A NOTE ON THE PROJECTIVE NORMALITY OF SPECIAL LINE BUNDLES ON ABELIAN VARIETIES

By

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Dedicated to Professor Yukihiro Kodama on his 60th birthday

Introduction.

Let $L$ be an ample line bundle on an abelian variety $A$ of dimension $g$ defined over an algebraically closed field $k$. It is well known that $L^\otimes 2$ is base point free and $L^\otimes 3$ is very ample and projectively normal. Moreover we know that

$$\Gamma(A, L^\otimes 2) \otimes \Gamma(A, L^\otimes 3) \longrightarrow \Gamma(A, L^\otimes (a+b))$$

is surjective if $a \geq 2$ and $b \geq 3$ (Koizumi [3], Sekiguchi [8], [9]). But in the case of $a=b=2$, this map is not surjective in general. In this paper we determine the condition of projective normality of $L^\otimes a$ for some ample line bundle $L$. Our result is as follows.

**Theorem.** If $L$ is a symmetric ample line bundle of separable type, $l(A, L)$ is odd and assume that $\text{char}(k) \neq 2$, then $L^\otimes a$ is projectively normal if and only if $B_\text{a}L \cap A[2] = \phi$.

In §1 we prove the above theorem for abelian varieties defined over $\mathbb{C}$. In §2 we give the Mumford's theory of a theta group (Mumford [4], [5]). In §3 we prove the above theorem in general by the theory in §2.

Notations.

$\text{char}(k)$: The characteristic of a field $k$

$f^*$: The pull back defined by a morphism $f$

$L$: The invertible sheaf associated to a line bundle $L$

$\mathcal{O}_A$: The invertible sheaf of a variety $A$

$(L^g)$: The self intersection number

$|L|$: The set of all effective Cartier divisors which define a line bundle $L$

$B_\text{s}\{L\}$: The set defined by $\bigcap_{D \in |L|} D$

$\Gamma(A, L)$: The set of global sections of a line bundle $L$

$l(A, L)$: The dimension of $\Gamma(A, L)$ as a vector space
$T_x$: The translation morphism on an abelian variety $A$ defined by $T_x(y) = x + y$ where $x$ and $y$ are elements of $A$

$K(L)$: The subgroup of an abelian variety $A$ defined by $\{ x \in A; T_x^* L \cong L \}$ where $L$ is a line bundle on $A$

$A[n]$: The set of all points of order $n$ of an abelian variety $A$

$Z$: The ring of integers

$R$: The field of real numbers

$C$: The field of complex numbers

§ 1. The $C$ case.

First we recall a definition of projective normality.

**Definition.** Let $M$ be an ample line bundle on an abelian variety $A$. We call that $M$ is projectively normal if

$$\Gamma(A, M)^{\otimes n} \twoheadrightarrow \Gamma(A, M^{\otimes n})$$

is surjective for every $n \geq 1$.

Next we define a theta function defined on $C^g$.

**Definition.** Let $m', m''$ be elements of $R^g$ and let $\tau$ be an element of a Siegel upper half space $H_g$. We define $\theta \begin{pmatrix} m' \\ m'' \end{pmatrix}(\tau, z)$ by

$$\theta \begin{pmatrix} m' \\ m'' \end{pmatrix}(\tau, z) = \sum_{\zeta \in z} e(1/2)(\zeta + m')(\zeta + m') + \zeta(\zeta + m')(x + m'')$$

where $e(x)$ means $e^{2\pi i x}$ and $z$ is contained in $C^g$.

Let $d_1, \cdots, d_g$ be positive integers with $d_1 | \cdots | d_g$. We define an integral matrix $e$ by

$$\begin{bmatrix} d_1 & 0 \\ 0 & \cdots & d_g \end{bmatrix}$$

For an element $\tau$ of $H_g$ we define an abelian variety $A$ by $C^g/\langle \tau, e \rangle$ where $\langle \tau, e \rangle$ is a lattice subgroup of $C^g$ defined by $\tau Z^g + e Z^g$. Let $A$ be a Riemann form on $\langle \tau, e \rangle$ defined by

$$\mathcal{A}(\tau x + ey, \tau x' + ey') = e^{xey - x'e'y}$$

where $x, x', y, y'$ are elements of $Z^g$. It is well known that this $A$ defines an
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algebraic equivalence class of line bundles on $A$. Now we take a line bundle $L$ on $A$ satisfying that $L$ is symmetric and the global sections of $L$ are generated by $\theta\left[ \begin{array}{c} \eta \\ 0 \end{array} \right](\tau, z)$ where $\eta$ runs over a complete system of representative of $(1/d_1)\mathbb{Z}/\mathbb{Z} \oplus \cdots \oplus (1/d_\delta)\mathbb{Z}/\mathbb{Z}$.

**Lemma 1.** The basis of $\Gamma(A, L^{\otimes n})$ is given by $\theta\left[ \begin{array}{c} \eta \\ 0 \end{array} \right](\tau, 2^sz)$ where $\eta$ runs over a complete system of representative of $(1/2^n d_1)\mathbb{Z}/\mathbb{Z} \oplus \cdots \oplus (1/2^n d_\delta)\mathbb{Z}/\mathbb{Z}$ ($n = 1, 2, \ldots$). Moreover $\Gamma(A, L^{\otimes n})$ is generated by $\theta\left[ \begin{array}{c} \xi \\ 0 \end{array} \right](\tau, 2z)$ where $\xi$ runs over a complete system of representative of $(1/2 d_1)\mathbb{Z}/\mathbb{Z} \oplus \cdots \oplus (1/2 d_\delta)\mathbb{Z}/\mathbb{Z}$ and $\sigma$ runs over a complete system of representative of $(1/2)\mathbb{Z}/\mathbb{Z}$.

**Proof.** This is well known fact (cf. Igusa [2], p. 72, Theorem 4, and p. 84, Theorem 6).

**Lemma 2** (Multiplication formula). If $\eta', \eta'', \xi', \xi''$ are contained in $R^s$ and $\tau$ is contained in $H_s$, then

$$\theta\left[ \begin{array}{c} \eta' \\ \eta'' \end{array} \right](\tau, z)\theta\left[ \begin{array}{c} \xi' \\ \xi'' \end{array} \right](\tau, z) = (1/2^s) \sum_{\alpha \in (1/2)\mathbb{Z}/\mathbb{Z}} e(-2^s \eta' a^\alpha) \cdot \theta\left[ \begin{array}{c} \eta' + \xi' \\ \eta'' + \xi'' + 2a^\alpha \end{array} \right](\tau/2, (z_1 + z_2)/2) \cdot \theta\left[ \begin{array}{c} \eta' - \xi' \\ \eta'' - \xi'' + 2a^\alpha \end{array} \right](\tau/2, (z_1 - z_2)/2)$$

where $z_1$ and $z_2$ are contained in $C^s$.

**Proof.** This is also well known fact (cf. Igusa [2], p. 139, Theorem 2).

**Lemma 3.** If $\eta, \eta'$ are elements of $(1/d_1)\mathbb{Z}/\mathbb{Z} \oplus \cdots \oplus (1/d_\delta)\mathbb{Z}/\mathbb{Z}$, $d_\delta$ is odd and $s, s'$ are contained in $Z^s$, then

$$\sum_{\epsilon \in (1/2)\mathbb{Z}/\mathbb{Z}, \epsilon} (-1)^{s \epsilon' s} \theta\left[ \begin{array}{c} \eta \pm (\sigma / 2) \\ 0 \end{array} \right](2\tau, 2z) \theta\left[ \begin{array}{c} \eta' \pm (\epsilon s + \epsilon / 2) \\ 0 \end{array} \right](2\tau, 2z) = \theta\left[ \begin{array}{c} \eta \pm \eta' \pm (s / 2) \\ \epsilon s' / 2 \end{array} \right](\tau, 2z) \theta\left[ \begin{array}{c} \eta - \eta' \pm (s / 2) \\ \epsilon s' / 2 \end{array} \right](\tau, 0).$$

**Proof.** By lemma 2, we obtain
Hence
\[
(1/2^e) \sum_{\sigma \in \mathbb{Z}/2 \mathbb{Z}} (-1)^{\sigma} \theta \left[ \frac{\eta + (\sigma/2)}{a^\sigma} \right] (\tau, 2\tau) \theta \left[ \frac{\eta^\prime + (\sigma + \varepsilon/2)}{a^\sigma} \right] (\tau, 2\tau)
\]
\[
= \sum_{\sigma \in \mathbb{Z}/2 \mathbb{Z}} (-1)^{\sigma} \theta \left[ \frac{\eta + (\sigma/2)}{a^\sigma} \right] (\tau, 2\tau) \theta \left[ \frac{\eta^\prime + (\sigma + \varepsilon/2)}{a^\sigma} \right] (\tau, 2\tau)
\]
\[
= \left( \frac{\eta + (\varepsilon/2)}{\varepsilon/2} \right) \theta \left[ \frac{\eta + (\varepsilon/2)}{\varepsilon/2} \right] (\tau, 2\tau) \theta \left[ \frac{\eta^\prime + (\varepsilon/2)}{\varepsilon/2} \right] (\tau, 2\tau).
\]
Therefore we obtain this lemma.

**Lemma 4.** If \( M \) is an ample line bundle on an abelian variety \( A \), then
\[
\Gamma(A, M^{a+b}) \rightarrow \Gamma(A, M^{a+b})
\]
is surjective for \( a \geq 2 \) and \( b \geq 3 \).

**Proof.** See Koizumi [3] or Sekiguchi [8], [9].

**Lemma 5.** Under the notation of lemma 3, if there exists some \( A, M^{a+b} \rightarrow \Gamma(A, M^{a+b}) \) for every \( \eta \in (1/d_1) \mathbb{Z}/\mathbb{Z} \cdots \oplus (1/d_n) \mathbb{Z}/\mathbb{Z} \) with \( \eta \in (1/d_1) \mathbb{Z}/\mathbb{Z} \cdots \oplus (1/d_n) \mathbb{Z}/\mathbb{Z} \) and \( \eta \in (1/d_1) \mathbb{Z}/\mathbb{Z} \cdots \oplus (1/d_n) \mathbb{Z}/\mathbb{Z} \), then \( \eta + (\varepsilon/2) \) \( \varepsilon/2 \) is in the image of \( \Gamma(A, L^{a+b}) \rightarrow \Gamma(A, L^{a+b}) \) for every \( \eta \in (1/d_1) \mathbb{Z}/\mathbb{Z} \cdots \oplus (1/d_n) \mathbb{Z}/\mathbb{Z} \).

**Proof.** Let \( \eta \) be an element of \( (1/d_1) \mathbb{Z}/\mathbb{Z} \cdots \oplus (1/d_n) \mathbb{Z}/\mathbb{Z} \). In this case, we obtain that
\[
\theta \left[ \frac{\eta + (\varepsilon/2)}{\varepsilon/2} \right] (\tau, 2\tau) \theta \left[ \frac{\eta + (\varepsilon/2)}{\varepsilon/2} \right] (\tau, 2\tau)
\]
is contained in the image of \( \Gamma(A, L^{a+b}) \rightarrow \Gamma(A, L^{a+b}) \) by lemma 3. Hence \( \theta \left[ \frac{\eta + (\varepsilon/2)}{\varepsilon/2} \right] (\tau, 2\tau) \) is contained in the image of \( \Gamma(A, L^{a+b}) \rightarrow \Gamma(A, L^{a+b}) \). As
$d_z$ is odd, therefore the set $\{2\gamma_1+\eta; \gamma_1\in (1/d_1)\mathbb{Z}/\mathbb{Z} \oplus \cdots \oplus (1/d_z)\mathbb{Z}/\mathbb{Z}\}$ is equal to $$(1/d_1)\mathbb{Z}/\mathbb{Z} \oplus \cdots \oplus (1/d_z)\mathbb{Z}/\mathbb{Z}.$$ 
Therefore we obtain this lemma.

**Lemma 6.** Under the assumption of lemma 3, the following conditions are equivalent:

a) For every $s, \varepsilon' \in \mathbb{Z}^g$, there exists some $\eta \in (1/d_1)\mathbb{Z}/\mathbb{Z} \oplus \cdots \oplus (1/d_z)\mathbb{Z}/\mathbb{Z}$ with $\theta \left[ \begin{array}{c} \eta + (\varepsilon/2) \\ \varepsilon'/2 \end{array} \right] \left( \tau, 0 \right) \neq 0$;

b) $B_1 | L \cap A[2] = \emptyset$.

**Proof.** As

$$\theta \left[ \begin{array}{c} m' + \xi' \\ m'' + \xi'' \end{array} \right] (\tau, x) = e((1/2)\xi' + \xi''(x + x + m'')) \theta \left[ \begin{array}{c} m' \\ m'' \end{array} \right] (\tau, \tau + x + x + m'')$$

(cf. Igusa [2], p. 50, (1.3)), therefore the condition $\theta \left[ \begin{array}{c} \eta + (\varepsilon/2) \\ \varepsilon'/2 \end{array} \right] \left( \tau, 0 \right) \neq 0$ is equivalent to $\theta \left[ \begin{array}{c} \eta \\ 0 \end{array} \right] \left( \tau, (\tau + x)/2 \right) \neq 0$. Hence this lemma is clear because $A = C^e/\langle \varepsilon', e \rangle$.

**Theorem.** If $(A, L)$ is odd, then $L^0$ is projectively normal if and only if $B_1 | L \cap A[2] = \emptyset$.

**Proof.** By lemma 1, a basis of $\Gamma(A, L^0)$ consists of $\theta \left[ \begin{array}{c} \eta \\ \sigma \end{array} \right] (\tau, 2x)$ where $\eta$ runs over a complete system of representative of $(1/2d_1)\mathbb{Z}/\mathbb{Z} \oplus \cdots \oplus (1/2d_z)\mathbb{Z}/\mathbb{Z}$ and $\sigma$ runs over a complete system of representative of $(1/2)\mathbb{Z}/\mathbb{Z}^g$. Hence $\Gamma(A, L^0) 
\rightarrow \Gamma(A, L^0)$ is surjective if and only if for every $s, \varepsilon' \in \mathbb{Z}^g$, there exists some $\eta \in (1/d_1)\mathbb{Z}/\mathbb{Z} \oplus \cdots \oplus (1/d_z)\mathbb{Z}/\mathbb{Z}$ such that $\theta \left[ \begin{array}{c} \eta + (\varepsilon/2) \\ \varepsilon'/2 \end{array} \right] (\tau, 0) \neq 0$ by lemma 5. Hence we obtain this theorem by lemma 4 and lemma 6.

§ 2. **Review of a theta group.**

In this section we recall the Mumford's theory for a theta group (cf. Mumford [4], [5]). Let $A$ be an abelian variety of dimension $g$ defined over an algebraically closed field $k$ with char $(k) \neq 2$. We fix these notations.

**Definition.** Let $M$ be an ample line bundle on $A$. We call that $M$ is of separable type if char $(k) \nmid l(A, M)$.
**Definition.** Let $M$ be an ample line bundle on $A$ and be of separable type. We define a theta group $G(M)$ by

$$\{(x, \phi); x \in K(M) \text{ and } \phi: M \xrightarrow{\sim} T_x^*M\}.$$ 

This $G(M)$ is a group by the following way:

$$(x, \phi) \cdot (y, \rho) = (x + y, T_x^*\phi \cdot \rho).$$

It is well known that $K(M)$ is isomorphic to the following abelian group via Weil pairing:

$$K(M) = \mathbb{Z}/d_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_x \mathbb{Z} \oplus (\mathbb{Z}/d_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_x \mathbb{Z})$$

where $d_1 \cdots d_x$ and $\bar{G}$ means Hom($G$, $k^*$) for a group $G$. Here we denote by $k^* = k \setminus \{0\}$. In this situation, we set $\delta_M = (d_1, \cdots, d_x)$ and put $H(\delta_M) = \mathbb{Z}/d_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_x \mathbb{Z}$. We define a Heisenberg group $G(\delta_M)$.

**Definition.** In the above notations, we define a Heisenberg group $G(\delta_M)$ by

$$G(\delta_M) = k^* \times H(\delta_M) \times H(\delta_M).$$

This $G(\delta_M)$ is a group by the following way:

$$(t, (x, m)) \cdot (t', (x', m')) = (tt'm'(x), (x + x', m + m'))$$

where $x, x' \in H(\delta_M)$, $m, m' \in H(\delta_M)$ and $t, t' \in k^*$. The following theorem is fundamental.

**Theorem.** In the above notations, the following two horizontal sequences are exact and isomorphic:

\[
\begin{array}{cccccc}
0 & \rightarrow & k^* & \rightarrow & G(M) & \rightarrow & K(M) & \rightarrow & 0 \\
\Vert & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & k^* & \rightarrow & G(\delta_M) & \rightarrow & H(\delta_M) \times H(\delta_M) & \rightarrow & 0. 
\end{array}
\]

**Proof.** See Mumford [4], p. 294, Corollary of Th. 1.

**Definition.** Let $z = (x, \phi)$ be an element of $G(M)$. We define a map $U_z$ as follows:

$$U_z : \Gamma(A, M) \xrightarrow{\Gamma(\phi)} \Gamma(A, T_x^*M) \xrightarrow{T_{-x}^*} \Gamma(A, M).$$

It is clear that $\Gamma(A, M)$ is a $G(M)$-module by this way. Next we define a vector space $V(\delta_M)$ and its $G(\delta_M)$-module structure.
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DEFINITION. The vector space $V(\delta_M)$ is defined as follows:

$V(\delta_M)$ is the set of all maps from $H(\delta_M)$ to $k$.

Let $(t, (x, m))$ be an element of $G(\delta_M)$. We define an automorphism $U_{(t, (x, m))}$ of $V(\delta_M)$ as follows:

$$U_{(t, (x, m))}(f)(y) = t m(y) f(x + y)$$

where $f \in V(\delta_M)$ and $y \in H(\delta_M)$.

The following theorem is also fundamental.

THEOREM. If $\alpha : G(M) \cong G(\delta_M)$ is an isomorphism given in the above theorem, then we obtain an isomorphism

$$\Gamma(A, M) \cong V(\delta_M)$$

as $G(M) \cong G(\delta_M)$-modules.


DEFINITION. Let $x$ be an element of $H(\delta_M)$. We define $\delta_x \in V(\delta_M)$ by $\delta_x(y) = 1$ if $y = x$ and $\delta_x(y) = 0$ if $y \neq x$.

It is clear that $U_{(t, (x, m))}(\delta_u) = t m(u - x) \delta_{u - x}$.

§ 3. The general case.

Let $L$ be an ample line bundle on an abelian variety $A$ of dimension $g$. Throughout of this section, we assume that $L$ is of separable type and $l(A, L)$ is odd and $L$ is symmetric. We fix an isomorphism $G(L^{\omega}) \cong G(\delta_L)$ and identify two vector spaces $\Gamma(A, L^{\omega})$ and $V(\delta_L)$ by the isomorphism in § 2.

LEMMA 1. Let $f$ be an element of $V(\delta_L) \cong \Gamma(A, L^{\omega})$ defined by $f = \sum_{u \in H(\delta_L) \text{ and } 2u = 0} \delta_u$. Then $f$ is in the image of $2_{\alpha}^* : \Gamma(A, L) \rightarrow \Gamma(A, L^{\omega})$ for some isomorphism $2_{\alpha}^* L \cong L^{\omega}$ where $2_{\alpha}(x) = 2x$ ($x \in A$).

PROOF. This lemma is trivial.

By the above lemma, we obtain $\theta \in \Gamma(A, L)$ with $2_{\alpha}^* \theta = f$. We fix these notations through this section.

DEFINITION. Let $x$ be an element of $H(\delta_L)$ and $\sigma$ be an element of $H(\delta_L)$. We define an element of $\theta \begin{bmatrix} x \\ \sigma \end{bmatrix}$ of $V(\delta_L)$ by
\[ \theta \left[ \begin{array}{c} x \\ \sigma \end{array} \right] = U \left( 2^*_\sigma \theta \right) \]

where \( z \) is an element of \( G(L^{\infty}) \) corresponding to \((1, (x, \sigma))\) which is an element of \( G(4d_L) \).

**Lemma 2.** Let \( x, u \) be elements of \( H(4d_L) \) and \( \sigma, u^* \) be elements of \( H(\tilde{d}_L) \). If \( 2u=0 \) and \( 2u^*=0 \), then \( \theta \left[ \begin{array}{c} x+u \\ \sigma+u^* \end{array} \right] = u^*(x) \theta \left[ \begin{array}{c} x \\ \sigma \end{array} \right] \).

**Proof.** By the definition, we obtain that
\[ \theta \left[ \begin{array}{c} x \\ \sigma \end{array} \right] = \sum_{z=0}^{2z} \sigma(z-x) \delta_{z-x}. \]
Therefore
\[ \theta \left[ \begin{array}{c} x+u \\ \sigma+u^* \end{array} \right] = \sum_{z=0}^{2z} (\sigma+u^*)(z-x-u) \delta_{z-x-u} \]
\[ = \sum_{z=0}^{2z} (\sigma+u^*)(z-x) \delta_{z-x} \]
\[ = \sum_{z=0}^{2z} u^*(z-x) \sigma(z-x) \delta_{z-x}. \]
As \( 2u^*=0 \), hence \( u^*(z)=1 \) for every \( z \in H(4d_L) \) with \( 2z=0 \). Therefore
\[ \theta \left[ \begin{array}{c} x+u \\ \sigma+u^* \end{array} \right] = u^*(-x) \sum_{z=0}^{2z} \sigma(z-x) \delta_{z-x} \]
\[ = u^*(x) \theta \left[ \begin{array}{c} x \\ \sigma \end{array} \right]. \]
So we obtain this lemma.

**Lemma 3.** The vector space \( \Gamma(A, L^{\infty}) \) is spanned by the elements \( \theta \left[ \begin{array}{c} x \\ \sigma \end{array} \right] \) where \( x \in H(4d_L) \) and \( \sigma \in (\mathbb{Z}/4\mathbb{Z})^* \) which is regarded the subgroup of order of \( H(4d_L) \).

**Proof.** We regard that \( H(d_L) \) and \( (\mathbb{Z}/4\mathbb{Z})^* \) are the subgroups of \( H(4d_L) \) in the canonical way. For every \( \xi \in H(d_L) \), \( \tau \in (\mathbb{Z}/4\mathbb{Z})^* \) and \( \sigma \in (\mathbb{Z}/4\mathbb{Z})^* \), we obtain
\[ \theta \left[ \begin{array}{c} \xi+\tau \\ \sigma \end{array} \right] = \sum_{z=0}^{2z} \sigma(z-\xi-\tau) \delta_{z-\xi-\tau} \]
\[ = \sum_{z=0}^{2z} \sigma(z-\tau) \delta_{z-\xi-\tau} \]
because as \( l(A, L) \) is odd, \( \sigma(\xi)=1 \). Therefore
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\[ \sum_{\sigma \in (\mathbb{Z}/2)^{2}} \sigma(\tau) \theta \left[ \xi + \tau \right] = \sum_{\sigma \in (\mathbb{Z}/2)^{2}} \sum_{\xi = 0} \sigma(\tau) \delta_{\xi - \xi - \tau} \]

= \sum_{\xi = 0} \left( \sum_{\sigma \in (\mathbb{Z}/2)^{2}} \sigma(\xi) \right) \delta_{\xi - \xi - \tau} 

= 4^2 \delta_{\xi - \xi - \tau}.

Therefore we obtain this lemma.

Let \( x \) be a closed point of \( A \) and \( M \) be an ample line bundle on \( A \) of separable type. We define \( M(x) \) by

\[ M(x) = M_x \otimes k(x) \]

where \( M_x \) and \( \mathcal{O}_{A,x} \) are the stalk of \( M \) and \( \mathcal{O}_A \) at \( x \) respectively, and \( k(x) \) is a residue field of \( \mathcal{O}_{A,x} \). It is clear that \( M(x) \cong k \). We choose an isomorphism \( \lambda_0 : M(0) \cong k \). We fix an isomorphism \( G(M) \cong G(\mathcal{O}_M) \). For every \( w \in K(M) \), we take \( (w, \phi_w) \in G(M) \) which is corresponding to an element of \( G(\mathcal{O}_M) \) with a type \((1, (x, m))\) by the above isomorphism.

**Definition.** We define \( \lambda_w : M(w) \to k \) by

\[ \lambda_w : M(w) \leftarrow (T_x^* M(0)) \leftarrow \phi_w(0) \rightarrow k \]

where \( w \in K(M) \) and \( \phi_w(0) \) is given by \( \phi_w \).

**Definition.** Under the above notations, we define \( q^w \) by

\[ q^w : \Gamma(A, M) \rightarrow M(w) \rightarrow k \]

**Remark.** For any \( z = (x, \phi) \in G(M) \) and any \( s \in \Gamma(A, M) \), the conditions \( q^w(U_z(s)) = 0 \) and \( q^w_{\omega + 1}(s) = 0 \) are equivalent.

**Remark.** The condition that \( q^w_\omega(s) = 0 \) for every \( s \in \Gamma(A, M) \) implies that \( w \) is contained in \( Bs\lfloor M \rfloor \).

**Remark.** If \( M \) is a symmetric ample line bundle on \( A \), then the conditions \( q^w_\omega(2A^* s) = 0 \) and \( q^w_\omega(s) = 0 \) are equivalent for any \( s \in \Gamma(A, M) \).

**Definition.** We define \( q_{L^\infty}(x) \) by

\[ q_{L^\infty}(x) = q_0^L \cdot (\delta_x) \]

where \( x \in K(L^\infty) \) and \( \delta_x \in V(4\delta_L \cdot \Gamma(A, L^\infty)) \). Moreover we define \( q \left[ \begin{matrix} x \\ \sigma \end{matrix} \right] \) by

\[ q \left[ \begin{matrix} x \\ \sigma \end{matrix} \right] = q_0^{L^\infty} \left( \theta \left[ \begin{matrix} x \\ \sigma \end{matrix} \right] \right) \]
where \( x \in H(4\delta_L) \) and \( \sigma \in (\mathbb{Z}/4\mathbb{Z})^\delta \).

Now the isomorphism \( G(L^\oplus) \cong G(4\delta_L) \) induces an isomorphism \( G(L^\oplus) \cong G(2\delta_L) \); these isomorphisms define the symmetric theta structure for \( (L^\oplus, L^\oplus) \) (cf. Mumford [4], p. 317). We identify the two vector spaces \( \Gamma(A, L^\oplus) \) and \( V(2\delta_L) \) by means of the isomorphism \( G(L^\oplus) \cong G(2\delta_L) \).

**Lemma 4 (Multiplication formula).** If \( \delta_x \) and \( \delta_x' \) are elements of \( \Gamma(A, L^\oplus) \), then

\[
\delta_x \cdot \delta_x' = \sum_{\xi \in H(L^\oplus) \text{ and } \xi' \in H(L^\oplus)} q_{L^\oplus}(x - x' + \xi - \xi') \delta_{x+x'-\xi},
\]

where \( \cdot \) is a canonical map \( \Gamma(A, L^\oplus) \to \Gamma(A, L^\oplus) \) and \( x, x' \in H(4\delta_L) \) satisfying \( 2x = x, 2x' = x' \). Here we regard \( H(2\delta_L) \) as a subgroup of \( H(4\delta_L) \) in the canonical way.

**Proof.** See Mumford [4], p. 330.

Let \( x, x' \) be elements of \( H(\delta_L) \), and \( \xi, \xi' \) be elements of \( (\mathbb{Z}/2\mathbb{Z})^\delta \). We take \( x, x' \in H(\delta_L) \) and \( \xi, \xi' \in (\mathbb{Z}/4\mathbb{Z})^\delta \) satisfying \( 2x = x, 2x' = x' \), \( 2\xi = \xi \) and \( 2\xi' = \xi' \).

**Lemma 5.** Under the above notations,

\[
\delta_{x+\xi} \cdot \delta_{x'+\xi'} = (1/4^\delta) \sum_{\sigma \in (\mathbb{Z}/4\mathbb{Z})^\delta} \sigma(\xi) \sigma(\xi') \theta\left[ \begin{array}{c} -x + x' - \xi + \xi' \\ \sigma \\ \sigma \end{array} \right].
\]

**Proof.** For \( \delta_{x+\xi} \) and \( \delta_{x'+\xi'} \), we obtain that

\[
\delta_{x+\xi} \cdot \delta_{x'+\xi'} = \sum_{\xi' \in H(L^\oplus) \text{ and } \xi' \in H(L^\oplus)} q_{L^\oplus}(x - x' + \xi - \xi') \delta_{x+x'-\xi+\xi'}
\]

\[
= \sum_{\xi' \in H(L^\oplus) \text{ and } \xi' \in H(L^\oplus)} q_{L^\oplus}(x - x' + \xi - \xi') \delta_{x+x'-\xi+\xi'}
\]

\[
\cdot \theta\left[ \begin{array}{c} -x + x' - \xi + \xi' \\ \sigma \\ \sigma \end{array} \right] (1/4^\delta) \sum_{\sigma \in (\mathbb{Z}/4\mathbb{Z})^\delta} \sigma(\xi) \sigma(\xi') q_{L^\oplus}(x - x' + \xi - \xi')
\]

\[
\cdot \theta\left[ \begin{array}{c} -x + x' - \xi + \xi' \\ \sigma \\ \sigma \end{array} \right].
\]

On the other hand, \( \theta\left[ \begin{array}{c} x + u, u^* \\ \sigma, \sigma^* \end{array} \right] = u^*(x) \theta\left[ \begin{array}{c} x \\ \sigma \end{array} \right] \) for \( 2u = 0 \) and \( 2u^* = 0 \). Moreover in above situation, \( \sigma(x) = \sigma(x') \). Hence
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\[ \delta_{x', \delta} \cdot \delta_{x', \delta'} = \left( \frac{1}{4^2} \sum_{\sigma} \sigma(-2\xi) \theta \left[ -x - x' - \xi - \xi' \right] \right) \sum_{\zeta} \sigma(x - x' + \xi - \xi' + \zeta) \]

Therefore we obtain this lemma.

\textbf{THEOREM.} Under the above notations, \( L^{\otimes 2} \) is projectively normal if and only if \( Bs|L| \cap A[2] = \emptyset \).

\textbf{PROOF.} Replacing the lemmas for the theorem in §1 by the above lemmas, the proof of the theorem in §1 still works in general case.

\textbf{COROLLARY.} If \( M \) is an ample line bundle and is of separable type on an abelian variety \( A \), then \( Bs|M| = \emptyset \) and \( l(A, M) = \text{odd} \) imply that \( M^{\otimes 2} \) is projectively normal.

To conclude this section, we give an easy criterion for the base point freeness of a line bundle \( M \) on an abelian variety \( A \). We assume that \( M \) is of separable type. Let \( \alpha : G(M) \to G(\delta_M) \) be an isomorphism. As \( \alpha \) induces \( \tilde{\alpha} : K(M) \to H(\delta_M) \otimes H(\delta_M) \), we put \( H(M) \) by \( \tilde{\alpha}^{-1}(H(\delta_M)) \). Let \( B \) be an abelian variety defined by \( A/H(M) \) and \( \pi : A \to B \) the canonical morphism. In this situation, the line bundle \( M \) is given by \( M \cong \pi^*N \) where \( N \) is a principal line bundle on \( B \). In this notations, we obtain the following proposition.

\textbf{PROPOSITION.} \( Bs|M| = \pi^{-1}( \bigcap_{x \in \pi(K(M))} T_x \cdot \theta ) \) where \( \theta \equiv \mid N \mid \).

\textbf{PROOF.} As there exists a canonical isomorphism

\[ \Gamma(A, M) \cong \bigoplus_{x \in \pi(K(M))} \Gamma(B, T_x \cdot N) \]

therefore this proposition is clear.

The following proposition is also clear.

\textbf{PROPOSITION.} Let \( M \) be as in above. If \( Bs|M| \) is finite and \( (M^\xi) \rangle (g \eta)^\ast \), then \( Bs|M| = \emptyset \).
Proof. If $Bs | M | \neq \emptyset$, then there is a point $q \in Bs | M |$. By the definition of $K(M)$, $q + K(M)$ is also contained in $Bs | M |$. As

the cardinality of $Bs | M | \subseteq (M^*)$,

hence

the order of $K(M) = ((M^*)/g!)^q \subseteq (M^*)$.

Therefore we obtain $(M^*) \subseteq (g!)^q$.

References


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