SOME ALMOST-HOMOGENEOUS COMPLEX STRUCTURES
ON $P^2 \times P^2$

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1. Introduction

It is well known that on $P^1(C) \times P^1(C)$ there exists an infinite sequence of different complex structures, namely the Hirzebruch surfaces $\Sigma_{lm}$, $m \in N$. These surfaces are of the form $P(\mathcal{O}_m(\mathcal{O}_m(-m))$ and are all almost-homogeneous (see [H]). In generalization of this, Brieskorn has studied $P^n$-bundles over $P^1$ and has proved that all complex structures on $P^1 \times P^n$ satisfying some supplementary conditions (see [Br], (5.3)) are such $P^n$-bundles. All these structures are almost-homogeneous.

Motivated by these results, it is natural to consider complex structures on $P^2 \times P^2$ of the form $P(E)$, where $E$ is a topologically trivial holomorphic vector bundle of rank 3 on $P^2$. In contrast with the situation on $P^1$, a complete classification of such bundles is not known, however Bănică has classified all topologically trivial rank 2 vector bundles on $P^2$ (see [B], §2). In particular these bundles do not depend only on discrete parameters, but also on “continuous” moduli. Using rank 3 vector bundles on $P^2$ of the form $E := F \oplus \mathcal{O}_{p^1}$, with $F$ topologically trivial of rank 2, one can easily construct complex structures on $P^2 \times P^2$, depending on “continuous” moduli, which are not almost-homogeneous.

Here we study some examples of almost-homogeneous complex structures on $P^2 \times P^2$ of the form $P(E)$, for homogeneous and almost-homogeneous $E$. In §2 are studied the cases when $E$ is $\mathcal{T}_{p^n(-1)} \oplus \mathcal{O}_{p^n(-1)}$ or its dual (together with $\mathcal{O}_{p^n}$), these are the only topologically trivial homogeneous rank 3 vector bundles on $P^2$, see for example [M]. It turns out that the automorphism group of $X_1 := P(\mathcal{T}_{p^n(-1)} \oplus \mathcal{O}_{p^n(-1)})$ has an open orbit, whose complement is an irreducible homogeneous hypersurface (hence $X_1$ gives an example of the manifolds classified by Ahiezer [Ah]), while the automorphism group of $X_2 := P(\mathcal{T}_{p^n(-2)} \oplus \mathcal{O}_{p^n(1)})$ has an open orbit, whose complement is irreducible and homogeneous of codimension 2. In §3 we consider the complex manifold $X := P(F \oplus \mathcal{O}_{p^n})$ with $F$ a topologically trivial rank 2 vector bundle on $P^2$ of generic splitting type $(-1, 1)$ and we prove that the automorphism group of $X$ has an open orbit, whose complement is an irreducible hypersurface, which contains a whole fiber of $P(E)$.
Section 1.

In this section we introduce some notations and preliminary material.

Here a vector bundle is always a holomorphic vector bundle and we often identify vector bundles and locally free sheaves. Following the notations of [OSS] for a vector bundle $E \to S$ we denote by $E(x)$ the fiber over a point $x \in S$.

Let $S$ be a complex manifold, $E \to S$ a rank $m$ vector bundle on $S$, and let $P(E) \to S$ be the corresponding projective bundle. Set $X := P(E)$. We denote by

$\operatorname{Aut}(E)$ the group of all biholomorphic maps $E \to E$, which transform fibers in fibers and are linear on fibers;

$\operatorname{Aut}(P(E))$ the group of all biholomorphic maps of $P(E)$ into itself, which transform fibers in fibers;

$\operatorname{Auts}(E), \operatorname{Auts}(P(E))$ the subgroups of all elements of $\operatorname{Aut}(E)$ and $\operatorname{Aut}(P(E))$ respectively, which induce the identity on $S$;

$\operatorname{PGL}(E)$ the subgroup of all elements of $\operatorname{Auts}(P(E))$, which are induced by elements of $\operatorname{Auts}(E)$;

$\operatorname{Aut}(X)$ the group of all biholomorphic maps of $X$ onto itself.

(1.1) Proposition.

With the same notations as above, let us suppose $S$ simply connected. Then

$$\operatorname{Auts}(P(E)) = \operatorname{PGL}(E).$$

Proof: Let $U$ be a simply connected open subset of $S$ such that $E|U$ is trivial. An element $\phi \in \operatorname{Aut}_U(P(E|U))$ can be regarded as a holomorphic map $\tilde{\phi}: U \to \operatorname{PGL}(m)$. Since $U$ is simply connected and $\text{SL}(m)$ is a covering space of $\operatorname{PGL}(m)$, the map $\tilde{\phi}$ can be lifted to a holomorphic map $\phi: U \to \text{SL}(m)$, and this gives an element $\phi \in \operatorname{Auts}(P(E|U))$, which induces $\phi$.

Now let $\phi \in \operatorname{Auts}(P(E))$ and let $\{U_i\}_{i \in I}$ be an open covering of $S$ such that every $U_i$ is simply connected and $E|U_i$ is trivial. Then for each $U_i$ there exists $\Phi_i \in \operatorname{Aut}(E|U_i)$ with $\det \Phi_i = 1$, which induces $\phi$. On $U_i \cap U_j$ the two matrices $\Phi_i$ and $\Phi_j$ induce the same projective automorphism $\phi$, therefore they coincide up to the multiplication by an $m$-th root of unity. Thus we obtain an element of $H^1(S, \mu_m)$, where $\mu_m$ is the locally constant sheaf of $m$-th roots of unity. Since $S$ is simply connected $H^1(S, \mu_m) = 0$, hence $\Phi_i \cdot \Phi_j^{-1} = \lambda_i \cdot \lambda_j^{-1}$ where $\lambda_i, \lambda_j$ are $m$-th roots of unity. Thus the $\lambda_i^{-1} \cdot \Phi_i$ can be glued together to give an element $\phi \in \operatorname{Auts}(E)$ which induces $\phi$.

We recall the following

(1.2) Definition.

Let $S$ be a homogeneous complex manifold and let $E \to S$ be a vector bundle on $S$. 

We say that $E$ is homogeneous if for all $g \in \text{Aut}(S)$ one has $g^* E \cong E$.

We say that $E$ is almost-homogeneous if there exists a subgroup $G$ of $\text{Aut}(S)$, one of whose orbits is a Zariski open dense subset of $S$, such that $g^* E \cong E$ for all $g \in G$.

(1.3) Corollary.
Let $S$ be a homogeneous simply connected complex manifold, $E \rightarrow S$ a homogeneous vector bundle on $S$ and let $P(E) \rightarrow S$ be the corresponding projective bundle. Then there is the exact sequence

$$0 \rightarrow \text{PGL}(E) \rightarrow \text{Aut}(P(E)) \rightarrow \text{Aut}(S) \rightarrow 0.$$ 

Proof: This is an obvious consequence of (1.1).

Section 2.
In this section we show that the complex manifolds $X_1 := P(T_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2})$ and $X_2 := P(T_{\mathbb{P}^2}(-3) \oplus \mathcal{O}_{\mathbb{P}^2})$ are almost-homogeneous and we determine the orbits with respect to the action of the group of automorphisms.

It is well known (see [A], th. 3) that in the decomposition $E := E_1 \oplus \cdots \oplus E_n$ of a vector bundle over a compact variety into direct sum of indecomposable bundles, the bundles $E_i$ are uniquely determined up to order and isomorphy; however in general the bundles $E_i$ are not uniquely determined as subbundles of $E$.

(2.1) Lemma.
Let $E_1 := \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}$. In this decomposition only the vector bundle $F_1 := \mathcal{O}_{\mathbb{P}^2} \oplus 0$ is uniquely determined as subbundle of $E_1$.

Proof: We first observe that the vector bundle $G_1 := 0 \oplus \mathcal{O}_{\mathbb{P}^2}$ is not uniquely determined as subbundle of $E$, since it is not invariant under an automorphism $\phi \in \text{Aut}_S(E_1)$ of the form

$$\begin{pmatrix}
\text{id}_{\mathcal{O}_{\mathbb{P}^2}} & \psi \\
0 & \text{id}_{\mathcal{O}_{\mathbb{P}^2}}
\end{pmatrix}$$

with $\psi \in \text{Hom}(\mathcal{O}_{\mathbb{P}^2}, T_{\mathbb{P}^2})$, $\psi \neq 0$.

On the other hand, from the Euler sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \rightarrow T_{\mathbb{P}^2}(-1) \rightarrow 0,$$

it follows that the vector bundle $T_{\mathbb{P}^2}(-1)$ is generated by global sections, therefore $F_1(-1)$ is the subbundle of $E_1(-1)$ generated by $\Gamma(\mathbb{P}^2, E_1(-1))$ and this characterizes $F_1(-1)$ as subbundle of $E_1(-1)$. 
(2.2) Corollary. Let $E_2 := T_p(-3) \oplus \mathcal{O}_p$. In this decomposition only the vector bundle $G_2 := 0 \oplus \mathcal{O}_p$ is uniquely determined as subbundle of $E_2$.

Proof: Since $E_2$ is the dual bundle of $E_1$ and $G_2$ consists of the linear forms on $E_1$, which are zero on $F_1 := T_p \oplus 0$, the assertion follows from (2.1).

(2.3) Theorem. Let $E_1 := T_p \oplus \mathcal{O}_p$ and let $X_1 := \mathbb{P}(E_1)$. The group $\text{Aut}(X_1)$ has exactly two orbits: $A_1 := \mathbb{P}(T_p \oplus 0)$ and $A_2 := X_1 - A_1$.

Proof: We first prove that $A_1$ is transformed into itself by all $\phi \in \text{Aut}(X_1)$. By [S], th. $A_1$, $\text{Aut}(X_1) = \text{Aut}(\mathbb{P}(E_1))$, so $\phi$ determines an automorphism $\tilde{\phi} \in \text{Aut}(S)$ and an isomorphism $\mathbb{P}(E_1) \cong \tilde{\phi}^* \mathbb{P}(E_1)$, which induces the identity on $\mathbb{P}^2$ and which can be identified with $\phi$. Therefore $E_1$ is isomorphic to $\tilde{\phi}^* E_1 \cong \mathbb{P}(E_1)$ and calculating the first Chern classes one sees that $k = 0$. Thus $\phi$ induces an isomorphism $\Phi: E_1 \cong \tilde{\phi}^* E_1$, which must transform $(T_p \oplus 0)(x)$ into $(\tilde{\phi}^* (T_p \oplus 0))(x) = (T_p \oplus 0)(\tilde{\phi}(x))$ for all $x \in \mathbb{P}^2$. Therefore $\phi(T_p \oplus 0) = \mathbb{P}(T_p \oplus 0)$.

Now we prove that the action of $\text{Aut}(X_1)$ is transitive on both $A_1$ and $A_2$, by showing that for all $x \in \mathbb{P}^2$ the subgroup of $\text{Aut}(X_1)$, which fixes the fiber $\mathbb{P}(E_1)_x$, acts transitively on $A_1 \cap \mathbb{P}(E_1)_x$, and on $A_2 \cap \mathbb{P}(E_1)_x$. They correspond to lines $r$, $r'$ of $\mathbb{P}^2$ through the point $x$. Let $\alpha \in \text{Aut}(\mathbb{P}^2)$ be such that $\alpha(x) = x$ and $\alpha(r) = r'$ and take an element $\phi \in \text{Aut}(\mathbb{P}(E_1))$ such that $\tilde{\phi} = \alpha$. It is easy to show that $\phi(x) = x'$.

Since $\text{End} \ E_1 = \left( \begin{array}{c|c} \text{End} \ T_p & \text{Hom}(\mathcal{O}_p, T_p) \\ \hline \text{Hom}(T_p, \mathcal{O}_p) & \text{End} \ \mathcal{O}_p \end{array} \right)$

and since $\text{Aut}_p(T_p) \cong \text{Aut}_p(\mathcal{O}_p) \cong \mathbb{C}^*$, the action on $\mathbb{P}(E_1)_x$ of an element of $\text{PGL}(E_1)$ can be thought as the action on $\mathbb{P}^2$, with projective coordinates $(a_1 : x : x_3)$, of a matrix like

\[
\begin{pmatrix}
\lambda & 0 & a_1 \\
0 & \lambda & a_2 \\
0 & 0 & \mu
\end{pmatrix}
\]

with $\lambda, \mu \in \mathbb{C}^*$, $a_1, a_2 \in \mathbb{C}$.

In this $\mathbb{P}^2$, $A_1 \cap \mathbb{P}(E_1)_x$, can be identified with the line $x_3 = 0$ and $A_2 \cap \mathbb{P}(E_1)_x$, with the complement of such a line; therefore it is clear that $\text{PGL}(E_1)$ acts transitively on $A_2 \cap \mathbb{P}(E_1)_x$.

(2.4) Theorem. Let $F_2 := T_p(-3) \oplus \mathcal{O}_p$ and let $X_2 := \mathbb{P}(E_2)$. The orbits of $X_2$ with respect to the action of $\text{Aut}(X_2)$ are exactly $B_1 := \mathbb{P}(0 \oplus \mathcal{O}_p)$ and $B_2 := X_2 - B_1$.

Proof: With an argument similar to the one used in prop. (2.3) and using (2.2), one has that all $\phi \in \text{Aut}(X_2) = \text{Aut}(\mathbb{P}(E_2))$ transform $B_1$ into itself.
Since for all \( x \in \mathbb{P}^2 \), \( B_1 \cap \mathbf{P}(E_2) \), consists exactly of one point, we have only to prove that for all \( x \in \mathbb{P}^2 \) the action of the subgroup \( \Sigma \) of \( \text{Aut} \ (X_2) \), containing all automorphisms, which fix the fiber \( \mathbf{P}(E_2)_x \), is transitive on \( B_2 \cap \mathbf{P}(E_2)_x \).

Let \( (x_1;x_2;x_3) \) and \( (y_1;y_2;y_3) \) be two points in \( B_2 \cap \mathbf{P}(E_2)_x \). With an argument similar to the one used in proving prop. (2.3), there exists \( \phi \in \Sigma \), which transforms \( (x_1;x_2;x_3) \) into \( (y_1;y_2;y_3) \).

Now there exists an element \( \psi \) in \( \text{PGL} \ (E_2) \) (whose action on \( \mathbf{P}(E_2)_x \) can be thought as the action on \( \mathbb{P}^2 \) of a matrix like

\[
\begin{pmatrix}
\lambda & 0 & 0 \\
0 & \lambda & 0 \\
a_1 & a_2 & \mu
\end{pmatrix}
\]

with \( \lambda, \mu \in \mathbb{C}^*, a_1, a_2 \in \mathbb{C} \), which transforms \( (y_1;y_2;y_3) \) in \( (y_1;y_2;y_3) \).

Section 3.

In this section we show that the complex manifold \( \mathbf{P}(F \oplus \mathcal{O}_{\mathbb{P}^2}) \), where \( F \) is a rank 2 topologically trivial vector bundle on \( \mathbb{P}^2 \) of generic splitting type \((-1,1)\), is almost-homogeneous.

(3.1) Proposition.

Let \( F \) be a rank 2 topologically trivial vector bundle on \( \mathbb{P}^2 \) of generic splitting type \((-1,1)\). Then

i) there is an exact sequence:

\[
0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(1) \rightarrow F \rightarrow g_{\mathcal{Z}}(-1) \rightarrow 0,
\]

where \( Z \) is a simple point of \( \mathbb{P}^2 \), which determines the bundle \( F \) up to isomorphy;

ii) \( F \cong F^\vee \) (that is \( F \) is self-dual);

iii) \( F \) is almost-homogeneous.

Proof: i) The existence of the exact sequence (\( \ast \)) has been proved by Bânîcâ (see [B], lemma 4).

Now let \( F' \) be a vector bundle on \( \mathbb{P}^2 \), which makes exact the sequence

\[
0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(1) \rightarrow F' \rightarrow g_{\mathcal{Z}}(-1) \rightarrow 0.
\]

Both \( F \) and \( F' \) correspond to elements \( \eta, \eta' \in \text{Ext}^1(g_{\mathcal{Z}}(-1), \mathcal{O}_{\mathbb{P}^2}(1)) \), which are not zero, since the trivial extension \( \mathcal{O}_{\mathbb{P}^2}(1) \oplus g_{\mathcal{Z}}(-1) \) is not a vector bundle. But, from [B], §2, \( \dim \text{Ext}^1(g_{\mathcal{Z}}(-1), \mathcal{O}_{\mathbb{P}^2}(1)) = 1 \), therefore \( \eta = \alpha \eta' \) with \( \alpha \in \mathbb{C}^* \), hence \( F \cong F' \).

ii) Since \( F \) has rank 2, we have \( F^\vee \cong F \otimes \det F \cong F \).

iii) Let \( G := \{ g \in \text{Aut} \ (\mathbb{P}^2) \mid g(Z) = Z \} \), and let \( g \in G \).

The vector bundle \( g^*F \) makes exact the sequence

\[
0 \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow g^*F \rightarrow g_{\mathcal{Z}}(-1) \rightarrow 0
\]
and with the same argument used in (i), \( g^*F \simeq F \).

(3.2) Lemma.

Let \( E := F \oplus \mathcal{O}_{\mathbb{P}^2} \), with \( F \) as in (3.1), and let \( \mathcal{V}_1 := \alpha(\mathcal{O}_{\mathbb{P}^2}(1)) \oplus 0 \), \( \mathcal{V} := \alpha(\mathcal{O}_{\mathbb{P}^2}(1)) \oplus \mathcal{O}_{\mathbb{P}^2} \). The filtration \( \mathcal{V}_1 \subset \mathcal{V} \subset E \) is invariant with respect to \( \text{Aut}(E) \).

Proof: From the exact sequence \((*)\), one has \( \Gamma(P^2, F) = \Gamma(P^2, \alpha(\mathcal{O}_{\mathbb{P}^2}(1))) \). It follows \( \mathcal{V}_1 = \mathcal{O}_{\mathbb{P}^2} \cdot \Gamma(P^2, \mathcal{O}_{\mathbb{P}^2}(1)) \), hence \( \mathcal{V}_1 = \mathcal{O}_{\mathbb{P}^2} \cdot \Gamma(P^2, E(-1)) \).

(3.3) Theorem.

Let \( E := F \oplus \mathcal{O}_{\mathbb{P}^2} \), with \( F \) as in (3.1), and let \( X := P(E) \). The action of \( \text{Aut}(X) \) on \( X \) has an open orbit, whose complement is an irreducible hypersurface \( H \subset X \), which can be described as follows: let \( V \) be the subbundle of \( E(1) \), defined by \( V := (\alpha(\mathcal{O}_{\mathbb{P}^2}(1)) \oplus \mathcal{O}_{\mathbb{P}^2})(1) \). Then \( H = P(V) \cup P(E) \).

Proof: We first observe that, from the fact that \( Z \) is characterized by the property that every non-zero section of \( E(-1) \) vanishes exactly on \( Z \) (see the proof of (3.2)), it follows that every \( \phi \in \text{Aut}(X) \) transforms \( P(E) \) into itself. By lemma (3.2), with an argument similar to the one used in proving prop. (2.3), one has that every \( \phi \in \text{Aut}(X) \) transforms also \( P(V) \) into itself.

Now we show that \( \text{Aut}(X) \) acts transitively on \( A := X - (P(V) \cup P(E)) \), by proving that for all \( x \in P^2 - Z \) the subgroup \( \text{PGL}(E) \) of \( \text{Aut}(X) \) acts transitively on \( A \setminus P(E) \).

We observe that \( \text{End} \ E = \left( \begin{array}{c|c} \text{End} F & \text{Hom}(\mathcal{O}_{\mathbb{P}^2}, F) \\ \hline \text{Hom}(F, \mathcal{O}_{\mathbb{P}^2}) & \text{End} \mathcal{O}_{\mathbb{P}^2} \end{array} \right) \).

From the exact sequence \((\ast)\) we get the exact sequence

\[ 0 \to \text{Hom}(F, \mathcal{O}_{\mathbb{P}^2}(1)) \to \text{End} F \to \text{Hom}(F, \mathcal{O}_{\mathbb{P}^2}(1)) \to \cdots \]

Since the endomorphisms of \( F \), which are in \( \text{Im} \sigma = \text{Ker} \tau \), cannot be surjective, \( \text{id}_F \not\in \text{Ker} \tau \), hence \( \dim(\text{Hom}(F, \mathcal{O}_{\mathbb{P}^2}(1))) \geq 1 \). On the other hand, from \((\ast)\) we have also the exact sequence

\[ 0 \to \text{Hom}(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(1)) \to \text{Hom}(F, \mathcal{O}_{\mathbb{P}^2}(1)) \to \text{Hom}(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(1)) \to \cdots \]

where \( \text{Hom}(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(1)) = 0 \). Therefore \( \text{Hom}(F, \mathcal{O}_{\mathbb{P}^2}(1)) \simeq \text{Hom}(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(1)) \) and the last, by Riemann's extension theorem, is isomorphic to \( \text{Hom}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}) \simeq S \). Therefore, \( \text{Hom}(F, \mathcal{O}_{\mathbb{P}^2}(1)) \simeq C \), and, since \( \tau(\text{id}_F) \neq 0 \), the morphism \( \tau \) of \((\ast)\) is surjective and \((\ast)\) becomes

\[ 0 \to \text{Hom}(F, \mathcal{O}_{\mathbb{P}^2}(1)) \to \text{End} F \to \text{Hom}(F, \mathcal{O}_{\mathbb{P}^2}(1)) \to 0. \]
Again from ($\ast$), we have the exact sequence

$$0 \to \text{Hom}(\mathcal{O}_Z(1), \mathcal{O}_p(1)) \to \text{Hom}(F, \mathcal{O}_p(1)) \to \text{End}(\mathcal{O}_p(1)) \to 0,$$

since $H^1(\text{Hom}(\mathcal{O}_Z(-1), \mathcal{O}_p(1)) \cong H^1(\mathcal{O}_p(2)) = 0$

as codim $Z = 2$. Moreover, since $\text{Hom}(\mathcal{O}_Z(-1), \mathcal{O}_p(1)) \cong \text{Hom}(\mathcal{O}_p(-1), \mathcal{O}_p(1)) \cong \mathcal{O}_p(2)$ and $\text{End}(\mathcal{O}_p(1)) \cong \mathcal{O}_p$ are globally generated, for any point $x$ in $P^2 - Z$ every homomorphism of $F(x)$ into $(\mathcal{O}_p(1))(x)$ is induced by an element of $\text{Hom}(F, \mathcal{O}_p(1))$. Now we fix a point $x$ in $P^2 - Z$ and a base $v_1, v_2$ of $F(x)$ such that $v_1 \in (\alpha(\mathcal{O}_p(1)))(x)$. With respect to such a base the automorphisms of $F(x)$ induced by global automorphisms of $F$ can be represented by matrices of the type

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

with $a, c \in \mathbb{C}^*$, $b \in \mathbb{C}$. From ($\ast$) we have the exact sequence

$$0 \to \text{Hom}(\mathcal{O}_p, \mathcal{O}_p(1)) \to \text{Hom}(\mathcal{O}_p, F) \to \text{Hom}(\mathcal{O}_p, \mathcal{O}_Z(-1)) = 0,$$

hence $\text{Hom}(\mathcal{O}_p, F) = \text{Hom}(\mathcal{O}_p, \mathcal{O}_p(1))$, that is every homomorphism of $\mathcal{O}_p$ into $F$ has values in $\alpha(\mathcal{O}_p(1))$. From ($\ast$) we have also

$$0 \to \text{Hom}(\mathcal{O}_p(-1), \mathcal{O}_p(1)) \to \text{Hom}(\mathcal{O}_p, F) \to \text{Hom}(\mathcal{O}_p, \mathcal{O}_Z(-1)) = 0,$$

hence $\text{Hom}(\mathcal{O}_p, F) = \text{Hom}(\mathcal{O}_p, \mathcal{O}_p(1))$, that is every homomorphism of $F$ into $\mathcal{O}_p$ is zero on $\alpha(\mathcal{O}_p(1))$. Now we complete the base $v_1, v_2$ of $F(x)$ to a base $v_1, v_2, v_3 \in E(x)$ by adding a vector $v_3 \in (0 \oplus \mathcal{O}_p)(x)$. With respect to such a base the automorphisms of $E(x)$ induced by global automorphisms of $E$ can be represented by a matrix of the type

$$\begin{pmatrix} a & b & d \\ 0 & c & 0 \\ 0 & e & f \end{pmatrix}$$

with $a, c, f \in \mathbb{C}^*$, $b, d, e \in \mathbb{C}$.

Since $A \cap P(E)_h$ can be identified with the complement of the line $x_2 = 0$, an easy computation shows that the action of $\text{PGL}(E)$ on $A \cap P(E)_h$ is transitive.

Now we prove that $H := P(V) \cup P(E)_2$ is an irreducible hypersurface in $X$, by showing that $H = P(V)$. Let $U$ be an open neighbourhood of $Z$, over which the bundles $F$ and $\mathcal{O}_p(1)$ are trivial and let $(e_1, e_2)$ and $e$ bases of $F|U$ and $\mathcal{O}_p(1)|U$ respectively. With respect to these bases, the morphism $\alpha: \mathcal{O}_p(1) \to F$ can be described as $\alpha(e) = f_1 e_1 + f_2 e_2$, where $f_1, f_2$ are holomorphic functions on $U$, which have exactly one common zero in the point $Z$. In $P(E)|U \cong U \times P^2$ we have

$$P(V)|U \times Z = \{(p; t_1: t_2: t_3) \in (U \times Z) \times P^2 | t_1 f_2(p) - t_2 f_1(p) = 0\},$$

hence

$$P(V)|U \times Z = \{(p; t_1: t_2: t_3) \in U \times P^2 | t_1 f_2(p) - t_2 f_1(p) = 0\} = P(V)|(U \times Z) \cup P(E)_2.$$
References


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