ON REPRESENTATION-FINITE ALGEBRAS WHOSE
AUSLANDER-REITEN QUIVER CONTAINS
A STABLE COMPLETE SLICE

By
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0. Introduction

Tilting modules and associated tilted algebras, introduced by Brenner and Butler in [7] and generalized by Happel and Ringel [12, 13] has been shown in [1, 8, 12, 13, 14, 16, 18, 19] to be of interest in representation theory. Recall [12] that a module $T_A$ over a finite-dimensional algebra $A$ is called a tilting module provided it satisfies the following three properties:

1. $\text{proj dim}_A(T_A) \leq 1$
2. $\text{Ext}_A^1(T_A, T_A) = 0$
3. There is an exact sequence $0 \to A_A \to T'_A \to T''_A \to 0$ with $T', T''$ being direct sums of summands of $T$.

An algebra $B$ is called a tilted algebra if there is an hereditary algebra $A$ and a tilting module $T_A$ such that $B = \text{End}(T_A)$. Tilted algebras together with recently developed covering techniques provide a rather general setting for dealing with arbitrary representation-finite algebras, that is, algebras with finitely many non-isomorphic finitely generated indecomposable modules. Happel and Ringel showed in [12] (see also [6, 15]) that representation-finite tilted algebra have the following nice characterization in the term of the associated Auslander-Reiten quiver: A connected representation-finite algebra $B$ is a tilted algebra if and only if the Auslander-Reiten quiver of $B$ contains a complete slice, that is, a set $S$ of indecomposable modules with the following properties

(i) Given any indecomposable module $X$, $S$ contains precisely one module from the orbit $\{\tau^r X; r \in \mathbb{Z}\}$ of $X$, where $\tau = DT\tau$ and $\tau^{-1} = TrD$ and $\tau^{-1} = TrD$ are the Auslander-Reiten operators [3].

(ii) If $X_0 \to X_1 \to X_2 \to \ldots \to X_r$ is a chain of non-zero maps and indecomposable modules, and $X_0$, $X_r$ belong to $S$, then all $X_i$ belong to $S$.

(iii) There is no oriented cycle of irreducible maps $U_0 \to U_1 \to \ldots \to U_r \to U_0$ with all $U_i$ in $S$.
Recently two interesting classes of representation-finite algebras, PHI algebras considered by Simson-Skowroński [18, 19] and trivial extension algebras investigated by Hughes-Waschbüsch [16] (see also [14]), have been completely classified by invariants involving only tilted algebras. In general the Auslander–Reiten quiver of such algebras contains no complete slice but the Auslander–Reiten quiver modulo projective-injectives has a complete slice of a Dynkin class.

In this paper we shall give a rather simple description of all algebras having this property. We use many ideas and extend results from [12, 16, 19].

We use the term algebra to mean finite-dimensional algebra over a fixed commutative field \( K \) and the term module to mean a finitely generated right module. Algebras, as is usual in representation theory, are assumed to be basic and connected. For any algebra \( A \) and an \( A \)-module \( M \) we shall denote by \( E_A(M) \) the \( A \)-injective envelope of \( M \), by \( P_A(M) \) the \( A \)-projective cover of \( M \), by \( \text{top}_A(M) \) the top of \( M \), by \( \text{soc}_A(M) \) the socle of \( M \), by \( \text{rad}(M) \) the radical of \( M \). For any indecomposable projective-injective \( A \)-module \( Q \), define \( a_A(\text{soc}_A(Q))=\text{top}_A(Q) \). Further, we will denote by \( \text{mod} \ A \) the category of (finite dimensional) \( A \)-modules and by \( \text{ind} \ A \) the full subcategory of \( \text{mod} \ A \) formed by the chosen representatives of the isomorphism classes of indecomposable modules. We will frequently ignore the distinction between the isomorphism class of a module and the module itself. Left modules will usually be regarded as right modules over the opposite algebra. We shall denote by \( D: \text{mod} \ A \longrightarrow \text{mod} \ A^{op} \) the usual duality \( \text{Hom}_K(-, K) \). We will use freely the properties of irreducible maps, almost split sequences, almost split morphisms, and the Auslander–Reiten operators \( \tau=DTr \) and \( \tau^{-1}=TrD \). For any algebra \( A \), we will denote by \( \Gamma_A \) the Auslander–Reiten quiver of \( A \) [10]. For definitions and further details we refer to [2, 3, 4, 5, 10]. Finally, for the definition of valued quivers and of the Cartan class of a valued quiver we refer to [11, 17].

1. Main result

In this section we formulate the main result of the paper. Let \( A \) be a connected basic algebra over a field \( K \) and let \( \mathcal{C} \) be a connected component of \( \Gamma_A \). Then a subquiver \( \mathcal{S} \) of \( \mathcal{C} \) is said to be path-complete if, whenever \( M \) and \( N \) are vertices of \( \mathcal{S} \) and there is a path \( M \longrightarrow \cdots \longrightarrow L \longrightarrow \cdots \longrightarrow N \) in \( \mathcal{C} \), \( L \) is a vertex of \( \mathcal{S} \). We say that a full subquiver \( \mathcal{S} \) of \( \mathcal{C} \) is a stable complete slice of \( \mathcal{C} \) if the following conditions are satisfied:

1. \( \mathcal{S} \) is path-complete.
2. There is no oriented cycles \( X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_r \longrightarrow X_0 \) with all \( X_i \) in \( \mathcal{S} \).
(3) \( S \) has no projective-injective modules.
(4) Given any non-projective-injective module \( X \) in \( \mathcal{C} \), \( S \) contains precisely one module from the orbit \( \{ \tau^r X; r \in \mathbb{Z} \} \) of \( X \).

It is easy to see that \( S \) is a stable complete slice in \( \mathcal{C} \) if and only if \( S \) is a complete slice of the full subquiver \( s\mathcal{C} \) of \( \mathcal{C} \) obtained by suppressing the vertices corresponding to projective-injective indecomposable modules.

A complete slice \( S \) of \( \mathcal{C} \) is of Dynkin class \( \mathcal{A} \) provided \( S \), considered as a nonoriented graph, is a Dynkin graph \( \mathcal{A} \). It follows from [9] that if \( A \) is a connected representation-finite hereditary algebra, then the vertices of \( \Gamma_A \) corresponding to the indecomposable projective \( A \)-modules form in \( \Gamma_A \) a complete slice of Dynkin class. If \( A \) is a hereditary representation-finite algebra, and \( T_A \) a tilting module, then the Cartan class of the tilted algebra \( B=\text{End}(T_A) \) is defined to be that of \( A \) (see [16]).

For any algebra \( A \), we will denote by \( F(A) \), the set of isomorphism classes of simple \( A \)-modules.

A system \( C \) of Dynkin class \( \mathcal{A} \) is defined to be \( C=(B, n, m, F_*, F'_*) \), where \( B \) is a tilted algebra of Dynkin class \( \mathcal{A} \), \( n \) and \( m \) are nonnegative integers, and \( F_*, F'_* \) are chains

\[
F_* : F(B) = F_0 \supseteq F_1 \supseteq \ldots \supseteq F_n
\]

\[
F'_* : F(B^\text{op}) = F'_1 \supseteq F'_2 \supseteq \ldots \supseteq F'_m
\]

of nonempty subsets of \( F(B) \) and \( F(B^\text{op}) \).

Then the algebra \( \mathcal{R}(C) \), for a given system \( C=(B, n, m, F_*, F'_*) \), is defined to be \( \mathcal{R}(C) = R(-m) \), where the sequence of algebras

\[
B = R(0), R(1), \ldots, R(n), R(-1), \ldots, R(-m)
\]

is obtained as follows:

\[
R(1) = \begin{pmatrix} E(1) & I(1) \\ 0 & R(0) \end{pmatrix}
\]

where \( I(1) = \bigoplus_{S \in F_i} E_S(S) \), \( E(1) = \text{End}_\mathbb{K}(I(1)) \), and \( I(1) \) has the canonical structure of \( E(1) - R(0) - \) bimodule. Let \( i \geq 1 \) and write \( \sigma_{R(i)} = \sigma_i \); similarly as in [19] one shows that the set \( F(R(i)) \) of \( R(i) \)-simples has a natural identification with the union of \( F(R(i-1)) \) and a new set of simples \( \bar{F}_i = \{ \sigma_1 \cdots \sigma_i(S); S \in F_i \} \). Then \( R(i+1) \), for \( i=1, \ldots, n-1 \), is the triangular matrix algebra

\[
R(i+1) = \begin{pmatrix} E(i+1) & I(i+1) \\ 0 & R(i) \end{pmatrix}
\]

where \( I(i+1) = \bigoplus_{S \in \bar{F}_i} E_{R(i)}(\sigma_1 \cdots \sigma_i(S)) \) and \( E(i+1) = \text{End}_{R(i)}(I(i+1)) \). Further, \( R(-1) \) is the triangular matrix algebra

\[
R(-1) = \begin{pmatrix} R(n) & I(-1) \\ 0 & E(-1) \end{pmatrix}
\]
where $I(-1) = \bigoplus_{S \in P_0} E_{R_{(-1); S}}^{op}(S)$ and $E(-1) = \text{End}_{R_{(-1); S}}^{op}(I(-1))$. Finally, for $-m \leq i \leq -2$, $R(i)$ is the triangular matrix algebra

$$R(i) = \begin{pmatrix} R(i+1), & I(i) \\ 0, & E(i) \end{pmatrix}$$

where $I(i) = \bigoplus_{S \in P_0} E_{R_{(-1); S}}^{op}(S)$ and $E(i) = \text{End}_{R_{(-1); S}}^{op}(I(i))$.

We can now formulate the main result of this paper.

**Theorem.** Let $A$ be a connected finite-dimensional basic algebra. Then $A$ is representation-finite and $\Gamma_A$ contains a stable complete slice of a Dynkin class if and only if $A$ is isomorphic to an algebra $\mathcal{R}(C)$ for some system $C$ of a Dynkin class.

2. **Proof of the theorem**

First we shall show that for any system $C = (B, n, m, F, F')$ of a Dynkin class, the algebra $\mathcal{R}(C)$ is representation-finite and $\Gamma_{\mathcal{R}(C)}$ contains a stable complete slice of a Dynkin class. We will apply results from [16].

Let $C = (B, n, m, F, F')$ be a system of a Dynkin class $\Delta$ and consider the doubly infinite matrix algebra without identity

$\hat{B} = \begin{pmatrix} B_{-1} & \cdots & M_{-1} \\ \vdots & \ddots & \vdots \\ B_n & \cdots & M_n \\ B_{n+1} & \cdots & M_{n+1} \end{pmatrix}$

in which matrices are assumed to have only finitely many entries different from zero, $B_n = B$ and $M_n = B \otimes D(B)$ for all integers $n$, all remaining entries are zero, and the multiplication is induced from the canonical maps $B \otimes B \rightarrow D(B)$, $D(B) \otimes D(B) \rightarrow D(B)$, and zero maps $D(B) \otimes D(B) \rightarrow 0$. Hughes and Waschbisch proved in [16] that mod $B$ has almost split sequences, the stable Auslander–Reiten quiver $\Gamma_B$ is isomorphic to $Z\Delta$ and that for any indecomposable projective $B$-module $P$, $\text{Hom}_B(P, X) \neq 0$ only for a finite number of nonisomorphic indecomposable $B$-modules $X$. It is not hard to see that all algebras $R(0) = B, R(1), \cdots, R(n), R(-1), \cdots, R(-m)$ occurring in the definition of $\mathcal{R}(C)$ are full finite subcategories of $\hat{B}$. Then from [2, §3], the algebras $R(i)$ are representation-finite and consequently $\mathcal{R}(C) = R(-m)$ is so. Now it suffices to prove that $\Gamma_{\mathcal{R}(C)}$ contains a stable complete slice of a Dynkin class. By assumption $\Gamma_B$ contains a complete slice $\mathcal{M}$. We shall prove that the modules from $\mathcal{M}$ being no projective-injective $\mathcal{R}(C)$-modules form a stable complete slice in $\Gamma_{\mathcal{R}(C)}$. First observe that the set $\mathcal{M}'$ of all modules from $\mathcal{M}$
being no projective-injective $B$-modules form a stable complete slice in $\Gamma_B$. Let $X$ be the set of all (isomorphism classes of) indecomposable projective $R(n)$-modules $Q \in X$ such that $\text{top}(Q) \in F(R(n)) \setminus F(B)$. From [20] we know that all modules from $X$ are also injective $R(n)$-modules. Choose a module $Q_i$ from $X$ such that $\text{rad}(Q_i)$ is not successor in $\Gamma_B$ of any module $\text{rad}(Q')$ for $Q' \in X$. Note that $\text{rad}(Q_i)$ is a $B$-module. Consequently, $\text{rad}(Q_i)$ is an injective $B$-module isomorphic to $\text{Hom}_{R(n)}(B, Q_i)$ and the algebras $T_i = \text{End}_{R(n)}(B \oplus Q_i)$ and

$$
\begin{pmatrix}
C_i & \text{rad}(Q_i) \\
0 & B
\end{pmatrix}
$$

where $C_i = \text{End}_R(\text{rad}(Q_i))$, are isomorphic. Algebra $T_i$ is representation-finite as a full subcategory of $B$. Moreover, just as in [16, 3.5, 3.6], we see that $\mathcal{M}'$ is a stable complete slice in $\Gamma_{T_i}$ provided $\text{rad}(Q_i)$ is not projective $B$-module. On the other hand, if $\text{rad}(Q_i)$ is projective-injective (as $B$-module), then $\text{rad}(Q_i)$ belongs to $\mathcal{M}$ and $\mathcal{M}' \cup \{\text{rad}(Q_i)\}$ forms a stable complete slice in $\Gamma_{T_i}$. Moreover, since all modules from $X$ are projective-injective $R(n)$-modules, if $Y = \text{rad}R(n)(Q)$, for $Q \in X$, is a projective-injective $T_i$-module, then $Y$ is a projective-injective $B$-module and so belongs to $\mathcal{M}$. Then we can repeat this procedure taking $T_i$ instead of $B$. Consequently, after a finite number of steps, we obtain $R(n)$ and the modules $Z$ from $\mathcal{M}$ being no projective-injective $R(n)$-modules form a stable complete slice in $\Gamma_{R(n)}$. Then the corresponding $R(n)^{op}$-modules $D(Z)$ form a stable complete slice in $\Gamma_{R(n)^{op}}$. Considering $\mathcal{R}(C)^{op}$-modules $Q$ whose tops belong to $F'((\mathcal{R}(C)) \setminus F'(R(n)))$, and applying above arguments, we conclude that the modules $D(Z)$, where $Z$ ranges over all modules $Z$ from $M$ being no projective-injective $\mathcal{R}(C)$-modules, form a stable complete slice in $\Gamma_{\mathcal{R}(C)^{op}}$. Consequently, $\Gamma_{\mathcal{R}(C)}$ contains a stable complete slice $\mathcal{S}$ being a connected subgraph of the complete slice $\mathcal{M}$ of $\Gamma_B$. Since $\mathcal{M}$ is of Dynkin class, $\mathcal{S}$ is so and we are done.

At the end of this paper we shall give an example showing that the graphs $\mathcal{M}$ and $\mathcal{S}$ can be different.

Now let $A$ be a representation-finite algebra and let $\Gamma_A$ contains a stable complete slice $\mathcal{M} = \{M_1, \ldots, M_l\}$ of Dynkin class $\mathcal{J}$. We shall show that $A$ is isomorphic to an algebra $\mathcal{R}(C)$ for some system $C = (B, n, m, F_s, F_\mu)$ of Dynkin class $\mathcal{J}$.

We start with the following lemma.

**Lemma 1.** Under the above assumption, $\Gamma_A$ has no oriented cycle.

**Proof.** Assume that $\Gamma_A$ has an oriented cycle

$$X_0 \rightarrow X_1 \rightarrow \ldots \rightarrow X_r \rightarrow X_0.$$
Since \( \Gamma_A \) has a stable complete slice, one of the modules \( X_0, X_1, \ldots, X_s \) is projective-injective. Indeed, in the opposite case, similarly as in [12, Prop. 8.1] one proves that there is an oriented cycle \( Y_0 \to Y_1 \to \cdots \to Y_s \to Y_0 \) with all modules \( Y_j \) from \( \mathcal{M} \), but this is a contradiction to the stable slice condition (2). Denote by \( \mathcal{D} \) the full subcategory of \text{ind} \ A formed by all non-projective-injective modules. From the stable slice condition (4), for each module \( X \) of \( \mathcal{D} \), there is exactly one module \( M_i \) from \( \mathcal{M} \) and one integer \( z \) such that \( X = \tau^z(M_i) \), and put \( z = z(X) \). Suppose that there is an irreducible map \( X = \tau(X)(M_i) \to Y = \tau(X)(M_j) \) between two objects \( X \) and \( Y \) from \( \mathcal{D} \). Then \( z(X) = z(Y) \) and there is an irreducible map \( M_i \to M_j \) or \( z(X) = z(Y) + 1 \) and there is an irreducible map \( M_i \to M_j \). Indeed, if \( z(X) = z(Y) \) then obviously there is an irreducible map \( M_i \to M_j \). If \( z(X) \leq 0 \) and \( z(Y) \geq 0 \), then there is a chain of irreducible maps \( M_i \to \cdots \to \tau^{(l)}(M_i) \to \cdots \to M_j \) and by the stable slice condition (4), \( z(X) = z(Y) = 0 \). Consider the case \( z(X) > z(Y) > 0 \). Then there is an irreducible map \( \tau(X)(M_i) \to M_j \), hence a chain of irreducible maps \( M_j \to \tau(X)(M_i) \to \cdots \to M_i \) and, by the stable slice condition (4), \( z(X) = z(Y) + 1 \). Similarly, if \( z(Y) > z(X) > 0 \), there is a chain of irreducible maps \( M_i \to \tau(X)(M_j) \to \cdots \to M_j \) and \( z(Y) = z(X) > 0 \), contrary to the stable slice conditions (1) and (4). Analogically one proves that \( z(X) \equiv z(Y) + 1 \) if \( z(X) \equiv z(Y) \), \( z(X) < 0 \), \( z(Y) < 0 \). Finally, if \( z(X) > 0 \), \( z(Y) \leq 0 \), then \( z(X) = 1 \), \( z(Y) = 0 \); and \( z(X) = 0 \), \( z(Y) = -1 \) in case \( z(X) > 0 \) and \( z(Y) < 0 \).

Consequently one of the modules in the cycle \( X_0 \to X_1 \to \cdots \to X_s \to X_0 \) is projective-injective. Without loss of generality we can assume that this is \( X_i \). If \( X_i \) is projective-injective, then \( X_i = \text{rad}(X_i) \), \( X_{i+1} = X_i / \text{soc}(X_i) \), \( X_{i+1} = \tau(X_{i+1}) \), and \( z(X_{i+1}) = z(X_{i+1}) + 1 \). Thus, from the above remarks, \( z(X_0) > z(X_i) \geq \cdots \geq z(X_s) \) and we get a contradiction. Therefore \( \Gamma_A \) has no oriented cycles and the lemma is proved.

Denote by \( \Psi_A \) (resp. \( \mathfrak{z}_A \)) the set of projective (resp. injective) modules in \( \text{ind} \ A \) and by \( \Sigma_A \) the sum \( \Psi_A \cup \mathfrak{z}_A \). Let us denote by \( \nu : \Sigma_A \to \Sigma_A \) and \( \nu^{-1} : \Sigma_A \to \Sigma_A \) two partial functions defined as follows: For each \( X \in \Sigma_A \), \( \nu(X) \) is defined iff \( X \in \Psi_A \) and then \( \nu(X) = E(\text{top}(X)) \); \( \nu^{-1}(X) \) is defined iff \( X \in \mathfrak{z}_A \), and then \( \nu^{-1}(X) = P(\text{soc}(X)) \). Then the set \( \{ \nu(X) ; z \in \mathbb{Z}, \nu(X) \text{ is defined} \} \) is said to be the \( \nu \)-orbit of \( X \in \Sigma_A \).

Let us denote by \( \mathcal{S} = \{ S_1, \ldots, S_r \} \) the set of all composition factors of modules \( M_1, \ldots, M_s \), and by \( B \) the algebra \( \text{End}_A(P_A(S_1) \oplus \cdots \oplus P_A(S_r)) \). As in [16, Lemmas 3.2, 3.3] one proves that any \( \nu \)-orbit in \( \Sigma_A \) contains exactly one module from the set \( \{ P_A(S_1), \ldots, P_A(S_r) \} \) and that the set \( \mathcal{M} \) considered as a set of \( B \)-modules is a complete slice of \( \Gamma_B \) of Dynkin class \( A \). In particular, \( B \) is a tilted algebra of Dynkin class \( D \). Moreover, any \( \nu \)-orbit in \( \Sigma_A \) is the \( \nu \)-orbit of some module \( P_A(S_j) \), \( j = 1, \ldots, r \), and we can define the function \( s : \Sigma_A \to \mathbb{Z} \) such that, for \( X \in \Sigma_A \), \( s(X) = i \) iff \( X = \nu^i(P_A(S_j)) \) for some \( j = 1, \ldots, r \). Thus, for \( X \in \Sigma_A \), \( s(X) \leq 0 \) implies \( X \in \Psi_A \),
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Let \( A_\delta = Q_1 \oplus \cdots \oplus Q_n \) be some decomposition as a direct sum of indecomposable projective \( A \)-modules, \( n = \max \{ s(X); X \in \Sigma_A \} \), \( m = - \min \{ s(X); X \in \Sigma_A \} \), and we let \( A_p = \oplus_{p-m} A_p \) and put for \( -m \leq p \leq n \), \( E_{p, q} = \operatorname{End}_A(\oplus_{k=p}^q A_k) \). We will write simply \( E_p \) instead of \( E_{0, p} \) and \( B_{q, 0} \) instead of \( E_{q, 0} \). Obviously the algebras \( B \) and \( E_0 \) are isomorphic.

In our proof an important role is played by the following lemma.

**Lemma 2.** In the above notation, \( \operatorname{Hom}_A(A_p, A_q) = 0 \) for \( p > q \) and \( q > p + 1 \).

**Proof.** Suppose that \( \operatorname{Hom}_A(A_p, A_q) \neq 0 \) for some \( p > q \). Then there are two indecomposable summands \( X \) of \( A_p \) and \( Y \) of \( A_q \) with \( \operatorname{Hom}(X, Y) \neq 0 \). First assume \( p > 0, q < 0 \). In this case \( X \) is projective-injective, there is a sequence of non-zero maps

\[
\begin{array}{c}
\oplus_{i=1}^q M_i \rightarrow \nu^{-p}(X) \rightarrow \cdots \rightarrow X \rightarrow Y \rightarrow \cdots \rightarrow \nu^{-q}(Y) \rightarrow \oplus_{i=1}^q M_i
\end{array}
\]

implying the corresponding sequence of irreducible maps, and we get a contradiction to the stable slice condition (3). If \( p = 0 \) and \( f : X \rightarrow Y \) is a non-zero map, then since \( \nu(X) \) is injective, there is a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & \operatorname{im}(f) \xrightarrow{g} Y \\
\downarrow{\alpha} & & \downarrow{\beta} \\
\topo(X) & \xleftarrow{\gamma} & \nu(X)
\end{array}
\]

where \( \alpha, \beta, \gamma, \sigma \) are canonical epimorphisms, \( \gamma, \sigma \) canonical monomorphisms, and obviously \( q \neq 0 \). Similarly, there is a non-zero map \( h : \nu(X) \rightarrow \nu(Y) \). But \( s(\nu(Y)) = q + 1 < 0 \), \( \nu(Y) \) is projective-injective, there is a sequence of irreducible maps

\[
\begin{array}{c}
\oplus_{i=1}^q M_i \rightarrow \nu^{-p}(X) \rightarrow \cdots \rightarrow \nu(X) \rightarrow \cdots \rightarrow \nu(Y) \rightarrow \oplus_{i=1}^q M_i
\end{array}
\]

and we get a contradiction to the stable slice conditions (1) and (3). If \( 0 > p > q \), then as above we conclude that \( \operatorname{Hom}(\nu^{-p}(X), \nu^{-q}(Y)) \neq 0 \), but this is impossible since \( s(\nu^{-p}(X)) = 0 \) and \( s(\nu^{-q}(Y)) = q - p < 0 \). Finally, in the case \( p > q > 0 \), similarly, as in [19, Lemma p. 60], we prove that \( \operatorname{Hom}(\nu^{-q}(X), \nu^{-q}(Y)) \neq 0 \). Since \( s(\nu^{-q}(X)) = p - q > 0, s(\nu^{-q}(Y)) = 0 \), from the first part of our proof, it is impossible. Consequently, \( \operatorname{Hom}(A_p, A_q) = 0 \) for \( p > q \).

Now assume that \( \operatorname{Hom}(A(X, Y)) \neq 0 \) for \( p < q - 1 \) and indecomposable direct summands \( X \) of \( A_p \) and \( Y \) of \( A_q \). If \( p > 0 \), as in [19, Lemma p. 60], \( \operatorname{Hom}(\nu^{-1}(Y), X) \neq 0 \), and since \( s(\nu^{-1}(Y)) = q - 1 > p \) we get a contradiction to the fact that \( \operatorname{Hom}(A_{q-1}, A_p) = 0 \). If \( p < 0 \), \( \nu(X) \) is projective-injective, and, as in the first part of the proof, we conclude that \( \operatorname{Hom}(Y, \nu(X)) = 0 \). This is a contradiction since
s(ν(X))=p+1<q=s(Y) and \(\text{Hom}_A(A_p, A_{p+1})=0\). Therefore, \(\text{Hom}_A(A_p, A_q)=0\) for \(p+1<q\) and the lemma is proved.

In our proof we shall need the following fact.

**Lemma 3.** For \(p<0\), \(\text{Hom}_A(A_p, A)\) is a projective-injective \(\Lambda^{op}\)-module.

**Proof.** Let \(X\) be an indecomposable direct summand of \(\text{Hom}_A(A_p, A)\). Then \(D(X)\equiv E(D(\text{top}_{\Lambda^{op}}(X))\equiv E(\text{top}_\Lambda(Y))=\nu(Y)\) for \(Y=\text{Hom}_A(\nu(X), A)\). Since \(Y\) is a direct summand of \(A_{p+1}\) and \(s(\nu(Y))=p+1\leq 0\), \(\nu(Y)\) is a projective-injective \(\Lambda\)-module. Hence \(X\equiv D(\nu(Y))\) is a projective-injective \(\Lambda^{op}\)-module and we are done.

Now we shall define a system \(C=(B, n, m, F_\ast, F_*\ast)\) where \(B=\text{End}_A(A_0)\), \(n=\max\{s(X)\mid X \in \Sigma_{\Lambda}\}, m=\min\{s(X)\mid X \in \Sigma_{\Lambda}\}\). The canonical action of \(B\) on \(\text{top}_\Lambda(A_0)\) (resp. \(\text{top}_{\Lambda^{op}}(\text{Hom}_A(A_0, A))\)) enabling us to identify the set \(F(B)\) (resp. \(F(\Lambda^{op})\)) with the set \(F_\ast\) (resp. \(F_*\ast\)) of simple \(\Lambda\)-module (resp. \(\Lambda^{op}\)-module) components of \(\text{top}_\Lambda(A_0)\) (resp. of to \(\text{top}_{\Lambda^{op}}(\text{Hom}_A(A_0, A))\)). Then \(F_\ast\) consists of the simple components of \(\text{soc}_\Lambda(A_1)\) (a summand of \(F_\ast=\text{top}(A_0)\)); for \(1\leq i<\ell, F_{i+1}\) consists of the simples \(S\) in \(F_\ast\) such that \(\sigma_\Lambda(S)\) is a component of \(\text{soc}_\Lambda(A_{i+1})\). Similarly, \(F_*\ast\) consists of the simples components of \(\text{soc}_{\Lambda^{op}}(\text{Hom}_A(A_{i-1}, A))\) (a summand of \(F_*\ast\)); for \(1\leq j<m, F_{j+1}\ast\), consists of the simples \(S\) in \(F_*\ast\) such that \(\sigma_{\Lambda^{op}}(S)\) is a component of \(\text{soc}_{\Lambda^{op}}(\text{Hom}_A(A_{j-1}, A))\).

From Lemma 2 it follows that \(A=\text{End}_A(A_{A})\) is isomorphic to the matrix algebra

\[
\begin{pmatrix}
E_n & nM_{n-1} & 0 \\
0 & E_{n-1} & n-1M_{n-2} & 0 \\
0 & \ddots & \ddots & \ddots \\
E_1 & sM_0 & 0 \\
0 & E_0 & sM_{-1} & 0 \\
0 & \ddots & \ddots & \ddots \\
E_{m+1} & -s+1M_{-m} \\
0 & \cdots & \cdots & E_{-m}
\end{pmatrix}
\]

where \(s+1M_i\) is the \(E_{i+1};E_{i}\)-bimodule \(\text{Hom}_A(A_i, A_{i+1})\). First we shall prove that the algebras \(B_i \) and \(R(i), i=0, \cdots, n\), are isomorphic. We shall proceed by induction, using [19, Proposition 2] and Lemma 2. For \(i=0\), \(B_0=R(0)\) by definition. Assume that for some \(i\geq 0\) there is an isomorphism \(h: B_i \to R(i)\). Observe that there is a canonical isomorphism of algebras
Remark that $A_{i-1} = \Hom_{E(U_i)}(i^{\bot}A_k, A_{i-1})$ is an injective $B_{i-1}$-module and $i^{\bot}M_i = \Hom_{E(U_i)}(i^{\bot}A_k, A_{i-1})$ is an injective $B_i$-module as the greatest $B_i$-submodule of $A_{i-1}$. Similarly as in [19, Proposition 2] we conclude that the algebras $E_{i-1}$ and $\End_{B_i}(i^{\bot}M_i)$ are isomorphic. By definition of $A_{i-1}$ and $I(i+1)$ it is not hard to see that $I(i+1) \cong \Hom_{R_k}(A_k, A_{i-1})$ is an injective $I(i+1)$-module as the greatest $I(i+1)$-submodule of $A_{i-1}$. Then $A \cong B_m \cong R(-m) = \mathcal{R}(C)$ and this completes the proof of the theorem.

We end the paper with an example illustrating previously considered questions.

Let $B$ be the tilted algebra of Dynkin class $D_4$ given by the bounden quiver algebra (see [10]) $KQ/I$, where

$$Q: \quad 4 \xrightarrow{\alpha} 3 \xrightarrow{\beta} 2 \xrightarrow{\gamma} 3' \xrightarrow{\xi} 4'$$

and $I$ is generated by the composed arrows $\alpha \beta$ and $\beta \gamma$. Consider the system $C = (B, 1, 1, F_*, F'_*)$ where $F_*$ consists of one simple $B$-module given by the vertex 4 and $F'_*$ consists of one simple $B^{\text{op}}$-module given by the vertex 3. Then it is easy to see that $\mathcal{R}(C)$ is the bounden quiver algebra $KQ'/I'$ where

$$Q': \quad 4' \xrightarrow{\sigma} 4 \xrightarrow{\tau} 3 \xrightarrow{\gamma} 3'$$

and $I'$ is generated by $\sigma \tau$, $\tau \gamma$, $\xi \alpha \beta$ and $\xi \alpha \gamma \beta$. Then a straightforward calculation shows that $I'_{\beta; \alpha \beta}$ is of the form

$$\begin{align*}
\text{rad}(P_3) &\xrightarrow{S_3} P_3 \xrightarrow{P_3/P_1} S_2 \xrightarrow{P_2/P_1} S_2 \\
P_3 &\xrightarrow{S_2} P_3 \xrightarrow{P_3/P_1} S_2 \xrightarrow{P_2/P_1} S_2 \\
P_3 &\xrightarrow{S_2} P_3 \xrightarrow{P_3/P_1} S_2 \xrightarrow{P_2/P_1} S_2 \\
P_3 &\xrightarrow{S_2} P_3 \xrightarrow{P_3/P_1} S_2 \xrightarrow{P_2/P_1} S_2
\end{align*}$$

where $P_1 = P(S_1)$ and $S_i$ denotes the simple module given by the vertex $i$. Here, $P_3$, $P_4$ and $P_5$ are projective-injective and the modules $S_i$, $S_2$ and $P_3/S_2$ form a stable complete slice of class $A_{2n}$, so different from the Dynkin class of $B$. On the other hand, $\mathcal{R}(C)$ is isomorphic to the algebra $\mathcal{R}(\tilde{C})$ where $\tilde{C}$ is the system $(\tilde{B}, 2, 1, \tilde{F}_*, \tilde{F}'_*)$ and $\tilde{B}$ is the path algebra of $1 \leftarrow 3 \rightarrow 2$, $\tilde{F}_* = \tilde{F}'_*$ (resp. $\tilde{F}'_*$) consists of the simple $\tilde{B}$-module (resp. $\tilde{B}^{\text{op}}$-module) given by the vertex 3.
References


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