THE GENERALIZATIONS OF FIRST COUNTABLE SPACES

By

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Abstract. In this paper we consider some generalizations of first countable spaces, called $w_\kappa$-spaces. When $\kappa=1$, $\omega_1$, $\omega$, the spaces are respectively Fréchet spaces, $w$-spaces in the sense of G. Gruenhage [5] and first countable spaces. We show that the $w_\kappa$-spaces are the images of metric spaces under certain kind of continuous maps, called $w_\kappa$-maps. For any cardinals $\kappa_1<\kappa_2$, we construct by forcing a model in which there is a countable space with character $\omega$, which is a $w_{\kappa_1}$-space but not $w_{\kappa_2}$-space.

1. Introduction

Generalizations of first countable spaces have been one of the traditional topics in general topology. G. Gruenhage [5] defined the class of $w$-spaces by topological games. P.L. Sharma [9] gave out a very useful characterization of $w$-spaces. In this paper we introduce $w_\kappa$-spaces which establish an interesting relationship among Fréchet spaces, $w$-spaces and first countable spaces.

It is well-known that Fréchet spaces and first countable spaces are respectively the images of metric spaces under pseudo-open and almost open maps (see [7]). The author [10] proved that $w$-spaces are the images of metric spaces under $w$-maps. Theorem 3.2 in this paper unifies all of these results.

Assuming $MA$, F. Galvin [4] constructed a $w$-space which is not a $c^*$-space, i.e. a space $X$ with countable tightness and every countable subspace of $X$ is first countable. In this paper we show that for any cardinals $\kappa_1<\kappa_2$, it is consistent that there is a countable space with character $\omega$, which is a $w_{\kappa_1}$-space but not $w_{\kappa_2}$-space.

2. Notations, Definitions and Basic Properties

All spaces considered are assumed to be Hausdorff and maps continuous onto. The notation $\{A_\alpha : \alpha < \kappa\}$ is not necessarily faithful. For the terminology

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and basic facts about forcing see [6], for the weak versions of Martin’s axiom see [3]. We use $\mathcal{G}$ to denote the least cardinality of a centered family $\mathcal{G}$ of subsets of $\omega$ such that there is no $A \subseteq [\omega]^\omega$ such that $A \subseteq \mathcal{G}$ for any $B \in \mathcal{G}$.

$\alpha, \beta, \ldots$ denote ordinals and $\kappa, \lambda, \ldots$ cardinals.

**Definition 2.1.** We call a map $f : X \to Y$ a $w_\kappa$-map, if for any $y \in Y$ and any open cover $\{U_\alpha : \alpha < \kappa\}$ of $f^{-1}(y)$, there exists an $\alpha$ such that $y \in \text{int}(f(U_\alpha))$. If a map is a $w_\kappa$-map for any $\kappa$, we call it a $w_\omega$-map. From now on, $\kappa$ is a nonzero cardinal or $\omega$.

It is obvious that the class of $w_\kappa$-maps equals to the class of pseudo-open maps. We can easily construct for any $\kappa_1 < \kappa_2$ a map which is a $w_{\kappa_2}$-map but not $w_{\kappa_1}$-map.

**Lemma 2.1.** Let $f : X \to Y$, then the following are equivalent:

1. $f$ is a $w_\omega$-map;
2. If $\{A_\alpha : \alpha < \kappa\}$ is a family of subsets of $Y$, $y \in \cap \{\text{cl}(A_\alpha) : \alpha < \kappa\}$, then there exists an $x \in f^{-1}(y)$, $x \in \cap \{\text{cl}(f^{-1}(A_\alpha)) : \alpha < \kappa\}$.

**Proof.** (1)$\Rightarrow$(2) Suppose that there exists a family $\{A_\alpha : \alpha < \kappa\}$ of subsets of $Y$ and $y \in \cap \{\text{cl}(A_\alpha) : \alpha < \kappa\}$ such that for any $x \in f^{-1}(y)$, $x \notin \cap \{\text{cl}(f^{-1}(A_\alpha)) : \alpha < \kappa\}$. Then if $x \in f^{-1}(y)$, there are an open neighbourhood $U_x$ of $x$ and an $\alpha_x < \kappa$ such that $U_x \cap f^{-1}(A_{\alpha_x}) = \emptyset$. Let $U_a = \cup \{U_x : x \in f^{-1}(y) \& \alpha_x = \alpha\}$ for any $\alpha < \kappa$. $\{U_\alpha : \alpha < \kappa\}$ is clearly an open cover of $f^{-1}(y)$. Since $f$ is a $w_\kappa$-map, there is a $U_a$ such that $y \in \text{int}(f(U_a))$. However, $U_a \cap f^{-1}(A_a) = \cup \{U_x \cap f^{-1}(A_\alpha) : x \in f^{-1}(y) \& \alpha_x = \alpha\} = \emptyset$. So $f(U_a) \cap A_a = \emptyset$, but $y \in \text{cl}(A_a)$. This is a contradiction.

(2)$\Rightarrow$(1) Suppose that $f$ is not a $w_\omega$-map, i.e. we have a $y_0 \in Y$ and a cover $\{U_\alpha : \alpha < \kappa\}$ of $f^{-1}(y_0)$ such that for any $\alpha$, $U_\alpha$ is open and $y_0 \notin \text{int}(f(U_\alpha))$. Therefore, we have $y_0 \in \cap \{\text{cl}(Y - f(U_\alpha)) : \alpha < \kappa\}$. By (2), there exists an $x \in f^{-1}(y_0)$ such that $x \in \cap \{\text{cl}(f^{-1}(Y - f(U_\alpha))) : \alpha < \kappa\}$. However, since $\{U_\alpha : \alpha < \kappa\}$ is a cover of $f^{-1}(y_0)$, there is a $U_\alpha$, $x \in U_\alpha$. Since $U_\alpha \cap f^{-1}(Y - f(U_\alpha)) = \emptyset$, $x \in \text{cl}(f^{-1}(Y - f(U_a)))$. This contradiction completes the proof.

**Definition [7] 2.2.** $f : X \to Y$ is called almost open, if for any $y \in Y$, there is an $x \in f^{-1}(y)$ such that for any neighbourhood $U$ of $x$, $f(U)$ is a neighbourhood of $y$.

**Theorem 2.1.** Let $f : X \to Y$. The following are equivalent:

1. $f$ is an almost open map;

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(2) \( f \) is a \( w_{\mu_1} \)-map, where \( \mu_1 = \sup \{ L(f^{-1}(y)) : y \in Y \} \), \( L \) denotes the Lindelöf degree;

(3) \( f \) is a \( w_{\mu_1} \)-map, where \( \mu_2 = 2^{\nu_1} \).

The proof is routine by the definitions and Lemma 2.1.

3. Theorems on \( w_\kappa \)-spaces

DEFINITION 3.1. A space \( Y \) is called a \( w_\kappa \)-space, if for any family \( \{ A_\alpha : \alpha < \kappa \} \) of subsets of \( Y \) and \( y \in \cap \{ \text{cl}(A_\alpha) : \alpha < \kappa \} \), there exists a decreasing sequence \( \{ F_\alpha : n \in \omega \} \) of subsets of \( Y \) satisfying that \( F_n \cap A_\alpha \neq 0 \) for any \( n \) and \( \alpha \) and for any open neighbourhood \( U \) of \( y \) there is an \( n \) such that \( F_m \subseteq U \) for any \( m > n \), i.e., \( \{ F_\alpha : n \in \omega \} \) converges to \( y \). What a \( w_\omega \)-space means is obvious.

We can see easily from the definition that when \( \kappa \) is finite, \( w_\kappa \)-spaces are exactly Fréchet spaces. By the trick of repeatedly enumerating, if necessary, we can see from [9] that \( w_\omega \)-spaces are exactly the \( w \)-spaces in the sense of G. Gruenhage [5].

THEOREM 3.1. Let \( Y \) be a space. The following are equivalent:

1. \( Y \) is a first countable space;
2. \( Y \) is a \( w_\omega \)-space;
3. \( Y \) is a \( w_{\omega_1} \)-space.

PROOF. We need only to proof (3) \( \rightarrow \) (1). Take \( y \in Y \). We enumerate \( \{ A : y \in \text{cl}(A) \cap \text{Ac}Y \} \) as \( \{ A_\alpha : \alpha < 2^{\nu_1} \} \). Since \( Y \) is a \( w_{\omega_1} \)-space, there must be a decreasing sequence \( \{ F_\alpha : n \in \omega \} \) converging to \( y \) such that \( F_n \cap A_\alpha \neq 0 \) for any \( n \) and \( \alpha \). Let \( U_n = \text{int}(F_n) \). Then \( \{ U_n : n \in \omega \} \) is a neighbourhood base at \( y \). \( \square \)

We generalize A.V. Arhangel'skii's sheaf (see [8]) to any cardinals. We need it in the proof of Theorem 3.2.

DEFINITION 3.2. If \( \{ r_\alpha : \alpha < \lambda \} \) is a family of convergent sequences with a common limit point \( y \), we call it \( \kappa \)-sheaf with the vertex \( y \). Let \( r_\alpha = \{ y_n : n \in \omega \} \). If for any neighbourhood \( U \) of \( y \), there is an \( n_0 \) such that \( y_n \in U \) for any \( n > n_0 \) and \( \alpha \), we call it a uniform \( \kappa \)-sheaf. If for any \( \kappa \)-sheaf \( \{ r_\alpha : \alpha < \kappa \} \) in \( Y \) there is a uniform \( \kappa \)-sheaf \( \{ r'_\alpha : \alpha < \kappa \} \) such that \( r'_\alpha \) is a subsequence of \( r_\alpha \), we call \( Y \) a \( \kappa \)-sheafed space.
Proposition 3.1. A space $Y$ is a $w_\kappa$-space if and only if $Y$ is a Fréchet $\kappa$-sheafed space. Consequently, $w_\kappa$-spaces are almost countably productive for any $\kappa \geq \omega$.

The last part of Proposition 3.1 follows from the fact that $w$-spaces are almost countably productive [8].

Theorem 3.2. A space $Y$ is a $w_\kappa$-space if and only if $Y$ is an image of a metric space under a $w_\kappa$-map.

Proof. On the part of "only if" needs to be proven here, since $w$-spaces are preserved by $w$-maps by Lemma 2.1.

Let \( \{R_\eta : \eta \in \Lambda \} \) be an enumeration of all uniform $\kappa$-sheaves in $Y$. For any $\eta \in \Lambda$ we construct a metric space $X_\eta$ as follows: Take $\kappa$ disjoint countable infinite sets \( \{s_{\eta \alpha} : \alpha < \kappa \} \) and \( x_\eta \not\in \bigcup \{s_{\eta \alpha} : \alpha < \kappa \} \). Let \( X_\eta = \bigcup \{s_{\eta \alpha} : \alpha < \kappa \} \cup \{x_\eta \} \) and \( s_{\eta \alpha} = \{x_{2n}^\alpha : n \in \omega \} \). We define
\[
\begin{align*}
    d_\eta(x_{2m}^\alpha, x_{2n}^\beta) &= \begin{cases} 1/m + 1/n & \alpha \neq \beta \\ 1/m - 1/n & \alpha = \beta \end{cases} \\
    d_\eta(x_{2m}^\alpha, x_\eta) &= 1/m.
\end{align*}
\]

Then \( (X_\eta, d_\eta) \) is a metric space. Let $X$ be the topological sum of \( \{X_\eta : \eta \in \Lambda \} \) and \( f : X \to Y \) be the map which maps $X_\eta$ onto $\bigcup R_\eta$ in a natural way. Now we want to show that $f$ is a $w_\kappa$-map. Suppose \( \{A_\alpha : \alpha < \kappa \} \) is a family of subsets of $Y$ and \( y \in \bigcap \text{cl}(A_\alpha) : \alpha < \kappa \). Since $Y$ is Fréchet, there is an $\eta \in \Lambda$ such that \( r_\eta \subset A_\eta \), where \( R_\eta = \{r_\alpha : \alpha < \kappa \} \), and the vertex of $R_\eta$ is $y$. It is easily seen from the definition of $f$ that \( s_{\eta \alpha} \subset f^{-1}(A_\alpha) \) and \( x_\eta \in f^{-1}(y) \). Therefore, \( x_\eta \in \text{cl}(f^{-1}(A_\eta)) \). By Lemma 2.1, $f$ is a $w_\kappa$-map. This completes the proof. $\square$

Theorem 3.3. Let $Y$ be a space with countable tightness and character less than $\kappa$. Then $Y$ is a $w_\kappa$-space. In particular, if $Y$ is countable, $Y$ is a $w_\kappa$-space for any $\kappa < \kappa$.

Proof. Let \( \mathcal{U} \) be a local base at $y \in Y$ with cardinality less than $\kappa$. Suppose that \( \{A_n : n \in \omega \} \) is a family of subsets of $Y$ such that $y \in A_n$. Since $Y$ has countable tightness, we can assume that $A_n$ is countable. Let \( \bigcup \{A_n : n \in \omega \} = \{y_n : n \in \omega \} \). We define \( P = \{(I, S) : I \in [\omega]^{<\omega} \land S \in [\mathcal{U}]^{<\omega} \} \) and \( (I', S') \leq (I, S) \) iff \( I' \supset I \), \( S' \supset S \) and \( I \setminus I' \subset \bigcap \{U : U \subset S \} \). It is easily seen that \( (P, \leq) \) is a \( \sigma \)-centered poset. The conclusion follows from the standard density arguments. $\square$
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Remark. It follows from Example 4.2 that it is consistent that $\omega_1 < \rho < 2^\omega$ and there is a countable space with character $\omega_1$ which is not a $w_p$-space.

4. Examples of countable $w_x$-spaces with character $\omega_1$

It follows from Theorem 3.1 that every countable $w_{\omega_1}$-space is first countable. Therefore, we are only interested in the models of $2^\omega > \omega_1$ in this section. We will construct some models of set theory in which there exist our desired examples.

Example 4.1. A countable space which is Fréchet but not a $w$-space.

Let $X$ be the quotient space of countably many copies of $\{0, 1/2, 1/3, \ldots\}$ with all limits adhering together. We adjoin $\omega_1$ dominating reals to any model of $2^\omega > \omega_1$. Then in this model, $X$ is a desired one.

Example 4.2. A countable space with character $\omega_1$ which is a $w_{\omega_1}$-space but not $w_{\omega_1}$-space, where $\omega \leq \kappa_1, \kappa_2 < 2^\omega$.

We can assume that $\kappa_2$ is regular. We start with a model $V$ of $MA + 2^\omega \geq \kappa_2$. Let $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ be a family of infinite subsets of $\omega$ and well-ordered by $\subset^*$. We define a finite supports iteration $\langle (P_\eta, Q_\eta) : \eta < \kappa_1 \rangle$ of ccc forcing in the following way:

In $V^{P_{\kappa_2}}$, we first take a ccc poset $Q'_\eta$ so that in $V^{P_{\kappa_2} * Q'_\eta}$ we have $MA + 2^\omega > \kappa_2$.

Now we work in $V^{P_{\kappa_2} * Q'_\eta}$. We define a poset $Q''_\eta = \{(a, S) : a \in \omega, S \subset [\omega]^\omega\}$ is finite and for any $\alpha < \omega_1, \cup S \subset A_\alpha$ where $(a', S') \subseteq (a, S)$ iff $a' \supset a, S' \supset S$ and $(a' \setminus a) \cap B = 0$ for any $B \subseteq S$. Let $D_{\alpha, \eta} = \{(a, S) : \text{there exists an } m > n \text{ such that } m \in a \cap A_\alpha\}$ for any $\alpha$ and $\eta$. It is easily seen that $D_{\alpha, \eta}$ is dense in $Q''_\eta$. So if $G''_\eta$ is a generic filter of $Q''_\eta$ then $B_\eta = \cup \{a : \text{there exists an } S \text{ with } (a, S) \in G''_\eta\}$ satisfies that $B_\eta \cap A_\alpha$ is infinite for any $\alpha < \omega_1$. By a similar density argument, if $B \in [\omega]^\omega \cap V^{P_{\kappa_2} * Q'_\eta}$ satisfies $B \subset A_\alpha$ for any $\alpha < \omega_1$, then $B \cap B_\eta$ is finite. Let $Q_{\kappa_2} = Q'_\kappa * Q''_\kappa$.

Let $G_{\kappa_2}$ be a generic filter of $P_{\kappa_2}$ over $V$. From now on, we work in $V[G_{\kappa_2}]$.

For any $U \subset [\omega]^\omega$ and $|U| < \kappa_2$ there is an $\alpha < \kappa_2$ such that $U \in V[G_\alpha]$. So if $U$ has the strong finite intersection property, there is a $W \in [\omega]^\omega$ such that $W \subset^* U$ for any $U \in \mathcal{U}$. Therefore, we have $\rho \geq \kappa_2$ in $V[G_{\kappa_2}]$.

On the other hand, since there is no $U \in [\omega]^\omega$ such that $U \subset^* A_\alpha$ and $U \cap B_\eta$ is infinite for any $\alpha < \omega_1$ and $\eta < \kappa_2$, we have $\rho \leq \kappa_2$ by Theorem 3.8 [2].

Now we begin to construct the countable space $X$ with character $\omega_1$ which
is a $w_{\kappa_1}$-space but not $w_{\kappa_2}$-space. Let $X=\omega$. We define the topology in the following way: If $x \neq 0$, $x$ is isolated; The neighbourhood base at 0 is \{(A_n \setminus s) \cup \{0\} : \alpha < \omega, \text{ and } s \subseteq [\omega]^\omega\}. By Theorem 3.3, $X$ is a $w_{\kappa_1}$-space since $p = \kappa_2$. However, we can take \{\overline{B'_i} : \eta < \kappa_2\} \subseteq [\omega]^\omega$ so that $B'_i \subseteq A_\alpha \cap B_\eta$ for any $\alpha < \omega$ and $\eta < \kappa_2$. It is obvious that $B'_i$ is a convergent sequence. Suppose that $X$ is a $w_{\kappa_2}$-space. Then there exist \{\overline{F_n} : n \subseteq \omega\} such that:

1. \overline{F_n} \subseteq [\omega]^\omega$ and \overline{F_n+1} \subseteq \overline{F_n}$ for any $n \subseteq \omega$;
2. For any $\alpha < \omega$, there is an $n$ such that \overline{F_n} \subseteq A_\alpha$;
3. \overline{F_n} \cap B_\eta \setminus m \neq 0$ for any $n, m \subseteq \omega$ and $\eta < \kappa_2$.

Therefore, there is an $n$ such that $\overline{F_n} \subseteq A_\alpha$ and $\overline{F_n} \cap B_\eta$ is infinite for any $\alpha < \omega$ and $\eta < \kappa_2$. This is impossible by our choice of \{\overline{B'_i} : \eta < \kappa_2\}.

**Question 4.1.** Is it consistent that every countable $w$-space is first countable? Moreover, is it consistent with $\neg \text{CH}$ that every countable Fréchet space with character less than $2^{\omega_1}$ is first countable?

**Remark.** A. Dow and J. Steprans [2] have constructed a model in which every countable Fréchet $\alpha_1$-space is first countable.

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