AUTOMORPHISMS OF CERTAIN ROOT LATTICES

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0. Introduction.

Let $\mathcal{A}$ be a reduced irreducible root system of type $X_i$ in a Euclidean space $V$, in the sense of Bourbaki [1]. Then $\mathcal{A}$ generates a lattice $\Gamma$ of rank $l$ in $V$. We fix the lattice $\Gamma$. Let $\mathcal{A}'$ be another reduced irreducible root system in $V$, generating $\Gamma'$, of type $X_i$. We investigated whether $\mathcal{A}'$ coincided with $\mathcal{A}$, and found out that only the case of $C_4$ is exceptional. If $X_i$ is not $C_4$ then $\mathcal{A}'$ is equal to $\mathcal{A}$. This means that $(V, \Gamma, X_i)$ determines $\mathcal{A}$ uniquely unless $X_i$ is $C_4$. In case $X_i$ is $C_4$, there are three root systems, generating $\Gamma$, of type $C_4$ in $V$. As we will explain afterward, these are verified by looking at the list of root systems in Bourbaki [1].

Let $W$ be the Weyl group of $\mathcal{A}$, and $O(\Gamma)$ the orthogonal group of $\Gamma$. Then $W \subseteq O(\Gamma)$. Let $D$ be the subgroup of $O(\Gamma)$ generated by all symmetries of the Dynkin diagram of $\mathcal{A}$. Put $\overline{W} = \langle W, D \rangle$, the subgroup of $O(\Gamma)$ generated by $W$ and $D$. Notice that $-I$ (minus identity) is contained in $\overline{W}$ (cf. [1], [5]). Then the fact in the previous paragraph can be described as follows. The group index $[O(\Gamma): \overline{W}]$ is 3 if $X_i = C_4$; 1 otherwise.

In this paper, we will calculate the index $[O(\Gamma): \overline{W}]$ in the case that $\mathcal{A}$ is the root system of a Kac-Moody Lie algebra of Euclidean type or of low rank hyperbolic type. Let $A$ be a generalized Cartan matrix of Euclidean type or of hyperbolic type, and $B$ the associated form. Let $\mathcal{A}$, $\Gamma$ and $O(\Gamma)$ be the root system of $A$, the root lattice of $\mathcal{A}$ and the orthogonal group of $\Gamma$ associated with $B$, respectively. We denote by $W$ (resp. $D$) the Weyl group (resp. the diagram automorphism group) of $A$. Put $\overline{W} = \langle W, D, -I \rangle$. It is known that the index $Ind(A) = [O(\Gamma): \overline{W}]$ is finite (cf. [1; Chap. 5, §4, Ex. 18], [11]). If $A$ is symmetric, then we get $Ind(A) = 1$ as a direct consequence of [7; Prop. 1.6] and [12; Theorem 2]. We will compute $Ind(A)$ explicitely when $A$ is of Euclidean type, of rank 2 hyperbolic type or of rank 3 hyperbolic type. The most interesting case is when $A = \begin{pmatrix} 2 & -3 & -1 \\ -1 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}$. In this case, we will observe that a certain

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subgroup of $\hat{W}$ acts on the infinite set of all solutions $(s, t, u, v)$ of the following Diophantine equation:
\[
\begin{align*}
& s^2 - 24t^2 = 1 \quad \text{(Pell's equation)} \\
& u^2 - 24v^2 = 1 \quad \text{(Pell's equation)} \\
& su - 24tv = -5
\end{align*}
\]
Furthermore this action is transitive. Using this fact, we can establish $\text{Ind}(A) = 2$.

In the appendix, we display the list of hyperbolic generalized Cartan matrices of rank $\geq 3$, which is already known but seems to be published explicitly nowhere. (cf. [1].)

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1. Finite type.

Let $\mathcal{A}$ denote a reduced irreducible root system in $V$, in the sense of Bourbaki [1]. Let $\Pi$ be a base of $\mathcal{A}$, and $\Gamma$ the root lattice. We denote by $A$ a Cartan matrix of $\mathcal{A}$. Put $\text{Ind}(A) = [O(\Gamma) : \hat{W}]$. Then we can determine $\text{Ind}(A)$ using the list of root systems in [1].

**Theorem 1.** If $A$ is of type $C_4$, then $\text{Ind}(A) = 3$. Otherwise $\text{Ind}(A) = 1$.

**Proof.** To show $\text{Ind}(A) = 1$, we prove that the elements of $\mathcal{A}$ are characterized by their lengths among the elements of $\Gamma$. If $A$ is symmetric (i.e. of type $A_n$, $D_n$ and $E_n$), $\mathcal{A}$ is the set of all the non-zero elements of minimal length in $\Gamma$ (e.g. see [7; Prop. 1.6]). The other cases are similarly proved by direct computation.

To treat the case of type $C_4$ and to show examples, we give the proof in the case of type $F_4$ and $C_4$.

$F_4$: $\mathcal{A}$ is
\[
\{\pm e_i \ (1 \leq i \leq 4), \ \pm e_i \pm e_j \ (1 \leq i < j \leq 4), \ \frac{1}{2} (\pm e_i \pm e_2 \pm e_3 \pm e_4)\}
\]
and $\Pi$ is
\[
\{e_2 - e_3, e_1 - e_4, e_1, \ \frac{1}{2} (e_1 - e_2 - e_3 - e_4)\} \quad \text{in $R^4$,}
\]
where $\{e_i\}$ is a standard orthonormal basis. It is easy to see that all elements of $\Gamma$ of length 1 or 2 are contained in $\mathcal{A}$. Therefore $O(\Gamma)$ coincides with the Weyl group $W$, which implies $\text{Ind}(F_4) = 1$. In particular, the order of $O(\Gamma)$ is $2^7 \cdot 3^4$. 
Automorphisms of certain root lattices\[C_4: \Delta \quad \text{is}\]
\[\{\pm 2e_i \mid 1 \leq i \leq 4\}, \pm e_i \pm e_j \mid 1 \leq i < j \leq 4\}\]
and \[\Pi \quad \text{is}\]
\[\{e_1 - e_2, e_2 - e_3, e_3 - e_4, 2e_4\}\quad \text{in} \quad \mathbb{R}^t.
\]
The dual root system \(\Delta(F_4)^\vee\) of type \(F_4\) is
\[\{\pm 2e_i, \pm e_i \pm e_j, \pm e_i \pm e_j \pm e_k\}.
\]
Therefore the root lattice \(\Gamma\) of \(\Delta\) is equal to that of \(\Delta(F_4)^\vee\). The \(\text{Ind}(C_4) = [O(\Gamma) : \mathbb{W}] = 2^3 \cdot 3^2 \cdot 7 = 360\)
\[
\text{Q.E.D.}
\]

2. Euclidean type and hyperbolic type.

An \(l \times l\) integral matrix \(A = (a_{ij})\) is called a generalized Cartan matrix if \(a_{ii} = 2\) \((1 \leq i \leq l)\), \(a_{ij} \leq 0\) \((1 \leq i < j \leq l)\) and \(a_{ij} = 0\) whenever \(a_{ij} = 0\). Cartan matrices arising from root systems in the sense of Bourbaki [1] are generalized Cartan matrices. Such generalized Cartan matrices are called of finite type. A generalized Cartan matrix \(A\) is called of Euclidean type if \(A\) is singular and possesses the property that removal of any row and the corresponding column leaves a Cartan matrix (i.e. a generalized Cartan matrix of finite type). A generalized Cartan matrix \(A\) is called indecomposable (resp. symmetrizable) if \(A\) cannot be expressed as \(\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\) under any permutations of indices (resp. if there are positive rational numbers \(q_1, \cdots, q_l\) such that \(q_i a_{ij} = q_j a_{ji}\) for any \(i, j = 1, \cdots, l\)). The generalized Cartan matrices of Euclidean type are indecomposable and symmetrizable. Of course, Cartan matrices are symmetrizable. A generalized Cartan matrix \(A\) is called of hyperbolic type if \(A\) is indecomposable, symmetrizable, not of finite type, not of Euclidean type and possesses the property that removal of any row and the corresponding column leaves a union of Cartan matrices and the generalized Cartan matrices of Euclidean type. The generalized Cartan matrices of Euclidean type and the generalized Cartan matrices of hyperbolic type have been classified (cf. Appendix, [1], [2], [6], [10], [13]).

From now on, we suppose that \(A\) is a generalized Cartan matrix of Euclidean type or of hyperbolic type. Then the root system \(\Delta = \Delta(A)\) of the Kac-Moody Lie algebra associated with \(A\) is described as follows. For Kac-Moody Lie algebras, we refer the reader, e.g. [8]. Let \(\Gamma = \bigoplus_{\alpha_i} \mathbb{Z} \alpha_i\) be a free abelian group with free generators \(\alpha_1, \cdots, \alpha_l\). We take an element \(w_i (1 \leq i \leq l) \) of \(GL(\Gamma)\) defined by \(w_i(a_j) = a_j - a_j a \alpha_i\) for all \(j = 1, \cdots, l\). The Weyl group of \(A\) is defined to be the subgroup \(W\) of \(GL(\Gamma)\) generated by \(w_i\) for all \(i = 1, \cdots, l\). Let \(B\) be a sym-
metric bilinear form on \( \Gamma \) satisfying \( B(\alpha_i, \alpha_j) = q_i a_{ij} \). This form \( B \) is \( W \)-invariant. Then the root system \( \mathcal{A} \) is a disjoint union of real roots, \( \mathcal{A}_r = \{ \alpha \in \Gamma | B(\alpha, \alpha) = 0 \} \), and imaginary roots, \( \mathcal{A}_i = \{ a \in \Gamma | B(\alpha, \alpha) > 0 \} \). Let

\[
O(\Gamma) = \{ g \in GL(\Gamma) | B(g\alpha, g\beta) = B(\alpha, \beta) \text{ for all } \alpha, \beta \in \Gamma \},
\]

and let \( D \) be the subgroup, called the diagram automorphism group of \( \mathcal{A} \), of \( O(\Gamma) \) generated by all symmetries which are induced by permutations on \( \{ \alpha_1, \cdots, \alpha_l \} \) preserving the form \( B \). Put \( \tilde{W} = \langle W, D, -I \rangle \subseteq O(\Gamma) \). We are interested in the index of \( \tilde{W} \) in \( O(\Gamma) \); denote it by \( \text{Ind}(\mathcal{A}) \).

\[ \text{Ind}(\mathcal{A}) = 1 \text{ if } \mathcal{A} = \mathcal{A}_1 \text{ or } \mathcal{A}_2 \]

\[ \text{Ind}(\mathcal{A}) = 2 \text{ if } \mathcal{A} = \mathcal{A}_3 \text{ or } \mathcal{A}_4 \]

\[ \text{Ind}(\mathcal{A}) = 3 \text{ if } \mathcal{A} = \mathcal{A}_5 \text{ or } \mathcal{A}_6 \]

\[ \text{Ind}(\mathcal{A}) = 4 \text{ if } \mathcal{A} = \mathcal{A}_7 \text{ or } \mathcal{A}_8 \]

\[ \text{Ind}(\mathcal{A}) = 6 \text{ if } \mathcal{A} = \mathcal{A}_9 \]

\[ \text{Ind}(\mathcal{A}) = 2n-1 \text{ if } \mathcal{A} = \mathcal{A}_{2n} \]

\[ \text{Ind}(\mathcal{A}) = 2^{n-1} \text{ if } \mathcal{A} = \mathcal{A}_{2n+1} \]

**Proof.** We can assume that \( \mathcal{A} = \left( \frac{A_0}{*} \right) \), where \( A_0 \) is of finite type \( X_n \)

(resp. \( B_n, C_n, D_n, F_4, G_2 \)) if \( \mathcal{A} \) is of type \( X_n \) (resp. \( A_n, D_n, E_6^{(2)}, D_4^{(2)}, F_4^{(2)} \)). For the convenience, we assume that \( \alpha_1 \) is a short root associated with \( A_0 \). As is well-known, \( \mathcal{A}_r = \{ \alpha \in \Gamma | B(\alpha, \alpha) = 0 \} = \text{Rad}(B) \) and \( \mathcal{A}_i \) is a free \( \mathbb{Z} \)-module of rank 1. Take a generator \( \xi \) of \( \mathcal{A}_i \), which is called a fundamental null root. Let \( \Gamma_0 = \sum_{i=1}^{l} \mathbb{Z} \alpha_i \), then \( \Gamma = \mathbb{Z} \xi \) (orthogonal sum). Take an element \( \sigma \in O(\Gamma) \). Since \( \sigma(\xi) = \pm \xi \), we can write \( \sigma = \left( \begin{array}{cc} \alpha \alpha \xi \xi \\ \alpha \alpha \end{array} \right) \), where \( \alpha \in O(\Gamma_0) \) and \( O(\Gamma_0) \) is embedded in \( O(\Gamma) \) by \( \sigma \rightarrow \left( \begin{array}{cc} \alpha \alpha \xi \xi \\ \alpha \alpha \end{array} \right) \).

Therefore \( \sigma \equiv \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \) modulo \( O(\Gamma_0) \times \langle -I \rangle \).

Set \( T = \left\{ \begin{array}{c} 1 \\ 0 \\ \vdots \\ s_1 \cdots s_{l-1} \end{array} \right\} \), then \( O(\Gamma) = \langle O(\Gamma_0) \times T \rangle \times \langle -I \rangle \). Let

\[ W_0 \text{ (resp. } W \text{) be the Weyl group of } A_0 \text{ (resp. } A \text{), and let } D_0 \text{ (resp. } D \text{) be the diagram automorphism group of } A_0 \text{ (resp. } A \text{). For each element } \alpha \text{ of } \mathcal{A}_0, \text{ we define an element } w_\alpha \text{ of } O(\Gamma) \text{ by } w_\alpha(x) = x - (2B(\alpha, x)/B(\alpha, \alpha))x \text{ for all } x \in \Gamma. \]

Set \( m_i = \min \{ m > 0 | \alpha_i + m \xi \in \mathcal{A}_\beta \} \) for \( i = 1, \cdots, l-1 \). For each \( i = 1, \cdots, l-1 \), an element \( h_i \) of \( W \) is defined to be \( w_{a_i + \epsilon_i} w_{a_i} \) if \( i = 1 \); \( w_{a_i + m_i \epsilon_i} w_{a_i} \) otherwise. Let \( H \) be the subgroup of \( W \) generated by \( h_1, \cdots, h_{l-1} \). Then
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$W = W_0 \times H$. We note that $\tilde{W} = (W \times D) \times \langle -I_1 \rangle$ and $\hat{W} = W_0 \times D_\alpha$. Hence we have

$[O(\Gamma) : \hat{W}] \leq \prod_{i=1}^{\ell} m_i (\det A_\alpha)^{\kappa}$, where $\kappa = \frac{1}{2}$ if $A = A_\alpha^0$; 1 otherwise. Then $[D : D_\alpha] = (\det A_\alpha)^{\kappa}$ (cf. [9; p. 96]). Therefore $\text{Ind}(A) = \text{Ind}(A_\alpha) \prod_{i=1}^{\ell} m_i$. By Theorem 1 and the structure of $\mathcal{A}$, we can compute the index $\text{Ind}(A)$.

Q. E. D.

Let $A$ be of rank 2 hyperbolic type. That is, $A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$, $ab > 4$. We put $q_1 = \frac{b}{a}$, $q_2 = \frac{a}{2}$, so the associated form $B$ is defined by $B(\alpha_1, \alpha_1) = b$, $B(\alpha_2, \alpha_2) = a$ and $B(\alpha_1, \alpha_2) = -\frac{ab}{2}$. The Weyl group $W$ is generated by $w_1 = \begin{pmatrix} -1 & a \\ 0 & 1 \end{pmatrix}$ and $w_2 = \begin{pmatrix} 1 & 0 \\ b & -1 \end{pmatrix}$. Let $\sigma \in O(\Gamma)$, and choose an element $\beta = n_1 \alpha_1 + n_2 \alpha_2$ of $W \sigma(\alpha_1)$, the $W$-orbit of $\sigma(\alpha_1)$, which satisfies the condition that $n_1^2 + n_2^2$ is minimal in this orbit. Since $w_1(\beta) = (-n_1 + a n_2) \alpha_1 + n_2 \alpha_2$ and $w_2(\beta) = n_1 \alpha_1 + (b n_1 - n_2) \alpha_2$, we have $n_2 a n_2 - 2 n_1 \geq 0$ and $n_1 (b n_1 - 2 n_2) \geq 0$ by the condition of $\beta$. If $n_1 > 0$, $n_2 > 0$ (resp. $n_1 < 0$, $n_2 < 0$), then $\frac{2}{a} n_1 \leq n_2 \leq \frac{b}{2} n_1$ (resp. $\frac{2}{a} n_1 \geq n_2 \geq \frac{b}{2} n_1$), which means $0 \geq B(\beta, \beta) = B(\alpha_1, \alpha_1) = b$, a contradiction. Thus, $n_1 n_2 \leq 0$. On the other hand, $b = b n_1^2 - a b n_2 + a n_2^2$ since $B(\beta, \beta) = B(\alpha_1, \alpha_1) = b$. Then $n_1 n_2 \leq 0$ implies that $(n_1, n_2) = (\pm 1, 0)$, or that $(n_1, n_2) = (0, \pm 1)$ and $a = b$. In the latter, $A$ is symmetric, so we already know $\sigma \in \hat{W}$. Therefore we can assume that there is an element $w \in \hat{W}$ satisfying $w \sigma(\alpha_1) = \alpha_1$. Write $w \sigma(\alpha_2) = k_1 \alpha_1 + k_2 \alpha_2$. Then $B(w \sigma(\alpha_2), w \sigma(\alpha_3)) = b k_1^2 - a b k_1 k_2 + a k_2^2 = -a b$ and $B(w \sigma(\alpha_3), \alpha_1) = b k_1 - \frac{a b}{2} k_2 = -\frac{a b}{2}$. Hence $(k_1, k_2) = (0, 1)$ or $(-a, -1)$. This leads to $w \sigma = \pm 1$ or $w \sigma = (-1) w_1$, and $\sigma \in \hat{W}$. Thus we have the following.

**Theorem 3.** Suppose that $A$ is of rank 2 hyperbolic type. Then $\text{Ind}(A) = 1$.

Next we treat the case that $A$ is of rank 3 hyperbolic type. We use here the classification of the generalized Cartan matrices of this type (cf. Appendix, [1], [2], [13]). Suppose that $A$ is none of

$\begin{pmatrix} 2 & -3 & -1 \\ -1 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -4 & -2 \\ -4 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$ or

$\begin{pmatrix} 2 & -3 & -1 \\ -1 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & -4 & -2 \\ -4 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$. Then $O(\Gamma) II \subseteq Z$, hence in particular $W \sigma(\alpha_i) \subseteq Z$ for all $\sigma \in O(\Gamma)$, $1 \leq i \leq 3$. Therefore $O(\Gamma) II = \Delta_3$ (cf. [7]). By [12; Theorem 2], we have $O(\Gamma) = \hat{W}$ and $\text{Ind}(A) = 1$. We shall consider the remaining three cases.
The case when $A = \begin{pmatrix} 2 & -3 & -1 \\ -1 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}$.

Let $\sigma$ be an endomorphism of $\mathcal{O}$ defined by

$$
\sigma(\alpha_1) = -\alpha_1, \quad \sigma(\alpha_2) = \alpha_1 - \alpha_3, \quad \sigma(\alpha_3) = -\alpha_1 - \alpha_2.
$$

Then $\sigma$ preserves the form $B$ and $\sigma \in O(\mathcal{O})$. Take an element $\sigma \in O(\mathcal{O})$. Since the elements $\alpha \in \mathcal{O}$ satisfying $B(\alpha, \alpha) = B(\alpha_1, \alpha_3)$ and $\alpha \in \mathcal{O}$ are $\pm(\alpha_1 - \alpha_3)$, there is an element $\omega \in \mathcal{O}$ such that $\omega \sigma(\alpha_3) = -\alpha_1 - \alpha_3$. (For $\sigma \in O(\mathcal{O})$, there is an element $\omega' \in \mathcal{O}$ such that $\omega' \sigma(\alpha_3) = -\alpha_1 - \alpha_3$ or $\alpha_2$. The latter induces an element $\omega'(\mathcal{O}) = \mathcal{O}$. But this leads to a contradiction.)

Therefore to consider $\mathcal{O} \setminus O(\mathcal{O})$ we can assume $\sigma(\alpha_3) = \alpha_1 - \alpha_3$. Write $\sigma(\alpha_1) = k_1 \alpha_1 + k_2 \alpha_2 + k_3 \alpha_3$ and $\sigma(\alpha_3) = l_1 \alpha_1 + l_2 \alpha_2 + l_3 \alpha_3$. We put $q_1 = \frac{1}{2}$, $q_2 = \frac{3}{2}$ and $q_3 = \frac{1}{2}$. Then we have:

$$
\begin{align*}
\begin{cases}
\frac{k_3^2 + 3k_1^2 + k_3^2 - 3k_1 k_2 - k_1 k_3 - 3k_3^2}{l_3^2 + 3l_1^2 + l_3^2 - 3l_1 l_2 - l_1 l_3 - 3l_3^2} = 1 \\
2k_1 l_1 + 6k_1 l_3 + 2k_2 l_3 - 3k_1 l_2 - k_2 l_3 - 3k_2 l_2 - 3k_3 l_3 - k_1 l_1 - 3k_3 l_2 = -1 \\
k_1 - k_3 = -1 \\
l_1 - l_3 = -1.
\end{cases}
\end{align*}
$$

(E_1)

Put $s = 2k_3 - 6k_1 + 1$, $t = k_5$, $u = 2l_1 - 6l_2 + 1$ and $v = l_3$. Then the Diophantine equation (E_1) implies the Diophantine equation

$$
\begin{align*}
\begin{cases}
s^2 - 24t^2 = 1 \\
u^2 - 24v^2 = 1 \\
su - 24tv = -5.
\end{cases}
\end{align*}
$$

(E_2)

Notice that $5 + \sqrt{24}$ is the dominant fundamental factor of the Pell’s equations $s^2 - 24t^2 = 1$ and $u^2 - 24v^2 = 1$ (cf. [3; P. 83], [4; P. 110]). Let

$$
S = \{(m, n; \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \mid m, n \in \mathbb{Z}_{\geq 0}, \ |m-n|=1, \ \varepsilon_4 = \pm 1, \ \varepsilon_1 \varepsilon_3 = \varepsilon_2 \varepsilon_4 = -1\}.
$$

Then the set of all solutions of the Diophantine equation (E_2) is parametrized by $S$. That is,

$$
\begin{align*}
s &= \varepsilon_1 (\xi^m + \xi^n)/2, \\
t &= \varepsilon_2 (\xi^m - \xi^n)/2 \sqrt{24}, \\
u &= \varepsilon_3 (\xi^m + \xi^n)/2, \\
v &= \varepsilon_4 (\xi^m - \xi^n)/2 \sqrt{24},
\end{align*}
$$

where $\xi = 5 \pm \sqrt{24}$. Here we will choose three elements of $\mathcal{O}$. Let $\rho_1 = w_3$, $\rho_2 = w_1 w_3 w_1$ and $\rho_3 = (-1) d$, where $d = [\alpha_1 \rightarrow \alpha_3, \alpha_2 \rightarrow \alpha_3, \alpha_3 \rightarrow \alpha_1]$, a non-trivial
diagram automorphism. Then these $\rho_i$'s fix $\alpha_i-\alpha_0$. Thus $\rho_i\sigma$ ($i=1, 2, 3$) gives a new solution of $(E_2)$. Since $\rho_i\sigma(\alpha_i)=k_i\alpha_i+(k_i-k_2+k_3)\alpha_2+k_3\alpha_3$ and $\rho_i\sigma(\alpha_3)=l_i\alpha_i+(l_i-l_2+l_3)\alpha_2+l_3\alpha_3$, we see that $\rho_1$ produces a new solution
\[ s'=-\varepsilon_i(\xi_{n+1}^m+\xi_{n+1}^{-1})/2 \quad \text{(resp. } s'=-\varepsilon_i(\xi_{n+1}^{m-1}+\xi_{n+1}^{-1})/2), \]
\[ t'=\varepsilon_i(\xi_{n+1}^m-\xi_{n+1}^{m-1})/2\sqrt{24} \quad \text{(resp. } t'=\varepsilon_i(\xi_{n+1}^{m-1}-\xi_{n+1}^{-1})/2\sqrt{24}), \]
\[ u'=\varepsilon_i(\xi_{n+1}^{m+1}+\xi_{n+1}^{-1})/2 \quad \text{(resp. } u'=-\varepsilon_i(\xi_{n+1}^{m+1}+\xi_{n+1}^{-1})/2), \]
\[ v'=\varepsilon_i(\xi_{n+1}^{m+1}-\xi_{n+1}^{m-1})/2\sqrt{24} \quad \text{(resp. } v'=-\varepsilon_i(\xi_{n+1}^{m-1}-\xi_{n+1}^{-1})/2\sqrt{24}) \]
of $(E_2)$ from an original solution $(s, t, u, v)$ if $\varepsilon_1\varepsilon_2>0$ (resp. $\varepsilon_1\varepsilon_2<0$). Since
\[ \rho_2\sigma(\alpha_2)=(6k_2-k_3)\alpha_3+k_2\alpha_2+(-k_2+6k_3)\alpha_3 \]
and
\[ \rho_2\sigma(\alpha_3)=(6l_3-l_2)\alpha_3+l_3\alpha_2+(-l_1+6l_2)\alpha_3, \]
the element $\rho_2$ produces a new solution $(-s, t, -u, v)$ from $(s, t, u, v)$. Since
\[ \rho_3\sigma(\alpha_2)=-k_2\alpha_1-k_3\alpha_3-k_3\alpha_3 \]
and
\[ \rho_3\sigma(\alpha_3)=-l_3\alpha_2-l_3\alpha_2-l_3\alpha_3, \]
the element $\rho_3$ produces a new solution $(-s, -t, -u, -v)$ from $(s, t, u, v)$. Hence the subgroup $G$ of $\tilde{W}$ generated by $\rho_i$, $\rho_2$ and $\rho_3$ transitively acts on the set of all solutions of $(E_2)$. This means $O(\Gamma)-\tilde{W}=\tilde{W}\sigma_0$, so $\{1, \sigma_0\}$ is the complete set of representatives of $\tilde{W}\setminus O(\Gamma)$.

(2) The case when $A=\begin{pmatrix} 2 & -4 & -2 \\ -1 & 2 & -1 \\ -2 & 4 & 2 \end{pmatrix}$.

In this case, we can take an element $\sigma_0$ of $O(\Gamma)-\tilde{W}$ defined by $\sigma_0(\alpha_2)=\alpha_2+\alpha_3$, $\sigma_0(\alpha_3)=\alpha_1-\alpha_2$ and $\sigma_0(\alpha_3)=\alpha_3$. For each element $\sigma\in O(\Gamma)-\tilde{W}$ there exists an element $w\in\tilde{W}$ such that $w\sigma(\alpha_2)=\alpha_1-\alpha_3$, since the elements $\alpha\in\Gamma$ satisfying $B(\alpha, \alpha)=B(\alpha_2, \alpha_3)$ and $\alpha\in Z$ are $(\alpha_1-\alpha_3)$. Then the elements $\tau\in O(\Gamma)-\tilde{W}$ with the property $\tau(\alpha_2)=\alpha_1-\alpha_3$ are $\tau_1$, $\tau_2$, $\tau_3$ and $\tau_4$, where
\[ \tau_1=[\alpha_1\mapsto \alpha_3, \alpha_2\mapsto \alpha_1-\alpha_3, \alpha_3\mapsto \alpha_2+\alpha_3], \]
\[ \tau_2=[\alpha_1\mapsto -\alpha_1, \alpha_2\mapsto \alpha_1-\alpha_3, \alpha_3\mapsto -\alpha_1-\alpha_2], \]
\[ \tau_3=[\alpha_1\mapsto \alpha_2+\alpha_3, \alpha_2\mapsto \alpha_1-\alpha_3, \alpha_3\mapsto \alpha_2], \]
\[ \tau_4=[\alpha_1\mapsto -\alpha_1-\alpha_2, \alpha_2\mapsto \alpha_1-\alpha_3, \alpha_3\mapsto \alpha_2]. \]
Put $d=[\alpha_1\mapsto \alpha_3, \alpha_2\mapsto \alpha_2, \alpha_3\mapsto \alpha_1]$, a nontrivial diagram automorphism. Then we have $(1)d\tau_1=\tau_2$, $(1)d\tau_3=\tau_4$, $w_1\tau_1=\tau_3$ and $w_2\tau_2=\tau_4$. Therefore $O(\Gamma)-\tilde{W}=\tilde{W}\sigma_0$,
so \([1, \sigma_3, \tau_0] \) is the complete set of representatives of \(\overline{W} \setminus O(\Gamma)\).

(3) The case when \(A = \begin{pmatrix} 2 & -1 & 0 \\ -4 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix} \).

In this case, the elements \(\alpha \in \Gamma\) satisfying \(B(\alpha, \alpha) = B(\alpha_3, \alpha_3)\) and \(\alpha \in \mathbb{Z}\) are \(\pm (\alpha_3 - \alpha_3)\) and \(\pm (\alpha_1 + \alpha_2 - \alpha_3)\). Let \(\sigma_3\) (resp. \(\tau_3\)) be the endomorphism of \(\Gamma\) defined by \(\sigma_3(\alpha_3) = -\alpha_3\) and \(\tau_3(\alpha_3) = -\alpha_3\) (resp. \(\tau_3(\alpha_3) = \alpha_3\) and \(\tau_3(\alpha_3) = \alpha_3\)). Then they belong to \(O(\Gamma) - \overline{W}\). For each element \(\sigma \in O(\Gamma) - \overline{W}\), there is an element \(w \in \overline{W}\) such that \(w \sigma(\alpha_3) = \alpha_3 - \alpha_3\) or \(\alpha_1 + \alpha_2 - \alpha_3\). Then the elements \(\tau\) of \(O(\Gamma) - \overline{W}\) having the property \(\tau(\alpha_3) = \alpha_3 - \alpha_3\) are \(\sigma_3, \tau_3, \alpha_3\) and \(\sigma_3\), where

\[
\sigma_3 = \left[\begin{array}{cccc} \alpha_1 - \alpha_1 - \alpha_3 & \alpha_2 - \alpha_3 & \alpha_3 - \alpha_3 & \alpha_4 - \alpha_3 \end{array}\right],
\]

and the elements \(\tau\) of \(O(\Gamma) - \overline{W}\) having the property \(\tau(\alpha_3) = \alpha_1 + \alpha_1 - \alpha_3\) are \(\tau_1, \tau_3, \tau_5\) and \(\tau_7\), where

\[
\tau_1 = \left[\begin{array}{cccc} \alpha_1 + \alpha_1 - \alpha_3 & \alpha_2 - \alpha_3 & \alpha_3 - \alpha_3 & \alpha_4 - \alpha_3 \end{array}\right],
\]

\[
\tau_3 = \left[\begin{array}{cccc} \alpha_1 + \alpha_1 - \alpha_3 & \alpha_2 - \alpha_3 & \alpha_3 - \alpha_3 & \alpha_4 - \alpha_3 \end{array}\right],
\]

\[
\tau_5 = \left[\begin{array}{cccc} \alpha_1 + \alpha_1 - \alpha_3 & \alpha_2 - \alpha_3 & \alpha_3 - \alpha_3 & \alpha_4 - \alpha_3 \end{array}\right],
\]

Furthermore \(w_1 w_2 w_3 \sigma_3 = \sigma_3\) and \(w_1 w_2 w_3 \sigma_3 = \sigma_3\) (1 \(\leq i \leq 4\)). Therefore \(O(\Gamma) - \overline{W} = \overline{W} \sigma_3\). On the other hand, \(\sigma_3 \tau_3^{-1}(\alpha_3) = \alpha_3 + \alpha_3 - \alpha_3 \in \Delta_B\), so \(\sigma_3 \tau_3^{-1} \in \mathbb{W}\). This means that \(\{1, \sigma_3, \tau_0\}\) is the complete set of representatives of \(\overline{W} \setminus O(\Gamma)\).

**Theorem 4.** Suppose that \(A\) is of rank 3 hyperbolic type. Then \(\text{Ind}(A) = 2\) if \(A = \begin{pmatrix} 2 & -3 & -1 \\ -1 & 2 & -1 \\ -1 & 3 & 2 \end{pmatrix}\) or \(\begin{pmatrix} 2 & -4 & -2 \\ -1 & 2 & -1 \\ -2 & 4 & 2 \end{pmatrix}\); \(\text{Ind}(A) = 3\) if \(A = \begin{pmatrix} 2 & -1 & 0 \\ -4 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}\); and \(\text{Ind}(A) = 1\) otherwise.
Automorphisms of certain root lattices

Appendix

Hyperbolic generalized Cartan matrices of rank \(\geq 3\)

\[
\begin{array}{c|c|c|c|c|c|c}
   a_{ij} & a_{ji} & i & j & a_{ij} & a_{ji} & i & j \\
   \hline
   0 & 0 & \bullet & \bullet & -1 & -3 & \circ & \circ \\
   -1 & -1 & \circ & \circ & -1 & -4 & \circ & \circ \\
   -1 & -2 & \circ & \circ & -2 & -2 & \circ & \circ \\
\end{array}
\]

Symmetric case.

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NON-SYMMETRIC CASE.
(1) rank 3

(2) rank 4
Automorphisms of certain root lattices

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(3) rank 5

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(4) rank 6

(2)
Automorphisms of certain root lattices

(5) rank 7

(6) rank 8

(7) rank 9

(8) rank 10

Note. The rank of a hyperbolic type generalized Cartan matrix is at most 10 (cf. [1]).

References

Loos, O., Symmetric spaces (vol. 2), Benjamin, New York, 1969.

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