ALMOST KÄHLER STRUCTURES ON THE Riemannian product of a 3-DIMENSIONAL HYPERBOLIC SPACE AND A REAL LINE

By

Takashi OGURO and Kouei SEKIGAWA

1. Introduction.

An almost Hermitian manifold $M = (M, J, g)$ is called an almost Kähler manifold if the Kähler form is closed (or equivalently $\bar{\partial} J = 0$ for $X, Y, Z \in \mathfrak{X}(M)$, where $\bar{\partial}$ and $\mathfrak{X}(M)$ denotes the cyclic sum and the Lie algebra of all differentiable vector fields on $M$ respectively). A Kähler manifold, which is defined by $\nabla J = 0$, is necessarily an almost Kähler manifold. A non-Kähler almost Kähler manifold is called a strictly almost Kähler manifold. It is well-known that an almost Kähler manifold with integrable almost complex structure is a Kähler manifold. Concerning the integrability of almost Kähler manifolds, the following conjecture by S. I. Goldberg is known ([1]):

**Conjecture.** A compact almost Kähler Einstein manifold is a Kähler manifold.

The second author has proved that the above conjecture is true for the case where the scalar curvature is nonnegative ([4]). However, the above conjecture is still open in the case where the scalar curvature is negative. Recently, the authors proved that a $2n(\geq 4)$-dimensional hyperbolic space $H^{2n}$ cannot admits (compatible) almost Kähler structure ([3]).

In the present paper, we consider about (compatible) almost Kähler structures on the Riemannian product $H^3 \times \mathbb{R}$ of a 3-dimensional hyperbolic space $H^3$ and a real line $\mathbb{R}$. We construct an example of strictly almost Kähler structure $(J, g)$ on the Riemannian product $H^3 \times \mathbb{R}$ and determine the automorphism group of the almost Kähler manifold $(H^3 \times \mathbb{R}, J, g)$. To our knowledge, this is the first example of strictly almost Kähler symmetric space. Moreover, we prove that the Riemannian product $H^3 \times \mathbb{R}$ provided with a (compatible) almost Kähler structure $(J, g)$ cannot be a universal (almost Hermitian) covering of any compact almost Kähler manifold (Theorem 2 in

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section 3).

2. Preliminaries.

Let \( H^3 \) be a 3-dimensional hyperbolic space of constant sectional curvature \(-1\). Then, the Riemannian product \( H^3 \times \mathbb{R} \) can be regarded as a Riemannian manifold \((\mathbb{R}^4, g)\) equipped with the Riemannian metric \( g \) defined by

\[
g = \frac{1}{x_1^2} \sum_{i=1}^{4} dx_i \otimes dx_i + dx_4 \otimes dx_4,
\]

where \( \mathbb{R}^4 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x_1 > 0}\).

We put \( X_i = x_i (\partial / \partial x_i), i = 1, 2, 3, \) and \( X_4 = \partial / \partial x_4 \). Then \( \{X_1, X_2, X_3, X_4\} \) forms a global orthonormal frame field on \( H^3 \times \mathbb{R} \). Direct calculation implies

(2.1) \nabla_{X_i} X_j = \sum_{k=1}^{4} \Gamma_{jk} X_k,

for \( i = 2, 3 \), and are otherwise zero. We set

(2.2) \Gamma_{ii} = -\Gamma_{ii} = 1

for \( i = 2, 3 \), and are otherwise zero.

Let \((J, g)\) be an almost Hermitian structure on \( H^3 \times \mathbb{R} \). We put

(2.3) \[ JX_i = \sum_{j=1}^{4} J_{ij} X_j, \]

for \( 1 \leq i \leq 4 \). Then we may easily observe that the \( 4 \times 4 \) matrix \((J_{ij})\) is a skew-symmetric orthogonal matrix, i.e. the equalities

\[ J_{ij} = -J_{ji}, \sum_{k=1}^{4} J_{ik} J_{jk} = \delta_{ij} \]

holds for \( 1 \leq i, j \leq 4 \), and, furthermore, that the matrix \((J_{ij})\) is of the form

\[
\begin{pmatrix}
0 & J_{12} & J_{13} & J_{14} \\
-J_{12} & 0 & J_{14} & -J_{13} \\
-J_{13} & -J_{14} & 0 & J_{12} \\
-J_{14} & J_{13} & -J_{12} & 0
\end{pmatrix}
\]

or
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\[
\begin{pmatrix}
0 & J_{12} & J_{13} & J_{14} \\
-J_{12} & 0 & -J_{14} & J_{13} \\
-J_{13} & J_{14} & 0 & -J_{12} \\
-J_{14} & -J_{13} & J_{12} & 0 \\
\end{pmatrix}
\]

with \( J_{12}^2 + J_{13}^2 + J_{14}^2 = 1 \).

3. An example of strictly almost Kähler structure on \( H^3 \times \mathbb{R} \).

The aim of this section is to construct an example of a strictly almost Kähler structure on the Riemannian product \((H^3 \times \mathbb{R}, g)\) and to show Theorem 2.

We assume that \((J, g)\) is an almost Kähler structure on the Riemannian product \((H^3 \times \mathbb{R}, g)\). Then, the almost Kähler condition \( \zeta_{i,j,k,l} \left( (\nabla_{\partial_i} J) X_j, X_k \right) = 0 \) and (2.2) yields the following system of first order partial differential equations:

\[
\begin{align*}
X_1 J_{23} - X_2 J_{13} + X_3 J_{12} - 2 J_{23} &= 0, \\
X_1 J_{24} - X_2 J_{14} + X_4 J_{12} - J_{24} &= 0, \\
X_1 J_{34} - X_3 J_{14} + X_4 J_{13} - J_{34} &= 0, \\
X_2 J_{34} - X_3 J_{24} + X_4 J_{23} &= 0.
\end{align*}
\]

We may regard the triple \((J_{12}, J_{13}, J_{14})\) as a unit vector in the 3-dimensional Euclidean space \( \mathbb{R}^3 \). First of all, we may observe that the unit vector \((J_{12}, J_{13}, J_{14})\) has the following property.

**Proposition 1.** The vector \((J_{12}, J_{13}, J_{14})\) varies with the variable \( x_4 \) on an open subset of \( H^3 \times \mathbb{R} \).

**Proof.** We assume that the vector \((J_{12}, J_{13}, J_{14})\) is independent on the variable \( x_4 \). Then, the system of partial differential equations (3.1) reduces to the following:

\[
\begin{align*}
X_1 J_{23} - X_2 J_{13} + X_3 J_{12} - 2 J_{23} &= 0, \\
X_1 J_{24} - X_2 J_{14} + X_4 J_{12} - J_{24} &= 0, \\
X_1 J_{34} - X_3 J_{14} + X_4 J_{13} - J_{34} &= 0, \\
X_2 J_{34} - X_3 J_{24} + X_4 J_{23} &= 0.
\end{align*}
\]

Now, we suppose that the matrix \((J_{ij})\) is of the form (I). Then, by (2.1), (2.2) and (3.2), we have
From (3.3), we have
\[
\begin{align*}
\Delta J_{12} - 2X_i J_{12} + 3J_{12} &= 0, \\
\Delta J_{13} - 2X_i J_{13} + 3J_{13} &= 0, \\
\Delta J_{14} - 2X_i J_{14} + 4J_{14} &= 0.
\end{align*}
\] 
and hence
\[
\sum_{i=1}^{4} ((X_i J_{12})^2 + (X_i J_{13})^2 + (X_i J_{14})^2) = 3 + J_{14}^2,
\]
since \(J_{12}^2 + J_{13}^2 + J_{14}^2 = 1\), and we conclude that
\[
\sum_{i,j,k=1}^{4} (X_{ij})^2 = 4(3 + J_{14}^2).
\]
Next, from the equality above, we have
\[
\sum_{i,j,k=1}^{4} (X_{ij})^2 = 4J_{14} X_{ij},
\]
for each \(X_i\). Thus, by a direct calculation, we obtain
\[
\begin{align*}
\sum_{i,j,k=1}^{4} (X_{ij})^2 &= 4 \sum_{i} (X_{i1})^2 + 4J_{14} \sum_{i} X_{i1} J_{14} - \sum_{i,j,k} (X_{ij})^2 X_{i1} X_{j1} J_{k1} \\
&= 4 \Delta J_{14} + 2X_i J_{14} \\
&= 0.
\end{align*}
\]
From (3.4) and (3.5), we find that \(\sum_{i,j,k} (X_{ij})^2\) and \(\sum_{i,j,k} (X_{ij})^2\) are both bounded. By applying the similar argument in [3] along \(x_i\)-curve, we can deduce a contradiction. More precisely, let \(\gamma_i\) be any integral curve of \(X_i\). Then, we
obtain

\begin{equation}
\lim_{x_i \to \infty} X_i J_{ij} = 0 \quad (1 \leq i, j \leq 4).
\end{equation}

along the geodesic \( \gamma \) (see [3]). We denote by \( \varphi_\alpha (a = 2, 3) \) isometries of \( H^3 \) such that \((\varphi_\alpha), X_i \) is orthogonal to \( X_i \) and, \((\varphi_\alpha), X_i \) and \((\varphi_\alpha), X_i \) are orthogonal to each other along \( \gamma \). Let \( \varphi_\alpha (x_1, x_2, x_3, x_4) = (\varphi_\alpha (x_1, x_2, x_3, x_j) (a = 2, 3) \) be the naturally induced isometries of \( H^3 \times R \), and we define almost complex structures \( J_{(a)} (a = 2, 3) \) on \( H^3 \times R \). Therefore, by similar argument as above, we obtain

\begin{equation}
\lim_{x_i \to \infty} X_i J_{(a)ij} = 0 \quad (1 \leq i, j \leq 4, a = 2, 3).
\end{equation}

along the geodesic \( \gamma \). Moreover, by semi-Kähler condition \( \sum_{a=1}^{4} \nabla a J_{aj} = 0 \)

\((j = 1, 2, 3, 4)\), we have

\begin{align*}
\sum_{i=1}^{4} \left\{(\nabla_i J_{12})^2 + (\nabla_i J_{13})^2 + (\nabla_i J_{14})^2\right\} \\
= \sum_i \left\{(X_i J_{12})^2 + (X_i J_{13})^2 + (X_i J_{14})^2\right\} + 1 + J_{14}^2 \\
+ 2(J_{23} X_2 J_{13} + J_{24} X_2 J_{14} + J_{32} X_3 J_{12} + J_{34} X_3 J_{14}) \\
= 4 + 2J_{14}^2 - 2(J_{13} X_2 J_{23} + J_{14} X_2 J_{24} + J_{12} X_3 J_{32} + J_{14} X_3 J_{34}) \\
= 2 - 2(J_{13} X_2 J_{23} + J_{14} X_2 J_{24} + J_{12} X_3 J_{32} + J_{14} X_3 J_{34}) \\
= 2 + 2(J_{13} \nabla_1 J_{13} + J_{14} (\nabla_1 J_{14} + \nabla_3 J_{32}) + J_{12} \nabla_1 J_{12} + J_{14} (\nabla_1 J_{14} + \nabla_2 J_{24})) \\
= 2,
\end{align*}

and hence, we have

\begin{equation}
\sum_{i,j,k=1}^{4} (\nabla_i J_{jk})^2 = 8,
\end{equation}

where \( \nabla_i J_{jk} = g((\nabla_i J) X_j, X_k) \). From (3.6), (3.7) and (3.8), we can derive a contradiction (see [3]).

In the case where \((J, g)\) is of the form (II), we also arrive at a contradiction. Now, we write down an example of strictly almost Kähler structure \((J, g)\) on \( H^3 \times R \).

**Example.** We define an almost complex structure \( J \) by
with respect to the orthonormal frame field \( \{X_i\}_{i=1,2,3,4} \) defined in the preceding section. Then, it is easy to verify that \((J, g)\) is a strictly almost Kähler structure on \( \mathbb{H}^3 \times \mathbb{R} \).

In the rest of this section, we shall prove the following Theorem 2. First of all, we recall an integral formula on a 4-dimensional compact almost Kähler manifold. Let \( \overline{M} = (\overline{M}, \overline{J}, \overline{g}) \) be a 4-dimensional compact almost Kähler manifold. Then, we see that the square of the first Chern class \( c_1(\overline{M}) \) is given by the following formula (cf. [2], [5]):

\[
(3.9) \quad c_1(\overline{M})^2 = \frac{1}{16\pi^2} \int_{\overline{M}} \left\{ (\overline{\tau})^2 - 2 \| \overline{\rho}^{*\text{sym}} \|^2 + 2 \| \overline{\rho}^{*\text{skew}} \|^2 - \frac{1}{4} (\overline{\tau} + \overline{\bar{\tau}}) \| \overline{\nabla J} \|^2 + (\overline{\rho}, \overline{D}) \right\} d\overline{M},
\]

where \( \overline{\rho}^*, \overline{\tau}, \overline{\bar{\tau}}, \overline{\nabla, d\overline{M}} \) denote the Ricci \(*\)-tensor, \(*\)-scalar curvature, scalar curvature, Levi-Civita connection on \( \overline{M} \) and the volume element of \( \overline{M} \) respectively, and \( \overline{\rho}^{*\text{sym}} \) (resp. \( \overline{\rho}^{*\text{skew}} \)) the symmetric (resp. skew-symmetric) part of \( \overline{\rho}^* \), and \( (\overline{\rho}, \overline{D}) = \sum_{a,b,i,j=1}^4 \overline{\rho}_{ab}(\overline{\nabla}_a J_i)\overline{\nabla}_b J_j \). Here we put \( \overline{\nabla}_a J_j = \overline{g}(\overline{\nabla}_a \overline{J}_j, \overline{X}_i, \overline{X}_j) \) and \( \overline{\rho}_{ab} = \overline{\rho}(\overline{X}_a, \overline{X}_b) \) for a local orthonormal frame field \( \{\overline{X}_1, \overline{X}_2, \overline{X}_3, \overline{X}_4\} \) on \( \overline{M} \).

THEOREM 2. Let \((g, J)\) be a (compatible) almost Kähler structure on the Riemannian product \( \mathbb{H}^3 \times \mathbb{R} \). Then, the almost Kähler manifold \((\mathbb{H}^3 \times \mathbb{R}, J, g)\) cannot be a universal (almost Hermitian) covering of any compact almost Kähler manifold.

PROOF. Let \((g, J)\) be a (compatible) almost Kähler structure on the Riemannian product \( \mathbb{H}^3 \times \mathbb{R} \). We assume that there exists a compact almost Kähler manifold \((\overline{M}, \overline{J}, \overline{g})\) whose universal (almost Hermitian) covering is the almost Kähler manifold \((\mathbb{H}^3 \times \mathbb{R}, J, g)\). We denote by \( p \) the covering map from \( \mathbb{H}^3 \times \mathbb{R} \) onto \( \overline{M} \). For any point \( \overline{p} \in \overline{M} \), we may choose a local orthonormal frame field \( \{\overline{X}_1, \overline{X}_2, \overline{X}_3, \overline{X}_4\} \) near \( \overline{p} \) in such a way that \( p_*(X_i) = \overline{X}_i, (i = 1, 2, 3, 4) \). We set \( \overline{J} \overline{X}_i = \sum_{j=1}^4 \overline{J}_j \overline{X}_j (i = 1, 2, 3, 4) \). For the proof, without loss of essentially, it is sufficient to consider the case where \((J, g)\) (and hence \((\overline{J}, \overline{g})\)) is of the form (I). We may easily observe that \( \sum_{i,j=1}^4 (\overline{X}_i \overline{J}_j)^2 \) gives rise to a differentiable function
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Since $\overline{M}$ is a locally product Riemannian manifold of 3-dimensional hyperbolic space and a real line, it follows that the Euler class $\chi(\overline{M})$ of $\overline{M}$ vanishes. Further, since $\overline{M}$ is conformally flat, it follows that the first Pontrjagin class $p_1(\overline{M})$ of $\overline{M}$ also vanishes. Thus, by the Wu's theorem ([6]), we have $c_1(\overline{M})^2 = 0$. On one hand, by direct calculation, from (3.9), we see that

$$c_1(\overline{M})^2 = \frac{1}{8\pi^2} \int_{\overline{M}} \left( \|\nabla \mathcal{J} \|^2 - \sum_{i,j=1}^{4} (\overline{\nabla} \mathcal{J}_ij)^2 \right) d\overline{M}$$

$$= \frac{1}{8\pi^2} \int_{\overline{M}} \sum_{i,j=1}^{4} (\overline{\nabla} \mathcal{J}_ij)^2 d\overline{M}.$$ 

Thus, it must follows that $\nabla \mathcal{J}_ij = 0$ (1 ≤ $i,j ≤ 4$) everywhere on $\overline{M}$, and hence, $\nabla \mathcal{J}_ij = X_4J_{ij} = 0$ (1 ≤ $i,j ≤ 4$) everywhere on $M$. But this contradicts Proposition 1, which completes the proof of theorem.

4. Automorphisms of the example of almost Kähler manifold $(\mathbb{H}^3 \times \mathbb{R}, J, g)$.

A differentiable transformation $\varphi$ on an almost hermitian manifold $(M, J, g)$ is called an automorphism if $\varphi$ is a isometry and pseudo-holomorphic transformation, that is, $\varphi$ satisfies

$$\varphi^* g = g \text{ and } \varphi^* \circ J = J \circ \varphi^*,$$

where $\varphi^*$ denotes the differential map of $\varphi$ and $\varphi^*$ its dual map. We denote by $\text{Aut}_M(J, g)$ the set of all automorphisms on $(M, J, g)$. It is obvious that the set $\text{Aut}_m(J, g)$ is a group under the composition of maps, and we call it the automorphism group on $(M, J, g)$. In this section, we shall determine the automorphism group $\text{Aut}_{\mathbb{H}^3 \times \mathbb{R}}(J, g)$ of the example of strictly almost Kähler manifold $(\mathbb{H}^3 \times \mathbb{R}, J, g)$ constructed in the preceding section.

Let $\varphi \in \text{Aut}_{\mathbb{H}^3 \times \mathbb{R}}(J, g)$. We set $\varphi_*(X_i) = \sum_{j=1}^{4} \varphi_jX_j$ for $i = 1, 2, 3, 4$. Then, we see that 4×4 matrix $(\varphi_j)$ is of the following form

$$(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = \begin{pmatrix}
\varphi_{11} & \varphi_{12} & \varphi_{13} & 0 \\
\varphi_{21} & \varphi_{22} & \varphi_{23} & 0 \\
\varphi_{31} & \varphi_{32} & \varphi_{33} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \text{ with } (\varphi_j)_{15, 153} \in SO(3),$$

(4.1)

with $(\varphi_j)_{15, 153} \in SO(3)$,
since $\varphi$ is an orientation-preserving isometry. We notice that $\varphi_{ij}(1 \leq i, j \leq 3)$ are independent on the variable $x_4$. Since $\varphi$ satisfies $\varphi^{-1} \circ J \circ \varphi = J$, we have, in particular,

$$(\varphi^{-1} \circ J \circ \varphi_*)X_i = JX_i.$$  

We now suppose that the matrix $(\varphi_{ij})$ is of the form (4.1). Then, by a direct calculation, we have

$$(\varphi^{-1} \circ J \circ \varphi_*)X_i = \begin{pmatrix}
\varphi_{33} & -\varphi_{32} & \varphi_{31} \\
-\varphi_{23} & \varphi_{22} & -\varphi_{21} \\
\varphi_{13} & -\varphi_{12} & \varphi_{11}
\end{pmatrix}
\begin{pmatrix}
\cos(x_4 + c_4) \\
\sin(x_4 + c_4) \\
0
\end{pmatrix} = \begin{pmatrix}
\cos x_4 \\
\sin x_4 \\
0
\end{pmatrix},$$

for some constant $c_4 \in \mathbb{R}$, and hence

$$(\varphi^{-1} \circ J \circ \varphi_*)X_i = JX_i, \quad (i = 2, 3, 4)$$

implies the same equality (4.3). From (4.3), it follows that

$$ (\varphi_{ij}) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos c_4 & \sin c_4 & 0 \\
0 & -\sin c_4 & \cos c_4 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$  

Thus, the automorphism $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ satisfies the following system of first order partial differential equations:

$$\frac{\partial \varphi_1}{\partial x_1} = \frac{1}{x_1} \varphi_1, \quad \frac{\partial \varphi_1}{\partial x_2} = \frac{\partial \varphi_1}{\partial x_3} = \frac{\partial \varphi_1}{\partial x_4} = 0.$$
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\[ \frac{\partial \varphi_1}{\partial x_2} = \frac{1}{x_1} \varphi_1 \cos c_4, \quad \frac{\partial \varphi_2}{\partial x_3} = -\frac{1}{x_1} \varphi_1 \sin c_4, \quad \frac{\partial \varphi_3}{\partial x_4} = \frac{\partial \varphi_4}{\partial x_1} = 0, \]

\[ \frac{\partial \varphi_1}{\partial x_2} = \frac{1}{x_1} \varphi_1 \sin c_4, \quad \frac{\partial \varphi_3}{\partial x_3} = \frac{1}{x_1} \varphi_1 \cos c_4, \quad \frac{\partial \varphi_1}{\partial x_4} = \frac{\partial \varphi_3}{\partial x_1} = 0, \]

\[ \frac{\partial \varphi_4}{\partial x_4} = 1, \quad \frac{\partial \varphi_1}{\partial x_2} = \frac{\partial \varphi_3}{\partial x_2} = \frac{\partial \varphi_4}{\partial x_3} = 0. \]

Solving this system of differential equations, we find that the automorphism \( \varphi \) can be express as the form

\[ \varphi(x_1, x_2, x_3, x_4) = (e^{\epsilon_1} x_1, e^{\epsilon_1} ((\cos c_4) x_2 - (\sin c_4) x_3) + c_2), \]

\[ e^{\epsilon_1} ((\sin c_4) x_2 + (\cos c_4) x_3) + c_3, x_4 + c_4) \]

for \( c_i \in \mathbb{R}, i = 1, 2, 3, 4. \)

Next, we suppose that the matrix \( (\varphi_{ij}) \) is of the form (4.2). Then, in the same way, we have

\[
\begin{pmatrix}
-\varphi_{33} & \varphi_{32} & -\varphi_{31} \\
\varphi_{23} & -\varphi_{22} & \varphi_{21} \\
-\varphi_{13} & \varphi_{12} & -\varphi_{11}
\end{pmatrix}
\begin{pmatrix}
\cos(-x_4 + c_4) \\
\sin(-x_4 + c_4) \\
0
\end{pmatrix}
= \begin{pmatrix}
\cos x_4 \\
\sin x_4 \\
0
\end{pmatrix},
\]

which implies

\[
(\varphi_{ij}) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos c_4 & \sin c_4 & 0 \\
0 & \sin c_4 & -\cos c_4 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},
\]

and hence, we have

\[ \varphi(x_1, x_2, x_3, x_4) = (e^{\epsilon_1} x_1, e^{\epsilon_1} ((\cos c_4) x_2 + (\sin c_4) x_3) + c_2), \]

\[ e^{\epsilon_1} ((\sin c_4) x_2 - (\cos c_4) x_3) + c_3, -x_4 + c_4) \]

for \( c_i \in \mathbb{R}, i = 1, 2, 3, 4. \)

We can summarize the above arguments as follows.

**PROPOSITION 3.** The automorphism group \( \text{Aut}_{H^1 \times R} (J, g) \) of the almost Kähler manifold \((H^1 \times R, J, g)\) is isomorphic to a solvable subgroup of affine transformation group \( GL(4, R) \times R^4 \) (embedded in \( GL(5, R) \)), which consists of the elements
From Proposition 3, we may easily see that the group $\text{Aut}_{H^3 \times R}(J,g)$ acts transitivity on $H^3 \times R$ and that the identity component $\text{Aut}_{H^3 \times R}(J,g)_0$ of $\text{Aut}_{H^3 \times R}(J,g)$ is a subgroup consists of the elements of the form (4.4) and acts simply transitively on $H^3 \times R$. Taking account of Theorem 2, we see that there does not exist a discrete uniform subgroup of $\text{Aut}_{H^3 \times R}(J,g)_0$ (i.e. discrete subgroup $\Gamma$ of $\text{Aut}_{H^3 \times R}(J,g)_0$ such that the orbit space $\Gamma \backslash H^3 \times R$ is compact).

REMARK 1. We may easily find that the system of differential equations $V_x J = 0 (i = 1, 2, 3, 4)$ has no solution, and thus the Riemannian product $(H^3 \times R, g)$ can not admit (a compatible) Kähler structure.

REMARK 2. Let $\psi$ be an isometry on $(H^3 \times R, g)$ and $(J, g)$ be an almost Kähler structure constructed in the preceding section. Then, almost Hermitian structure $(\psi(J), g)$ is also an almost Kähler structure on $(H^3 \times R, g)$, where $\psi(J)$ is defined by $\psi^{-1} \circ J \circ \psi$. The automorphism group $\text{Aut}_{H^3 \times R}(\psi(J), g)$ is determined by the automorphism group $\text{Aut}_{H^3 \times R}(J, g)$. Indeed, the map $\text{Aut}_{H^3 \times R}(J, g) \ni \phi \mapsto \psi^{-1} \circ \phi \circ \psi \in \text{Aut}_{H^3 \times R}(\psi(J), g)$ is an isomorphism.

References


T. Oguro
Department of Mathematical Science,
Graduate School of Science and Technology,
Niigata University,
Niigata, 950-21,
Japan

K. Sekigawa
Department of Mathematics,
Faculty of Science,
Niigata University,
Niigata, 950-21,
Japan