Almost Kähler manifolds

\[ T_{12} = \sum_{a,b=1}^{4} N''_{ab}N''_{ba} = \sum_{a=1}^{4} (N''_{31}N''_{32} + N''_{41}N''_{42}) \]

\[ = g(N(e_3,e_1),N(e_3,Je_1)) + g(N(e_4,e_1),N(e_4,Je_1)) \]

\[ = -g(N(e_3,e_1),JN(e_3,e_1)) - g(N(e_4,e_1),JN(e_4,e_1)) \]

\[ = 0. \]

Similarly, we have

\[ T_{22} = T_{33} = T_{44} = \frac{1}{4} \|N\|^2, \]

and

\[ T_{13} = T_{14} = T_{23} = T_{24} = T_{34} = 0. \]

Consequently, we have

\[ T_{ij} = \frac{1}{4} \|N\|^2 g_{ij}. \]

By Proposition 3.1, we see that \( M \) is an Einstein and weakly \(*\)-Einstein manifold. Since \( c \) and \( \tau \) are constant on \( M \), \( \tau^* \) is also constant by (3.9), that is, \( M \) is \(*\)-Einstein. Then, taking account of the theorem of Sekigawa and Vanhecke [10], we can conclude that \( M \) is Kählerian.

REMARK. From the result of U. K. Kim, I-B. Kim and J-B. Jun [3], it will be also obtained that \( M \) is Einstein and weakly \(*\)-Einstein.

COROLLARY 3.4. Let \( M \) be a 4-dimensional compact almost Kähler manifold of constant holomorphic sectional curvature satisfying

\[ \rho - \rho^* = \frac{\tau - \tau^*}{4} g \]

Then \( M \) is a Kähler manifold.

PROOF. This follows from Theorem 3.2 and Theorem 3.3.

COROLLARY 3.5. Let \( M \) be a 4-dimensional compact almost Kähler manifold of pointwise constant holomorphic sectional curvature. If \( M \) satisfies the condition (b), then \( M \) is a Kähler manifold.

PROOF. Under the condition (b), we can see that the function \( c \) is constant on \( M \) ([6]). Hence this follows immediately from Theorem 3.3.
References


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ON AUTOMORPHISMS OF A CHARACTER RING

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1. Introduction

Throughout this paper G, Z(G) and C denote a finite group, the center of G and the field of complex numbers respectively. For a finite set S, we denote the number of elements in S by |S|.

Let Irr(G) be the full set of irreducible C-characters of G and X(G) be the character ring of G. If R is any subring of C, we write RX(G) to denote the R-algebra of R-linear combinations of irreducible C-characters of G.

Suppose G and H are finite groups. Weidman showed that if X(G) is isomorphic to X(H), then G and H have the same character table.

In addition Saksonov proved the following theorem, which is a strengthened version of Weidman's theorem.

THEOREM 1.1. (Saksonov) Suppose R is the ring of all algebraic integers and there exists an R-algebra isomorphism \( \phi \) from \( RX(G) \) onto \( RX(H) \). If \( \text{Irr}(G) = \{ \chi_1, \cdots, \chi_s \} \) and \( \text{Irr}(H) = \{ \psi_1, \cdots, \psi_h \} \), then the following holds:

(i) The character tables of G and H are the same.
(ii) \( \phi(\chi_i) = \epsilon_i \psi_{i'} \) (\( i = 1, \cdots, h \)) where the \( \epsilon_i \) are roots of unity and \( i \to i' \) is a permutation.

From now on we assume that R is the ring of all algebraic integers. Then in this paper we intend to prove the following theorem.

THEOREM 1.2. Suppose G and H are finite groups. Then we have

(i) If \( u \) is a central element in G and \( \tau_u : RX(G) \to RX(G) \) is the map defined by \( \chi \to (\chi(u)/\chi(1))\chi \) where \( \chi \in \text{Irr}(G) \) and 1 is the identity element of G, then \( \tau_u \) is an R-automorphism of \( RX(G) \). Furthermore the map \( u \to \tau_u \) is a group isomorphism of \( Z(G) \) onto a subgroup \( T = \{ \tau_u | u \in Z(G) \} \) of Aut(RX(G)).
ii) Every $R$-isomorphism $\phi: RX(G) \to RX(H)$ is the composition of an $R$-isomorphism $\Theta$ that maps $\text{Irr}(G)$ onto $\text{Irr}(H)$ with an automorphism of $RX(H)$ of the form $\tau_u$ for some element $u$ in $Z(H)$.

(iii) The full group $A = \text{Aut}(RX(G))$ is the product of the subgroup $T$ of part (i) above, which is normal, with the subgroup $P$ consisting of those automorphisms that map $\text{Irr}(G)$ onto $\text{Irr}(G)$.

2. Proof of Theorem 1.2

In order to prove Theorem 1.2 we prove a basic lemma concerning the roots of unity which appear in Saksonov's Theorem.

Lemma 2.1. Suppose for each character $\chi$ in $\text{Irr}(G)$, there is a root of unity $e(\chi)$ such that each product $e(\chi)e(\psi)\psi$ for $\chi, \psi$ in $\text{Irr}(G)$ is a non-negative integer linear combination of $e(\xi)\xi$, as $\xi$ runs over $\text{Irr}(G)$. Then there exists $u$ in $Z(G)$ such that $e(\chi) = \chi(u)/\chi(1)$ for every character $\chi$ in $\text{Irr}(G)$.

Proof. If we are given $\chi$ and $\psi$ in $\text{Irr}(G)$, then we assume that

$$\chi\psi = \sum_{\xi \in \text{Irr}(G)} m_\xi \xi$$

and $e(\chi)e(\psi)\psi = \sum_{\xi \in \text{Irr}(G)} n_\xi e(\xi)\xi$

where the coefficients $m_\xi$ and $n_\xi$ are non-negative integers. Then it follows easily that $m_\xi = n_\xi$ for all characters $\xi$ in $\text{Irr}(G)$ and thus the map $\phi: \chi \to e(\chi)\chi$ defines an automorphism of the algebra $CX(G)$. In particular the map $\phi$ permutes the primitive idempotents of this C-algebra (See the proof of Lemma 2.3 in [3]) and so it carries the characteristic class function of the identity to the characteristic class function of some other conjugacy class, say the class $K$. Therefore we have

$$(1/|G|) \sum_{\chi \in \text{Irr}(G)} e(\chi)\chi(1)\chi = (1/|C_G(v)|) \sum_{\chi \in \text{Irr}(G)} \overline{\chi(v)}\chi$$

where $v$ is an element in $K$. It follows that for each irreducible character $\chi$ in $\text{Irr}(G)$ we have $\chi(1)e(\chi) = |K|\chi(u)$ where $u = v^{-1}$. Applying this where $\chi$ is the principal character yields that $|K|$ is a root of unity and so $u$ is a central element in $G$. Thus for every character $\chi$ in $\text{Irr}(G)$, $e(\chi) = \chi(u)/\chi(1)$ for some element $u$ in $Z(G)$, as claimed. Q.E.D.

Proof of Theorem 1.2. (i) Suppose $u$ is a central element in $G$. Then for each character $\chi$ in $\text{Irr}(G)$ we denote by $e(\chi)$ and $T(\chi)$ the root of unity given by $\chi(u)/\chi(1)$ and the irreducible matrix representation of $G$ which affords $\chi$ respectively. We assume further that for $\chi, \psi$ in $\text{Irr}(G)$, $\chi\psi = \sum_{\xi \in \text{Irr}(G)} m_\xi \xi$ where
On automorphisms of a character ring

the \( m_i \) are non-negative integers. Then we show \( \varepsilon(\xi) = \varepsilon(\chi)\varepsilon(\psi) \) for \( m_i \neq 0 \).

Indeed \( T(\chi)(u) = \text{diag}(\varepsilon(\chi), \ldots, \varepsilon(\chi)) \) and \( T(\psi)(u) = \text{diag}(\varepsilon(\psi), \ldots, \varepsilon(\psi)) \) which have diagonals of lengths \( \chi(1) \) and \( \psi(1) \) respectively. Hence

\[
T(\chi)(u) \otimes T(\psi)(u) = \text{diag}(\varepsilon(\chi)\varepsilon(\psi), \ldots, \varepsilon(\chi)\varepsilon(\psi))
\]

where \( T(\chi) \otimes T(\psi) \) is the Kronecker product of \( T(\chi) \) and \( T(\psi) \). Since \( T(\chi) \otimes T(\psi) \) is the representation of \( G \) which affords \( \chi\psi \), we have \( \varepsilon(\xi) = \varepsilon(\chi)\varepsilon(\psi) \) for \( m_i \neq 0 \), as claimed. Therefore we have \( \varepsilon(\chi)\chi\varepsilon(\psi)\psi = \sum_{\xi \in \text{Irr}(G)} m_i \varepsilon(\xi)\xi \).

Thus the map \( \tau_\phi \) defined by \( \varepsilon(\chi)\chi\varepsilon(\psi)\psi = \sum_{\xi \in \text{Irr}(G)} m_i \varepsilon(\xi)\xi \) is an \( \text{R}\)-automorphism of \( \text{RX}(G) \).

The fact that \( Z(G) \equiv T \) is easy to prove and so we omit its proof.

(ii) Now we can easily observe that Saksonov's result guarantees that the image of \( \text{Irr}(G) \) under \( \phi \) satisfies the hypotheses of Lemma 2.1 for \( H \). Hence we may write \( \phi(\chi_i) = \varepsilon(\psi_i)\psi_i, \varepsilon(\psi_i) = \psi_i(\chi_i, 1) \) for some element \( u \) in \( Z(H) \), \( i = 1, \ldots, h \) where \( \text{Irr}(G) = \{\chi_1, \ldots, \chi_h\} \), \( \text{Irr}(H) = \{\psi_1, \ldots, \psi_h\} \) and \( i \rightarrow i' \) is a permutation.

Therefore the map \( \tau_\phi \) defined by \( \varepsilon(\chi)\chi\varepsilon(\psi)\psi \) is an \( \text{R}\)-automorphism of \( \text{RX}(H) \) from fact (i) above. If we put \( \theta = \tau_\phi^{-1} \), then \( \theta(\chi_i) = \tau_\phi^{-1}(\phi(\chi_i)) = \psi_i, \varepsilon(\psi_i) = \psi_i(\chi_i, 1) \) and so \( \theta \) maps \( \text{Irr}(G) \) onto \( \text{Irr}(H) \). Hence we have \( \phi = \tau_\phi \theta \), as required.

(iii) Fact (iii) follows since fact (ii) tells us that \( A = TP \) and it is clear from fact (ii) that \( A \) induces a permutation action on \( \text{Irr}(G) \) and \( T \) is the kernel of this action. This completes the proof of the theorem.

Q.E.D.

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References


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