A CHARACTERIZATION OF ABSOLUTE NEIGHBORHOOD RETRACTS IN GENERAL SPACES

Dedicated to Professor Keio Nagami on his 60th birthday

By

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Some characterizations of absolute neighborhood retracts were established in separable metric spaces by Hanner [4]. Hanner's characterizations were easily extended to the metric case. For this, see Hu [5] Chapter IV. In this paper, we shall extend one of Hanner's characterizations to more general spaces, especially, stratifiable spaces, spaces with a $\sigma$-almost locally finite base and paracomplexes. For the ANR theory of these spaces, we refer Cauty [2], Miwa [8] and Hyman [6], respectively.

Throughout this paper, all spaces are assumed to be paracompact normal spaces and all maps to be continuous. $I$ and $S$ denote the closed unit interval $[0,1]$ and the class of all stratifiable spaces, respectively. $ANR(Q)$ (resp. $ANE(Q)$) is the abbreviation for absolute neighborhood retract (resp. extensor) for the class $Q$. For these definitions, see [5].

In this paper, all theorems are proved in the class $S$. But these theorems can be proved in some other classes. For instance, see Remark 2.3.

1. Preliminaries.

DEFINITION 1.1 ([3]). A space $Y$ is equiconnected if there is a map $F: Y \times Y \times I \rightarrow Y$ such that $F(x, y, 0) = x, F(x, y, 1) = y$ and $F(x, x, t) = x$ for all $(x, y) \in Y \times Y$ and $t \in I$. The space $Y$ is said to be locally equiconnected if $F$ is defined only on $U \times I$, for some neighborhood $U$ of the diagonal of $Y \times Y$.

DEFINITION 1.2 ([4]). Let $f, g: Y \rightarrow X$ be two maps. If $X$ is covered by $\mathcal{U} = \{U_a\}$, $f$ and $g$ are called $\mathcal{U}$-near if for each $y \in Y$ there is a $U_a \in \mathcal{U}$ such that $f(y) \in U_a$, $g(y) \in U_a$.

DEFINITION 1.3 ([4]). Let $h_t: Y \rightarrow X$ be a homotopy. If $X$ is covered by $\mathcal{U} = \{U_a\}$, $h_t$ is called a $\mathcal{U}$-homotopy if for each $y \in Y$ there is a $U_a \in \mathcal{U}$ such that $h_t(y) \in U_a$ for all $t \in I$. 

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The following proposition is easily verified. For instance, see Cauty [2]. But for completeness, we state the proof.

**Proposition 1.4.** If $Y$ is an ANR$(S)$, then $Y$ is locally equiconnected.

**Proof.** Let $A = Y \times Y \times \{0, 1\} \cup \Delta \times I$, where $\Delta$ is the diagonal of $Y \times Y$. We define a function $f : A \to Y$ as follows: $f(x, y, 0) = x$, $f(x, y, 1) = y$ and $f(x, x, t) = x$ for all $t \in I$. Then $f$ is continuous. Since $Y$ is an ANR$(S)$ by [1] Corollary 6.3, there is a neighborhood $U$ of $\Delta$ in $Y \times Y$ and a map $F : U \times I \to Y$ such that $F|A = f$. Therefore $Y$ is locally equiconnected.

2. **Main theorems.**

In this section, we extend Hanner's theorems [4] Theorem 4.1 and 4.2 to stratifiable case. Each proof is simpler than Hanner's one.

**Theorem 2.1.** If $Y$ is an ANR$(S)$ and $\mathcal{U} = \{U_s\}$ a given open covering of $Y$, then there exists an open covering $\mathcal{W}$ of $Y$, which is refinement of $\mathcal{U}$, such that, for any two $\mathcal{W}$-near maps $f, g : X \to Y$ defined on a stratifiable space $X$ and any $\mathcal{W}$-homotopy $j_t : A \to Y$, $(0 \leq t \leq 1)$, defined on a closed subspace $A$ of $X$ with $j_0 = f|A$ and $j_1 = g|A$, there exists an $\mathcal{U}$-homotopy $h_t : X \to Y$, $(0 \leq t \leq 1)$, such that $h_0 = f$, $h_1 = g$ and $h_t|A = j_t$ for every $t \in I$.

**Proof.** Since $Y$ is locally equiconnected by Proposition 1.4, there exist a neighborhood $U$ of the diagonal of $Y \times Y$ and a map $F : U \times I \to Y$ such that $F(x, y, 0) = x$, $F(x, y, 1) = y$ and $F(x, x, t) = x$ for all $(x, y) \in U$ and $t \in I$. For any $y \in Y$, since $I$ is compact, there exists an open neighborhood $V_y$ of $y$ such that $V_y \times V_y \subset U$ and $F(V_y \times V_y \times I) \subset U$, for some $U \in \mathcal{U}$. Let $\mathcal{U}' = \{V_y : y \in Y\}$ and $\mathcal{U}''$ be a barycentric refinement of $\mathcal{U}$; i.e., the covering $\{St(y, \mathcal{U}'') : y \in Y\}$ refines $\mathcal{U}$. For any $y \in Y$, there exists an open neighborhood $W_y$ of $y$ such that $F(W_y \times W_y \times I) \subset V'$ for some $V' \in \mathcal{U}'$. Let $\mathcal{W}' = \{W_y : y \in Y\}$. Then it is obvious that $\mathcal{W}$ refines $\mathcal{U}'$ and $W_y \times W_y \subset U$ for each $y \in Y$.

Now, let $f, g : X \to Y$ be any two $\mathcal{W}$-near maps defined on a stratifiable space $X$ and let $j_t : A \to Y$, $(0 \leq t \leq 1)$, be given $\mathcal{W}$-homotopy defined on a closed subspace $A$ of $X$ with $j_0 = f|A$ and $j_1 = g|A$.

By using the map $F$, we can construct a $\mathcal{U}'$-homotopy $k_t : X \to Y$, $(0 \leq t \leq 1)$, by taking

$$k_t(x) = F(f(x), g(x), t) \quad \text{for } x \in X \text{ and } t \in I.$$  

Since $f, g$ are $\mathcal{W}$-near maps, it is clear that $k_t$ is a $\mathcal{U}'$-homotopy.
In the topological product $P = X \times I$, consider the closed subspace $Q = (X \times \{0, 1\}) \cup A \times I$ and define a map $m : Q \to Y$ by taking

$$m(x, t) = \begin{cases} f(x) & \text{(if } x \in X \text{ and } t = 0) \\ j_1(x) & \text{(if } x \in A \text{ and } t \in I) \\ g(x) & \text{(if } x \in X \text{ and } t = 1) \end{cases}$$

Since $Y$ is ANR($S$), it follows that $m$ has an extension $m' : N \to Y$ over neighborhood $N$ of $Q$ in $P$. Since $I$ is compact, there exists an open neighborhood $C$ of $A$ in $X$ such that $C \times I$ is contained in $N$ and that a homotopy $n_t : C \to Y$, $(0 \leq t \leq 1)$, defined by

$$n_t(x) = m'(x, t), \quad (x \in C, t \in I)$$

is a $W$-homotopy. Therefore of course $n_t$ is a $C\cup'$-homotopy.

Since $X$ is stratifiable, there exists an open subset $B$ in $X$ such that $A \subset B \subset \bar{B} \subset C$. By Urysohn's lemma, there exists a map $e : X \to I$ such that

$$e(x) = \begin{cases} 0, & \text{(if } x \in X - B) \\ 1, & \text{(if } x \in A) \end{cases}$$

Define a homotopy $h_t : X \to Y$, $(0 \leq t \leq 1)$, by taking

$$h_t(x) = \begin{cases} k_t(x) & \text{(if } x \in X - B) \\ F(k_t(x), n_t(x), e(x)) & \text{(if } x \in C) \end{cases}$$

Then $h_t$ is well-defined. Indeed, since $k_t$, $n_t$ are $C\cup'$-homotopies, for each $x \in C$ there exist some $V'_1 \in C\cup'$ and $V'_2 \in C\cup'$ such that $k_t(x) \in V'_1$ and $n_t(x) \in V'_2$ for any $t \in I$. By the fact $k_0(x) = n_0(x) = f(x)$, $V'_1 \cap V'_2 = \emptyset$. Therefore there is a $V_y \in C\cup$ with $V'_1 \cup V'_2 \subset V_y$ since $C\cup'$ is a barycentric refinement of $C\cup$. Thus for any $t \in I$, $(h_t(x), n_t(x)) \in V'_1 \times V'_2 \subset U$.

It can be easily verified that $h_t$ is a $C\cup$-homotopy satisfying the required properties. This completes the proof.

The following theorem is easy to see by Theorem 2.1 and the same method of Hanner [4] Theorem 4.2 (or Hu [5] p. 114 Theorem 1.3).

**Theorem 2.2.** A necessary and sufficient condition for a stratifiable space $Y$ to be an ANR($S$) is the existence of an open covering $W$ of $Y$ such that, for any two $W$-near maps $f, g : X \to Y$ defined on a stratifiable space $X$ and any $W$-homotopy $j_t : A \to Y$, $(0 \leq t \leq 1)$, defined on a closed subspace $A$ of $X$ with $j_0 = f|A$ and $j_1 = g|A$, there exists a homotopy $h_t : X \to Y$, $(0 \leq t \leq 1)$, with $h_0 = f$, $h_1 = g$ and $h_t|A = j_t$ for every $t \in I$.

**Remark 2.3.** In this paper, we considered exclusively in the class $S$. But
if we reconsider the proofs of Proposition 1.4, Theorem 2.1 and 2.2, it is found that, for each class $Q$ satisfying the following four conditions, Proposition 1.4, Theorem 2.1 and 2.2 are valid.

(1) Every $X \in Q$ is paracompact normal.
(2) If $A$ is a closed (resp. an open) subspace of $X \in Q$, then $A \in Q$.
(3) For $X \in Q$, $X^4 \in Q$.
(4) A space $X \in Q$ is an ANR($Q$) if and only if $X$ is an ANE($Q$).

Indeed, these conditions are used in the proofs of theorems as follows: The condition (1) has been used in the proof of Theorem 2.2 ("every local ANR($Q$) is an ANR($Q'$)) and the proof of Theorem 2.1 ("$C'$ is a barycentric refinement of $C'$"). The condition (2) has been used in the proof of Theorem 2.1 ("a closed subspace $A$ of $X \in Q$ is in $Q$ and $Q$ is in $Q'$") and in the proof of Theorem 2.2 ("$Q$ is open hereditary"). The condition (3) has been used in the proof of Proposition 1.4 ("$A \in Q$") and in the proof of Theorem 2.1 ("$X \times I \in Q$"; by $(X+I)^4 \in Q$ and the condition (2)). The condition (4) has been used in the proof of Proposition 1.4 ("$f$ has an extension $F$") and in the proof of Theorem 2.1 ("$m$ has an extension $m'$").

Of course, the class $S$ satisfies these conditions, and for instance, the following classes also satisfy these conditions: Paracomplex (Hyman [6]), space with a $\sigma$-almost locally finite base (Itô and Tamano [7] and Miwa [8]) and paracompact $\sigma$-space (Okuyama [9]).

References