\mathfrak{k}\). Then \(\mathfrak{h}\) is also a maximal Abelian subalgebra in \(\mathfrak{g}\) and the complexification \(\mathfrak{g}^c\) of \(\mathfrak{h}\) is a Cartan subalgebra of \(\mathfrak{g}^c\). Let \(\mathcal{A}\) denote the set of all non-zero roots of \(\mathfrak{g}^c\) with respect to \(\mathfrak{h}^c\). For each \(\alpha \in \mathcal{A}\), define a subspace \(\mathfrak{g}_\alpha\) of \(\mathfrak{g}^c\) by

\[
\mathfrak{g}_\alpha = \{X \in \mathfrak{g}^c ; [H, X] = \alpha(H)X\text{ for all }H \in \mathfrak{h}^c\}
\]

and choose a non-zero vector \(X_\alpha \in \mathfrak{g}_\alpha\) such that

\[
X_\alpha - X_{-\alpha}, \sqrt{-1}(X_\alpha + X_{-\alpha}) \in \mathfrak{g}_\alpha \quad \text{and} \quad [X_\alpha, X_{-\alpha}] = \frac{2}{\alpha(H_o)}H_o,
\]

where \(H_o\) in \(\mathfrak{h}^c\) is the dual vector of \(\alpha\) with respect to the Killing form \(\langle , \rangle\) of \(\mathfrak{g}^c\). The set of all compact [resp. noncompact] roots in \(\mathcal{A}\) is denoted by \(\mathcal{A}_c\) [resp. \(\mathcal{A}_n\)]

\[
\mathfrak{t}^c = \mathfrak{h}^c + \sum_{\alpha \in \mathcal{A}_c} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{m}^c = \sum_{\alpha \in \mathcal{A}_n} \mathfrak{g}_\alpha.
\]

Fix the lexicographic ordering in the dual space of the real vector space \(\sqrt{-1}\mathfrak{h}\) with respect to an ordered basis

\[
\sqrt{-1}H_o = Y_1, Y_2, \ldots, Y_m; m = \dim_R(\sqrt{-1}\mathfrak{h})
\]

in \(\sqrt{-1}\mathfrak{h}\). Let \(\mathcal{A}^+\) [resp. \(\mathcal{A}_n^+\)] denote the set of all positive roots in \(\mathcal{A}\) [resp. \(\mathcal{A}_n\)]. There is a direct sum decomposition of \(\mathfrak{m}\):

\[
\mathfrak{m} = \bigoplus_{\alpha \in \mathcal{A}_n^+} (\mathfrak{R}(X_\alpha - X_{-\alpha}) + \mathfrak{R}\sqrt{-1}(X_\alpha + X_{-\alpha})).
\]

According to Harish-Chandra [1, § 6], there exists a subset \(I = \{i_1, \ldots, i_r\}\) of \(\mathcal{A}_n^+\) such that \(i_i \pm i_j \notin \mathcal{A} \quad (1 \leq i, j \leq r)\) and

\[
a = \sum_{i=1}^r \mathfrak{R}\sqrt{-1}(X_{i_1} + X_{-i_1})
\]

is a maximal Abelian subspace of \(\mathfrak{m}\), where \(r\) is the rank of the symmetric pair \((G,K)\).

Consider the automorphism, so-called Cayley transformation,
\[ \nu = \exp \frac{\pi}{4} \text{ad} (\sum_{i=1}^{r} (X_{i} - X_{-i})) \]

of the Lie algebra \( \mathfrak{g}^c \). We have \( \nu(\mathfrak{a}) \subseteq \mathfrak{f} \), since

\[ \nu(\sqrt{\mathfrak{h}} (X_{i} + X_{-i})) = \frac{2\sqrt{\mathfrak{h}}}{\gamma(H_{i})} H_{i} \quad (1 \leq i \leq r) . \]

Let \( \mathfrak{h} \) denote the restriction of a linear form on \( \mathfrak{h}^c \) to \( \mathfrak{a}^c \). The sets of all non-zero elements in \( \mathcal{J}, \mathcal{J}^t, \mathcal{J}_c, \mathcal{J}_a, \) and \( \mathcal{J}_n^c \) are denoted by \( R, R^t, R_c, R_n, \) and \( R_n^c \) respectively. \( R \) is isomorphic to the restricted root system of the Hermitian symmetric pair \( (G, K) \). By Harish-Chandra [1, § 6], there are only two possibilities:

Case i) \( R \) is of type \( C \);

\[ R = \{ \pm \bar{\gamma}_i \} \cup \left\{ \pm \frac{1}{2} (\pm \bar{\gamma}_i + \bar{\gamma}_j) ; i \neq j \right\} , \]

\[ R_c = \left\{ \frac{1}{2} (\bar{\gamma}_i - \bar{\gamma}_j) ; i \neq j \right\} , \]

\[ R_n = \{ \pm \bar{\gamma}_i \} \cup \left\{ \pm \frac{1}{2} (\bar{\gamma}_i + \bar{\gamma}_j) ; i \neq j \right\} , \]

Case ii) \( R \) is of type \( BC \);

\[ R = \{ \pm \bar{\gamma}_i \} \cup \left\{ \pm \frac{1}{2} (\pm \bar{\gamma}_i + \bar{\gamma}_j) ; i \neq j \right\} \cup \left\{ \pm \frac{1}{2} \bar{\gamma}_i \right\} , \]

\[ R_c = \left\{ \frac{1}{2} (\bar{\gamma}_i - \bar{\gamma}_j) ; i \neq j \right\} \cup \left\{ \pm \frac{1}{2} \bar{\gamma}_i \right\} , \]

\[ R_n = \{ \pm \bar{\gamma}_i \} \cup \left\{ \pm \frac{1}{2} (\bar{\gamma}_i + \bar{\gamma}_j) ; i \neq j \right\} \cup \left\{ \pm \frac{1}{2} \bar{\gamma}_i \right\} . \]

Then our result is the following:

**Theorem.** Let \( M \) be an \( R \)-space associated with an irreducible Hermitian symmetric pair \( (G, K) \). Then the following two conditions are equivalent.

1) The action of \( K_\mathfrak{a} \) on \( M \) is transitive.

2) The restricted root system \( R \) of \( (G, K) \) is of type \( BC \) or there exists a \( \gamma_i \) in \( \Gamma \) such that \( \gamma_i (\nu(\mathfrak{g}^c \cap \mathfrak{a})) = \{0\} \).

In particular, \( K_\mathfrak{a} \) acts transitively on each of the associated \( R \)-spaces if and only if \( R \) is of type \( BC \).

**Remark.** Suppose that \( M \) is an \( R \)-space of the highest dimension among those associated with a given irreducible Hermitian symmetric pair \( (G, K) \), i.e., \( M \) is a maximum dimensional \( K \)-orbit of the linear isotropy representation of
Then $M \cap a$ contains a regular element $H$, which satisfies $\gamma_i(\nu(H)) \neq 0$ for all $i$. Then the transitivity of $K_\nu$ on $M$ is equivalent to the condition that the restricted root system $R$ is of type BC.

3. Proof of Theorem.

Fix an $\text{Ad}(G)$-invariant inner product on $\mathfrak{g}$, which is a negative multiple of the restriction of the Killing form $\langle , \rangle$ of $\mathfrak{g}^c$ to $\mathfrak{g}$.

Let $H$ be any fixed element of $M \cup a$ and $t_H$ denote the centralizer of $H$ in $\mathfrak{t}$:

$$t_H = \{ T \in \mathfrak{t}; [T, H] = 0 \}.$$  \hspace{1cm} (1)

The orthogonal complement of $t_H$ in $\mathfrak{t}$ is denoted by $t_H^\perp$.

Since $t$ is the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{t}$, the kernel of the orthogonal projection $\rho$ of $\mathfrak{t}$ to $\mathfrak{z}$ is equal to $t$.

Since $K$ and $K_\nu$ are compact and connected, the condition 1) in Theorem is equivalent to

$$\dim \mathfrak{t} - \dim t_H = \dim t - \dim (t_H \cap t),$$

that is,

$$\dim t_H = 1 + \dim (t_H \cap t),$$

which is equivalent to $\rho(t_H) = \mathfrak{h}$, because $\dim \mathfrak{z} = 1$.

On the other hand, $\rho(t_H) = \mathfrak{h}$ if and only if $t_H \subseteq \mathfrak{z}^\perp$, that is, $t_H \supseteq \mathfrak{h}$. If we take $H_1 \in t_H$ and $H_2 \in t_H^\perp$ such that

$$H_0 = H_1 + H_2,$$  \hspace{1cm} (2)

then $H_0 = 0$ is equivalent to $t_H \supseteq \mathfrak{h}$.

So the condition 1) in Theorem is equivalent to $H_0 \neq 0$ in the equation (2). Therefore the following lemma completes the proof of our theorem.

**Lemma.** $H_0 \neq 0$ if and only if either the restricted root system $R$ of $(G, K)$ is of type BC or there exists a $\gamma_i$ in $\Gamma'$ such that $\gamma_i(\nu(H)) = 0$.

**Proof of Lemma.** Let $\mathfrak{b}$ be the orthogonal complement of $\nu(a) = \sum_{i \in \Gamma} \mathbf{R} \sqrt{-1} H_{t_i}$ in $\mathfrak{h} = \mathfrak{n} + \sum_{i \in \Gamma} \mathbf{R} \sqrt{-1} H_{t_i}$.

Put $\Gamma_H = \{ \gamma_i \in \Gamma'; \gamma_i(\nu(H)) = 0 \}$, $a_H = \sum_{t_i \in \Gamma_H} \mathbf{R} \sqrt{-1} H_{t_i}$, and $a_H = \sum_{t_i \notin \Gamma_H} \mathbf{R} \sqrt{-1} H_{t_i}$. Then $a_H$ is the orthogonal complement of $a_H$ in $\nu(a)$. We have an orthogonal direct sum decomposition of $\mathfrak{h}$:

$$\mathfrak{h} = (\mathfrak{b} + a_H) + a_H^\perp.$$  \hspace{1cm} (3)
As the first step, we claim that the decomposition of $H_0$ with respect to the decomposition (3) is the same as the equation (2). In fact, $t_H \supseteq b + a_H$, since $[\nu (b + a_H), \nu (H)] = \{0\}$ by

$$\nu \left[ \frac{2\sqrt{-1}}{\gamma (H_{i_1})} \right] = -\sqrt{-1}(X_i + X_{-i}) \quad (1 \leq i \leq r),$$

$\nu |_h = \text{id}_h$ and $\nu (b + a) = 0$.

We also have $t_H \supseteq a_H$, since $\langle \nu (t_H), \nu (a_H) \rangle = 0$ by

$$\nu (t_H) \subseteq b + \sum_{\alpha \in \Delta^-} (\alpha + \alpha_{-\alpha}), \quad \nu (a_H) \subseteq \sum_{r \in \Delta^+} (\alpha_r + \alpha_{-\alpha_r}).$$

Therefore $\mathfrak{h} \cap t_H = b + a_H$ and $\mathfrak{h} \cap t_H = a_H$. In particular,

$$H_2 = -\sum_{r \in \Delta^+} \frac{\sqrt{-1}}{\gamma (H_{i_1})} H_{i_1},$$

(4)

because we have

$$\gamma (H_0) = -\sqrt{-1} \quad \text{for all } \gamma \in \Delta^+$$

by the definition of $\Delta^+$. As a result, we obtain

$$H_1 = H_0 + \sum_{r \in \Delta^+} \frac{\sqrt{-1}}{\gamma (H_{i_1})} H_{i_1},$$

(5)

As the second step, we claim that $H_1 \neq 0$ in the equation (5) if and only if either $R$ is of type BC or $\Gamma_H \neq \phi$. We may assume that $H \neq 0$. Then there exists $\gamma \in \Gamma - \Gamma_H$.

If $R$ is of type BC, then there is a compact root $\alpha$ such that

$$\bar{\alpha} = \frac{1}{2} \bar{\gamma}.$$

In this case, by the equation (4) and $\alpha (H_0) = 0$ for all $\alpha \in \Delta$, we have

$$\alpha (H_1) = \alpha (-H_0) = \frac{1}{2} \gamma (H_0) = -\sqrt{-1}/2 \neq 0,$$

especially $H_1 \neq 0$.

Now suppose that $R$ is of type C. If $\Gamma_H \neq \phi$, we can take $\gamma \in \Gamma_H$. There exists a compact root $\alpha$ such that

$$\bar{\alpha} = \frac{1}{2} (\bar{\gamma} - \bar{\delta}).$$

In this case, by the equation (4),
Hiroyuki Tasaki and Osami Yasukura

\[ \alpha(H_1) = \alpha(-H_2) = \frac{1}{2} \gamma(-H_2) = \frac{1}{2} \sqrt{-1} \neq 0, \]

especially \( H_1 \neq 0 \). Here we have used the fact

\[ \gamma(a_H) = \{0\}, \]

which follows from the orthogonality of elements in \( I' \). If \( I' = \phi \), then

\[ \beta(H_1) = \beta(H_0) + \beta(-H_2) = -\sqrt{-1} + \beta(-H_2) = 0 \]

for all \( \beta \in \mathcal{A}_c \), by the equation (4) and \( R^*_e = \left\{ \frac{1}{2}(\tilde{r}_p + \tilde{r}_q); 1 \leq p, q \leq r \right\} \). On the other hand

\[ \alpha(H_1) = \alpha(-H_2) = 0 \quad \text{for all} \quad \alpha \in \mathcal{A}_c, \]

by \( R_e = \left\{ \frac{1}{2}(\tilde{r}_p - \tilde{r}_q); p \neq q \right\} \). So \( H_1 = 0 \). This completes the proof of Lemma.

References


