**DTr-INVARINT MODULES**

By

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Throughout this paper, we shall work over a fixed basic artin algebra $A$ and deal only with finitely generated right modules. Let $X$ be an indecomposable module. We say that $X$ is $DTr$-invariant if $DTrX \cong X$. In [7], with some other conditions, the author has shown that $A$ is a local Nakayama algebra if there is a $DTr$-invariant module. The aim of this paper is to generalize this result.

Recall that an indecomposable module $X$ is said to be $DTr$-periodic if, more generally, $DTr^n X \cong X$ for some positive integer $n$. In Riedtmann [8], Todorov [10] and Happel-Preiser-Ringel [6], they have completely determined the Cartan class of a component of the stable Auslander-Reiten quiver containing a $DTr$-periodic module (see [6] for details). In [6], they have also shown that a component of the Auslander-Reiten quiver containing a $DTr$-periodic module is a quasi-serial component (in the sense of [9]) if it contains neither projective nor injective modules. It seems, however, that there has not been given any characterization of a component of the (not stable) Auslander-Reiten quiver containing a $DTr$-periodic module. In this paper, we shall investigate the case in which there is a $DTr$-invariant module and prove

**Theorem 1.** Suppose there is a $DTr$-invariant module $A$. Then either $A$ is a local Nakayama algebra or the component of the Auslander-Reiten quiver containing $A$ is a quasi-serial component (in the sense of [9]) consisting of only $DTr$-invariant modules.

Let $X$ be a $DTr$-invariant module and $Y$ an indecomposable summand of the middle term of the Auslander-Reiten sequence ending in $X$. Then there are irreducible maps both from $X$ to $Y$ and from $Y$ to $X$. The converse holds.

**Theorem 2.** Let $X, Y$ be indecomposable modules. Suppose there are irreducible maps both from $X$ to $Y$ and from $Y$ to $X$. Then either $X$ or $Y$ is $DTr$-invariant. Thus either $A$ is a local Nakayama algebra or the component of the Auslander-Reiten quiver containing $X, Y$ is a quasi-serial component (in the sense of...
of [9]) consisting of only $DTr$-invariant modules.

Recently, the author learned that the similar result of Theorem 2 was obtained by K. Bautista and S.O. Smalø [4].

It is well known that there is a quasi-serial component consisting of only $DTr$-invariant modules if $A$ is an hereditary algebra of tame representation type (see [5]).

The proof of Theorems 1, 2 will be performed by calculating composition lengths, and in that of Theorem 1 the work of Auslander [1, Theorem 6.5] will play an impotant roll (see also [10, Proposition 2.3]).

For an indecomposable module $X$, let $F(X) = \text{End}(X)/\text{Rad}(X)$, this is a division ring, and for two indecomposable modules $X, Y$, let $N(X, Y) = \text{Rad}(X, Y)/\text{Rad}^2(X, Y)$, this is an $F(Y) - F(X)$-bimodule called the bimodule of irreducible maps (see [8], [10] for details).

The Auslander-Reiten quiver has as vertices the isomorphism classes of the indecomposable modules, and there is an arrow $[X] \rightarrow [Y]$ if $N(X, Y) \neq 0$, which is endowed with the valuation $(d_{XY}, d'_{XY})$ such that $d_{XY} = \dim_{F(Y)} N(X, Y)$ and $d'_{XY} = \dim N(X, Y)_{F(X)}$. Two indecomposable modules $X, Y$ belong, by definition, to the same component if there is a sequence $X = X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_r = Y$ of indecomposable modules such that either $N(X_{i-1}, X_i) \neq 0$ or $N(X_i, X_{i-1}) \neq 0$ for all $i$.

We refer to [2], [3] for $DTr$, Auslander-Reiten sequences and so on, and shall freely use results of [2], [3].

In what follows, we denote by $\tau$ (resp. $\tau^{-1}$) $DTr$ (resp. $TrD$) and by $|X|$ the composition length of a module $X$.

1. Proof of Theorem 1.

Let $A$ be a $\tau$-invariant module and $0 \rightarrow A \rightarrow \bigoplus_{i=0}^r B_i \rightarrow A \rightarrow 0$ be the Auslander-Reiten sequence, where $B_i$'s are non-isomorphic indecomposable modules and $a_i = \dim_{F(B_i)} N(A, B_i)$ for all $i$. By induction, it is sufficient to show that the possible cases are the following:

1. Some $B_i$ is projective-injective. We get $\text{rad} B_i \cong A \cong B_i/\text{soc} B_i$, thus $\text{top}(\text{rad} B_i) \cong \text{top} B_i$, this means that $A$ is a local Nakayama algebra.
2. We have $r = 1$, $a_1 = 1$, and $B_1$ is $\tau$-invariant.
3. We have $r = 2$, $a_1 = a_2 = 1$, and each $B_i$ is $\tau$-invariant.

We have to exclude the other cases. Note that $\tau B_i \cong B_j$, $a_i = a_j$ for some $j$ if $B_i$ is not projective, and that $\tau^{-1} B_i \cong B_k$, $a_i = a_k$ for some $k$ if $B_i$ is not injective.
Consider, first, the case in which some $B_t$ is not $\tau$-periodic. Then $\tau^n B_t$ is projective for some non-negative integer $n$, and $\tau^m B_t$ is injective for some non-positive integer $m$. Since $2|A| = \sum_{i=1}^r a_i |B_i|$, we conclude that $n = m = 0$ and $B_t$ is projective-injective.

Next, assume that all $B_t$'s are $\tau$-periodic. Let $0 \to \tau B_t \to A^l \oplus C_t \to B_t \to 0$ be the Auslander-Reiten sequence for each $i$, where $a_i' = \dim N(A, B_t)_{F(A)}$. We get

$$a_i' |A| + |C_i| = |\tau B_t| + |B_t|,$$

hence

$$\left( \sum_{i=1}^r a_i a_i' \right) |A| + \sum_{i=1}^r a_i |C_i| = \sum_{i=1}^r a_i |\tau B_t| + \sum_{i=1}^r a_i |B_t|$$

$$= 2|A| + 2|A|$$

$$= 4|A|.$$ 

Therefore we conclude that $\sum_{i=1}^r a_i a_i' \leq 4$.

(c) Suppose $\sum_{i=1}^r a_i a_i' = 4$. Then $C_i = 0$ for all $i$. Hence we get a finite component $\{ A, B_t, \ldots, B_r \}$ consisting of only $\tau$-periodic modules, a contradiction (cf. [1, Theorem 6.5]).

(d) Suppose $r = 1$, $a_1 a_1' = 3$. By (b) we get $a_1 |C_1| = |A|$, and clearly $B_1$ is $\tau$-invariant. We get

$$2 |B_1| = a_1' |A| + |C_1|$$

$$= a_1 a_1' |C_1| + |C_1|$$

$$= 4 |C_1|.$$ 

Hence $C_1$ does not have a projective-injective summand, therefore by (b), (c) we get a contradiction.

(e) Suppose $r = 2$, $a_1 a_1' + a_2 a_2' = 3$. We may assume $a_1 a_1' = 2$, $a_2 a_2' = 1$. Clearly, each $B_i$ is $\tau$-invariant.

We prepare a lemma.

**Lemma 1.** Let $X$ be an indecomposable module such that $\tau^2 X \cong X$. Let $0 \to \tau X \to Y \oplus Z \to X \to 0$ be the Auslander-Reiten sequence with $Y$ indecomposable. Suppose $\tau^2 Y \cong Y$, $|X| < |Y|$, $|\tau X| < |Y|$, $|X| < |\tau Y|$ and $|\tau X| < |\tau Y|$. Then either $Z = 0$ or $Z$ is indecomposable with $\tau^2 Z \cong Z$.

**Proof.** We may assume $Z \neq 0$. Let $Z = \bigoplus_{i=1}^r Z_i^l$, where $Z_i$'s are non-isomorphic
indecomposable modules and \( d_i = \dim_{F[X]} N(\tau X, Z_i) \) for all \( i \). Let \( 0 \to X \to Y \oplus W \to Z \to 0 \) be the Auslander-Reiten sequence. Since \( |Z| < |X| \), \( |Z| < |\tau X| \), \( |W| < |X| \) and \( |W| < |\tau X| \), both \( Z \) and \( W \) have neither projective nor injective summands. Hence \( \tau Z \cong W \) and \( \tau^{-1} Z \cong W \). Let \( d'_i = \dim_{F[X]} N(\tau X, Z(\tau X)) \) for each \( i \). Using the Auslander-Reiten sequences ending in and starting from \( Z_i \) we get

\[
(\sum_{i=1}^s d_i d'_i) (|\tau X| + |X|) \leq 2 \sum_{i=1}^s d_i |Z_i| + \sum_{i=1}^s d_i |\tau Z_i| + \sum_{i=1}^s d'_i |\tau^{-1} Z_i|
\]

\[
= 2|Z| + |W| + |W| < 2(|\tau X| + |X|)
\]

Therefore we conclude that \( \sum_{i=1}^s d_i d'_i = 1 \). This finishes the proof.

(e') Suppose \( a_i = 2 \). Since \( 2|C_i| = |A| \), \( |C_i| < |A| \) for all \( i \). Suppose \( |A| < |B_i| \) for some \( i \), then we get \( |A| < |B_i| < |C_i| \), a contradiction. Hence \( |B_i| < |A| \), thus \( |C_i| < |B_i| < |A| \) for all \( i \). Suppose \( C_i \neq 0 \). By Lemma 1, \( C_i \) is indecomposable, and clearly \( \tau \)-invariant. Let \( 0 \to C_i \to B_i \oplus D_i \to C_i \to 0 \) be the Auslander-Reiten sequence. If \( D_i \neq 0 \), then again by Lemma 1, \( D_i \) is indecomposable and \( \tau \)-invariant with \( |D_i| < |C_i| \). Continuing these procedures, we get a finite component \( \{A, B_1, B_2, C_1, C_2, D_1, D_2, \ldots\} \) consisting of only \( \tau \)-invariant modules, a contradiction (cf. [1, Theorem 6.5]).

(e'') Suppose \( a_i = 2 \). We get \( |C_i| < |B_i| \), hence \( C_i \) does not have a projective-injective summand. Therefore by (b), (c) and (e') we get a contradiction.

(f) Suppose \( r = 1 \), \( a_i a_i' = 2 \). Clearly, \( B_i \) is \( \tau \)-invariant.

(f') If \( a_i = 2 \), then we get \( |A| = |B_i| \), a contradiction.

(f'') If \( a_i' = 2 \), then we get \( |A a_i'| = |B_i| \), a contradiction.

(g) Suppose \( r = 3 \), \( a_i a_i' = 1 \) for all \( i \). Put \( a i = j \) if \( \tau B_i \cong B_j \). Then \( \sigma \) is a permutation of the set \( \{1, 2, 3\} \). Note that \( \sum_{i=1}^r |B_i| = 2|A| \) and \( \sum_{i=2}^r |C_i| = |A| \).

(g') Suppose \( \sigma \) is cyclic. Suppose \( |A| < |B_i| \) for some \( i \). We get \( |B_i| + |B_{a_i'|} < |A| \). On the other hand, using the Auslander-Reiten sequence ending in \( B_{a_i'} \), we get \( |A| < |B_{a_i'|} + |B_{a_i'|} \), a contradiction. Hence \( |B_i| < |A| \), thus \( |C_i| < |B_i| \) for all \( i \). Suppose \( C_i = 0 \) for some \( i \). We get \( |A| = |B_i \oplus B_{a_i'}| \), a contradiction. Hence \( C_i \neq 0 \) for all \( i \). Clearly, each \( C_i \) does not have a projective summand. Let \( X \) be an indecomposable summand of \( C_i \). Using the Auslander-
Reiten sequences ending in $X, \tau X$ and $\tau^2 X$, we get
\[
2|A| = |B_{s\tau^2}| + |B_{s\tau^1}| + |B_1|
\]
\[
\leq (|X| + |\tau X|) + (|\tau X| + |\tau^2 X|) + (|\tau^2 X| + |\tau^3 X|)
\]
\[
\leq 2(|C_1| + |C_{s\tau^1}| + |C_{s\tau^2}|)
\]
\[
= 2|A|.
\]
Therefore each $C_i$ is indecomposable and the Auslander-Reiten sequence ending in $C_i$ is of the form $0 \to C_{e_1} \to B_{e_2} \to C_{e_3} \to 0$. Hence we get a finite component $\{A, B_1, B_2, B_3, C_1, C_2, C_3\}$ consisting of only $\tau$-periodic modules, a contradiction.

(g') Suppose $\sigma$ is not cyclic. Suppose $|A| < |B_i|$ for some $i$. We get $|B_{e_1}| < |C_i| \leq |A| < |B_i|$, thus $C_i \neq 0$ and $C_i$ does not have an injective summand. Let $X$ be an indecomposable summand of $C_i$. Using the Auslander-Reiten sequence starting from $X$, we get
\[
|A| < |B_i|
\]
\[
\leq |X| + |\tau^{-1} X|
\]
\[
\leq |C_1| + |C_{e_1}| + |C_{e_2}|
\]
\[
\leq |A|,
\]
a contradiction. Hence $|B_i| > |A|$ for all $i$. By Lemma 1, each $C_i$ is either zero or indecomposable with $|C_i| < |B_i|$. Therefore, as in (e'), we get a finite component $\{A, B_1, B_2, B_3, C_1, C_2, C_3, \ldots\}$ consisting of only $\tau$-periodic modules, a contradiction.

(h) Suppose $r = 2$, $a_1a_2 = a_4a_2 = 1$ and $\tau B_1 \cong B_2$. Note that $\tau^2 B_i \cong B_i$ and $|C_i| = |A|$ for all $i$. We claim that each $C_i$ is indecomposable.

**Lemma 2.** Let $X$ be an indecomposable module such that $\tau^2 X \cong X$. Let $0 \to \tau X \to Y \oplus Z \to X \to 0$ be the Auslander-Reiten sequence with $Y$ indecomposable. Suppose $\tau^2 Y \cong Y$, $|\tau Y| = |Y|$ and $|X| + |\tau X| = 2|Y|$. Then $Z$ is indecomposable with $\tau^2 Z \cong Z$.

**Proof.** We may assume $Z \neq Y$. First, assume $|\tau X| < |Y| < |X|$. Let $0 \to X \to \tau Y \oplus W \to \tau X \to 0$ be the Auslander-Reiten sequence. Since $|Z| = |W| < |X|$, $Z$ does not have an injective summand and $W$ does not have a projective summand. Hence $W \cong \tau^{-1} Z$. Let $Z = \bigoplus_{i=1}^t Z_i$, where $Z_i$'s are non-isomorphic indecomposable modules and $d_i = \dim_{F(Z_i)} N(\tau X, Z_i)$ for all $i$. Let $d_i = \dim_{F(Z_i)} N(\tau X, Z_i)$ for each $i$. Using the Auslander-Reiten sequence starting from $Z_i$, we get
hence
\[
\left( \sum_{i=1}^k d_i d'_i \right) |X| \leq \sum_{i=1}^k d_i |Z_i| + \sum_{i=1}^k d_i |\tau^{-1} Z_i| \\
= |Z| + |W| \\
< |X| .
\]

Therefore \( \sum_{i=1}^k d_i d'_i = 1 \), thus \( Z \) is indecomposable. Suppose \( Z \) is projective. Let \( 0 \to Z \to X \oplus E \to W \to 0 \) be the Auslander-Reiten sequence. Since \( |E| = |\tau X| < |Z| \), \( E \) does not have a projective summand. Let \( F \) be an indecomposable summand of \( E \). Using the Auslander-Reiten sequence ending in \( F \), we get
\[
|Z| \leq |F| + |\tau F| \\
\leq |E| + |\tau F| \\
= |\tau X| + |\tau F| .
\]

On the other hand, since \( \tau X \oplus \tau F \) is a summand of \( \text{rad} \ Z \), we get \( |\tau X| + |\tau F| < |Z| \), a contradiction. Therefore \( \tau Z \cong W \), thus \( \tau^2 Z \cong Z \). Exchanging \( W \) for \( Z \), the above arguments imply the case in which \( |X| < |Y| < |\tau X| \). This finishes the proof.

By Lemma 2, each \( C_i \) is indecomposable. Clearly, \( \tau C_i \cong C_i \) and \( \tau C_i \cong C_i \). Let \( 0 \to C \to X \oplus D_i \to C_i \to 0 \) be the Auslander-Reiten sequence for each \( i \). Clearly, \( |D_i| = |B_i| \) for all \( i \). We claim that each \( D_i \) is indecomposable with \( \tau^k D_i \cong D_i \).

**Lemmas 3.** Let \( X \) be an indecomposable module such that \( \tau X \cong X \) and \( |\tau X| = |X| \). Let \( 0 \to X \to Y \oplus Z \to X \to 0 \) be the Auslander-Reiten sequence with \( Y \) indecomposable. Suppose \( \tau^2 Y \cong Y \), \( |Y| + |\tau Y| = 2|X| \). Let \( Z = \bigoplus_{i=1}^k Z_i \), where \( Z_i \)'s are non-isomorphic indecomposable modules and \( d_i = \dim_{k \otimes \tau} N(\tau X, Z_i) \) for all \( i \). Let \( d'_i = \dim N(\tau X, Z_i) F(\tau X) \), for each \( i \). Then \( \sum_{i=1}^k d_i d'_i \leq 2 \):

1. If \( \sum_{i=1}^k d_i d'_i = 1 \), then \( Z \) is indecomposable with \( \tau^k Z \cong Z \).
2. If \( \sum_{i=1}^k d_i d'_i = 2 \), then each \( Z_i \) is neither projective nor injective and the Auslander-Reiten sequences ending in and starting from \( Z_i \) are of the form
\[
0 \to \tau Z_i \to \tau X \to Z_i \to 0, \\
0 \to Z_i \to X \to \tau Z_i \to 0
\]
respectively.

**Proof.** First, assume \( |\tau Y| < |X| < |Y| \). Let \( 0 \to X \to \tau Y \oplus W \to \tau X \to 0 \) be the Auslander-Reiten sequence. Since \( |Z| < |X| = |\tau X| \), each \( Z_i \) is neither projective
nor injective. Using the Auslander-Reiten sequence starting from $Z$, we get
$$d_i |X| \leq |Z_i| + |\tau^{-1}Z_i|,$$

hence
$$\left( \sum_{i=1}^{s} d_i d'_i \right) |X| \leq \sum_{i=1}^{s} d_i |Z_i| + \sum_{i=1}^{s} d_i |\tau^{-1}Z_i|$$
$$\leq |Z| + |W|$$
$$= 2|X|.$$ 

Therefore $\sum_{i=1}^{s} d_i d'_i \leq 2$. Suppose $\sum_{i=1}^{s} d_i d'_i = 2$. Then $\tau^{-1}Z \cong W$, thus $W$ does not have a projective summand and the Auslander-Reiten sequence starting from $Z_i$ is of the form
$$0 \longrightarrow Z_i \longrightarrow X_i \longrightarrow \tau^{-1}Z_i \longrightarrow 0$$
for all $i$. Using the Auslander-Reiten sequences ending in $Z_i$'s, we conclude also that if $\sum_{i=1}^{s} d_i d'_i = 2$, then $\tau Z \cong W$, thus $W$ does not have an injective summand and the Auslander-Reiten sequence ending in $Z_i$ is of the form
$$0 \longrightarrow \tau Z_i \longrightarrow \tau X_i \longrightarrow Z_i \longrightarrow 0$$
for all $i$. Assume $\sum_{i=1}^{s} d_i d'_i = 1$. Clearly, $Z$ is indecomposable. Suppose $\tau^i Z \neq Z$. Then $\tau Z$ is projective and $\tau^{-1}Z$ is injective, thus we get
$$2|X| = |X| + |\tau X|$$
$$< |\tau Z| + |\tau^{-1}Z|$$
$$\leq |W|$$
$$< 2|X|,$$

a contradiction. Hence $\tau^i Z \cong Z$. Suppose $\tau Z \neq W$ and let $W \cong \tau Z \oplus W'$. Then $W'$ is projective-injective, thus we get
$$|Z| + |\tau Z| = |\tau Y| + |\tau Z|$$
$$< |\tau X|.$$

On the other hand, using the Auslander-Reiten sequence ending in $Z$, we get $|\tau X| \leq |Z| + |\tau Z|$, a contradiction. Hence $\tau Z \cong W$. Exchanging $W$ for $Z$, the above arguments imply the case in which $|Y| < |X| < |\tau Y|$. This finishes the proof.

Let $D_i = \bigoplus_{i=1}^{s} E_j$, where $E_j$'s are non-isomorphic indecomposable modules and $d_j = \dim_{F(E_j)} N(C, E_j)$ for all $j$. Let $d'_j = \dim N(C, E_j)_{F(C)}$ for each $j$. Suppose $\sum_{i=1}^{s} d_j d'_j \neq 1$. Then by Lemma 3(2), we get a finite component $(A, B, C, D)$. 


$G, E_1, \cdots, E_n, \tau E_1, \cdots, \tau E_n$ consisting of only $\tau$-periodic modules, a contradiction. Therefore, by Lemma 3(1), $D_i$ is indecomposable with $\tau^i D \cong D_i$. Note that $D_2 \cong \tau D_1$, since, by Lemma 3, $D_1$ does not have an injective summand. Thus $D_2$ is also indecomposable with $\tau^2 D \cong D_2$. Therefore, by induction, we get a bounded length component $\{A, B_1, B_2, C_1, C_2, D_1, D_2, \cdots\}$ consisting of only $\tau$-periodic modules, a contradiction.

This finishes the proof of Theorem 1.

2. Proof of Theorem 2.

Let $X, Y$ be indecomposable modules such that $N(X, Y) \neq 0$ and $N(Y, X) \neq 0$. We claim that either $X$ or $Y$ is $\tau$-invariant. Note that $N(\tau X, \tau Y) \neq 0$ and $N(\tau Y, \tau X) \neq 0$ if neither $X$ nor $Y$ is projective, and that $N(\tau^{-1} X, \tau^{-1} Y) \neq 0$ and $N(\tau^{-1} Y, \tau^{-1} X) \neq 0$ if neither $X$ nor $Y$ is injective. Therefore, it is sufficient to consider the following three cases:

1. Either $X$ or $Y$ is projective.
2. Either $X$ or $Y$ is injective.
3. Both $X$ and $Y$ are stable. (Recall that an indecomposable module $X$ is said to be stable if for any integer $n$, $\tau^n X$ is neither projective nor injective).

Case 1. We may assume $X$ is projective. Then $Y$ is a summand of $\text{rad} X$, thus $|Y| < |X|$. Hence $Y$ is not projective. Using the Auslander-Reiten sequence ending in $Y$, we get $|X| \leq |\tau Y| + |Y|$. Suppose $Y$ is not $\tau$-invariant. Then $\tau Y \oplus Y$ is a summand of $\text{rad} X$, thus $|\tau Y| + |Y| < |X|$, a contradiction. Therefore $Y$ is $\tau$-invariant.

Case 2. By the dual arguments, we conclude that either $X$ or $Y$ is $\tau$-invariant.

Case 3. Suppose neither $X$ nor $Y$ is $\tau$-invariant. For any integer $n$, using the Auslander-Reiten sequence ending in $\tau^n X$, we get $|\tau^{n+1} Y| + |\tau^n Y| \leq |\tau^{n+1} X| + |\tau^n X|$, hence, by symmetry, $|\tau^{n+1} Y| + |\tau^n Y| = |\tau^{n+1} X| + |\tau^n X|$. Therefore, for any integer $n$ the Auslander-Reiten sequences ending in $\tau^n X, \tau^n Y$ are of the form

$$
0 \longrightarrow \tau^{n+1} X \longrightarrow \tau^{n+1} Y \oplus \tau^n Y \longrightarrow \tau^n X \longrightarrow 0,
$$

$$
0 \longrightarrow \tau^{n+1} Y \longrightarrow \tau^{n+1} X \oplus \tau^n X \longrightarrow \tau^n Y \longrightarrow 0
$$

respectively. We may assume $X$ is of minimal length in the component $\{\tau^n X, \tau^n Y | n, m \in \mathbb{Z} \}$. Let $f : \tau Y \rightarrow X$ be an irreducible map. Extending $f$ to the minimal right almost split map ending in $X$, we get the commutative diagram
where $\alpha'$, $\beta'$ and $f'$ are irreducible maps. Next, extending $f'$ to the minimal right almost split map ending in $Y$, we get the commutative diagram

\[
\begin{array}{c}
0 \to \ker f' \to \tau Y \xrightarrow{f'} X \to 0 \\
\uparrow \downarrow \quad \uparrow \downarrow \quad \uparrow \downarrow \\
0 \to \ker g \to \tau Y \xrightarrow{g} X \to 0,
\end{array}
\]

where $\alpha''$, $\beta''$ and $g$ are irreducible maps. Hence, putting $\alpha = \alpha'\alpha''$ and $\beta = \beta'\beta''$, we get the commutative diagram

\[
\begin{array}{c}
0 \to \ker f' \to \tau X \xrightarrow{f''} Y \to 0 \\
\uparrow \downarrow \quad \uparrow \downarrow \quad \uparrow \downarrow \\
0 \to \ker g \to \tau Y \xrightarrow{g} X \to 0,
\end{array}
\]

where $\alpha'' \in \text{rad End} (X)$, $\beta'' \in \text{rad End} (\tau Y)$ and $g$ is an irreducible map. Clearly, the above arguments hold for any irreducible maps from $\tau Y$ to $X$. Therefore, by induction, we conclude that for any positive integer $n$, there is an irreducible map $f_n: \tau Y \to X$ such that the following diagram commutes

\[
\begin{array}{c}
0 \to \ker f \to \tau Y \xrightarrow{f} X \to 0 \\
\uparrow \downarrow \quad \uparrow \downarrow \quad \uparrow \downarrow \\
0 \to \ker g \to \tau Y \xrightarrow{g} X \to 0,
\end{array}
\]

where $\alpha \in \text{rad End} (X)$, $\beta \in \text{rad End} (\tau Y)$ and $\alpha_n \in (\text{rad End} (X))^n$ and $\beta_n \in (\text{rad End} (\tau Y))^n$, this contradicts the fact that $\text{rad End} (X)$ and $\text{rad End} (\tau Y)$ are nilpotent.

This finishes the proof of Theorem 2.

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