CURVATURE BOUND AND TRAJECTORIES FOR MAGNETIC FIELDS ON A HADAMARD SURFACE

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Introduction.

On a complete oriented Riemannian manifold $M$, a closed 2-form $B$ is called a magnetic field. Let $\Omega$ denote the skew symmetric operator $\Omega : TM \to TM$ defined by $\langle u, \Omega(v) \rangle = B(u, v)$ for every $u, v \in TM$. We call a smooth curve $\gamma$ a trajectory for $B$ if it satisfies the equation $\nabla_\gamma \dot{\gamma} = \Omega(\dot{\gamma})$. Since $\Omega$ is skew symmetric, we find that every trajectory has constant speed and is defined for $-\infty < t < \infty$. We shall call a trajectory normal if it is parametrized by its arc length. When $\gamma$ is a trajectory for $B$, the curve $\sigma$ defined by $\sigma(t) = \gamma(\lambda t)$ with some constant $\lambda$ is a trajectory for $\lambda B$. We call the norm $\|B_x\|$ of the operator $B_x : T_x M \times T_x M \to \mathbb{R}$ the strength of the magnetic field at the point $x$. For the trivial magnetic field $B = 0$, the case without the force of a magnetic field, trajectories are nothing but geodesics. In term of physics it is a trajectory of a charged particle under the action of the magnetic field. For a classical treatment and physical meaning of magnetic fields see [8].

On a Riemann surface $M$ we can write down $B = f \cdot \text{Vol}_M$ with a smooth function $f$ and the volume form $\text{Vol}_M$ on $M$. When $f$ is a constant function, the case of constant strength, the magnetic field is called uniform. On surfaces of constant curvature the feature of trajectories are well-known for every uniform magnetic field $k \cdot \text{Vol}_M$. On a Euclidean plane $\mathbb{R}^2$ they are circles (in usual sense of Euclidean geometry) of radius $1/|k|$. On a sphere $S^2(c)$ they are small circles with prime period $2\pi/\sqrt{k^2 + c}$. In these cases all trajectories are closed. On a hyperbolic plane $H^2(-c)$ of constant curvature $-c$, the situation is different. In his paper [4] Comtet showed that the feature of trajectories changes according to the strength of a uniform magnetic field $k \cdot \text{Vol}_M$. When the strength $|k|$ is greater than $\sqrt{c}$, normal trajectories are still closed, hence bounded, but if $|k| \leq \sqrt{c}$ they are unbounded simple curves, in particular, if $|k| = \sqrt{c}$ they are horocycles. In the preceding paper [2] we studied trajectories for Kähler magnetic fields $k \cdot B_j$. 

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which are scalar multiples of the Kähler form $B_j$, on a manifold of complex space form. On a complex projective plane all trajectories for Kähler magnetic fields are closed. But on a complex hyperbolic space $CH^n(-c)$ of constant holomorphic sectional curvature $-c$, normal trajectories for Kähler magnetic fields have similar properties as of trajectories for uniform magnetic fields on a hyperbolic plane. Their feature depend on the strength of a Kähler magnetic field; trajectories are bounded, horocyclic, or unbounded according to the strength is greater, equal to, or smaller than $\sqrt{c}$. In this context it is quite natural to pose the following problem. Consider a Hadamard manifold, which is a simply connected complete Riemannian manifold of nonpositive curvature $-\beta^2 \leq \text{Riem}_M \leq -\alpha^2$, $\beta \geq \alpha \geq 0$. Are they true that all trajectories are unbounded if the strength is smaller than $\alpha$ and that all trajectories are bounded if the strength is greater than $\beta$? In this note we shall concerned with this problem on a Hadamard surface.

**THEOREM 1.** Let $B = f \cdot \text{Vol}_M$ be a magnetic field with $|f| \leq \alpha$ on a Hadamard surface $M$ of curvature $\text{Riem}_M \leq -\alpha^2$. Then every normal trajectory for $B$ is unbounded for both directions.

For Hadamard manifolds we have an important notion of ideal boundary. We denote by $\overline{M} = M \cup M(\infty)$ the compactification of a Hadamard surface $M$ with its ideal boundary $M(\infty)$. For a two-sides unbounded curve $\gamma$ on $M$, if $\lim_{t \to \infty} \gamma(t)$ and $\lim_{t \to -\infty} \gamma(t)$ exist in $\overline{M}$ we denote these points by $\gamma(\infty)$ and $\gamma(-\infty)$ respectively, and call that $\gamma$ has points of infinity. If we review the Comtet's result from this point of view, it assures the following. On $H^2(-c)$ every trajectory $\gamma$ for a uniform magnetic field $k \cdot \text{Vol}_{H^2(-c)}$ with $|k| \leq \sqrt{c}$ has points of infinity $\gamma(\infty), \gamma(-\infty)$. When $|k| = \pm \sqrt{c}$ they coincide $\gamma(\infty) = \gamma(-\infty)$, and they are distinct when $|k| < \sqrt{c}$. We show that a similar property holds for general Hadamard surfaces.

**THEOREM 2.** Let $B = f \cdot \text{Vol}_M$ be a magnetic field with $|f| \leq \alpha$ on a Hadamard surface $M$ of curvature $\text{Riem}_M \leq -\alpha^2 \leq 0$. Suppose either $f \leq 0$ or $f \geq 0$ except on a compact subset of $M$. We then have the following.

1. Every normal trajectory for $B$ has points of infinity.
2. If $|f| < \alpha$ except on a compact subset of $M$, every normal trajectory has two distinct points at infinity.
Curvature Bound and Trajectories

§1. A note on \( \gamma \)-Jacobi fields.

We shall show our theorems by applying the Rauch's comparison theorem. Let \( B = f \cdot \text{Vol}_M \) be a magnetic field on a oriented surface \( M \). We denote by \( \Omega_0 \) the skew symmetric operator associated with the uniform magnetic field \( \text{Vol}_M \).

Clearly the skew symmetric operator associated with \( B \) is of the form \( \Omega = f \cdot \Omega_0 \). For a normal trajectory \( \gamma \) for \( B \), we denote by \( V_t(s) \) the \( \gamma \)-Jacobi field along the geodesic \( s \to \sigma(t,s) = \exp_{\gamma(t)} s\Omega_0(\dot{\gamma}) \) with \( V_t(0) = \dot{\gamma}(t) \). This Jacobi field \( V_t \) is perpendicular to \( \sigma(t,\cdot) \) and is obtained by the variation \( \{\sigma(t+\varepsilon,\cdot)\} \) of geodesics;

\[
V_t(s) = \frac{\partial}{\partial t} \sigma(t,s).
\]

For the sake of reader's convenience, we recall the explicit formula for normal trajectories and \( \gamma \)-Jacobi fields for uniform magnetic fields on surfaces of constant curvature.

**Example 1.** On a Euclidean plane \( \mathbb{R}^2 \), trajectories for the uniform magnetic fields of strength \( k \) satisfy the following equation:

\[
\gamma(t) = \left( \frac{1}{k} \cos(kt-\theta), \frac{1}{k} \sin(kt-\theta) \right) + (\xi_1, \xi_2).
\]

The variation of geodesics is given by

\[
\sigma(t,s) = \left( \frac{1}{k} (1-ks) \cos(kt-\theta), \frac{1}{k} (1-ks) \sin(kt-\theta) \right) + (\xi_1, \xi_2)
\]

and the \( \gamma \)-Jacobi field is

\[
V_t(s) = (1-ks)\dot{\gamma}(t),
\]

hence it vanishes at \( s_0 = 1/k \). The point \( \sigma(t,1/k) = (\xi_1, \xi_2) \) is usually called the center of \( \gamma \).

**Example 2.** On a sphere \( S^2(c) = \{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 = 1 \} \) of sectional curvature \( c \), the trajectory \( \gamma \) for the uniform magnetic field of strength \( k \) satisfies the following equation when \( \gamma(0) = x \in S^2(c), \dot{\gamma}(0) = u \in U_x S^2(c) = \{ \xi \in \mathbb{R}^3 | \langle x, \xi \rangle = 0, \langle \xi, \xi \rangle = c \} : \)

\[
\gamma(t) = \frac{1}{k^2 + c} (k^2 + c \cdot \cos \sqrt{k^2 + ct}) \cdot x
\]

\[
+ \frac{1}{\sqrt{k^2 + c}} \sin \sqrt{k^2 + ct} \cdot u + \frac{k}{k^2 + c} (1 - \cos \sqrt{k^2 + ct}) \cdot \Omega_0(u).
\]
Since the variation of geodesics is given by
\[ \sigma(t, s) = \gamma(t) \cos \sqrt{cs} + \Omega_\gamma(\dot{\gamma}(t)) \cdot \frac{1}{\sqrt{c}} \sin \sqrt{cs} \]
hence
\[ V_\gamma(s) = \dot{\gamma}(t)(\cos \sqrt{cs} - \frac{k}{\sqrt{c}} \sin \sqrt{cs}) \cdot \frac{1}{\sqrt{c}} \sinh \sqrt{cs} \]
Therefore it vanishes at \( s_0 = \frac{1}{\sqrt{c}} \tan^{-1} \sqrt{c} / k \). The point \( \sigma(t, s_0) \) and the trajectory \( \gamma \) can be regarded as a pole and a latitude line of this sphere.

**Example 3.** On the hyperbolic plane \( H^2(-c) = \{ x = (x_0, x_1, x_2) \in \mathbb{R}^3 \mid \langle (x, x) \rangle = -x_0^2 + x_1^2 + x_2^2 = -1, x_0 \geq 1 \} \) of constant sectional curvature \(-c\), the trajectory of the uniform magnetic field of strength \( k \) satisfies the following equation if \( \gamma(0) = x \) and \( \dot{\gamma}(0) = u \in U_x H^2(-c) = \{ \xi \in \mathbb{R}^3 \mid \langle (x, \xi) \rangle = 0, \langle (\xi, \xi) \rangle = -c \} \):

\[
\gamma(t) = \frac{1}{c - k^2} (-k^2 + c \cdot \cosh \sqrt{c - k^2} t) \cdot x + \frac{1}{\sqrt{c - k^2}} \sinh \sqrt{c - k^2} t \cdot u \\
+ \frac{k}{c - k^2} (-1 + \cosh \sqrt{c - k^2} t) \cdot \Omega_0(u), \quad \text{when } 0 \leq k < \sqrt{c},
\]

\[
\gamma(t) = (1 + \frac{ct^2}{2}) x + tu + \sqrt{ct^2} \cdot \Omega_0(u), \quad \text{when } k = \sqrt{c},
\]

\[
\gamma(t) = \frac{1}{k^2 - c} (k^2 - c \cdot \cos \sqrt{k^2 - ct}) \cdot x + \frac{1}{\sqrt{k^2 - c}} \sin \sqrt{k^2 - ct} \cdot u \\
+ \frac{k}{k^2 - c} (1 - \cos \sqrt{k^2 - ct}) \cdot \Omega_0(u), \quad \text{when } k > \sqrt{c}.
\]

The variation of geodesics is given by
\[
\sigma(t, s) = \gamma(t) \cosh \sqrt{cs} + \Omega_\gamma(\dot{\gamma}(t)) \cdot \frac{1}{\sqrt{c}} \sinh \sqrt{cs}
\]
hence
\[ V_\gamma(s) = \dot{\gamma}(t)(\cosh \sqrt{cs} - \frac{k}{\sqrt{c}} \sin \sqrt{cs}) \cdot \frac{1}{\sqrt{c}} \sinh \sqrt{cs} \]
Therefore if \( |k| > \sqrt{c} \) the \( \gamma \)-Jacobi field vanishes at \( s_0 = \frac{1}{\sqrt{c}} \tan^{-1} \sqrt{c} / k = \frac{1}{2\sqrt{c}} \log \frac{k + \sqrt{c}}{k - \sqrt{c}} \). If \( |k| \leq \sqrt{c} \) it does not vanish. When \( k = \sqrt{c} \), the case that \( \gamma \) is a horocycle, the point \( \gamma(\infty) = \gamma(-\infty) \) on the ideal boundary can be regarded as the vanishing point of the \( \gamma \)-Jacobi field; \( \lim_{s \to \infty} V_\gamma(s) = 0 \).
§2. Proofs.

We are now in the position to prove theorems. Let $\gamma$ be a trajectory for the magnetic field $f \cdot \text{Vol}_{M}$ with $|f| \leq \alpha$ on a Hadamard surface $M$ of curvature $\text{Riem}_{M} \leq -\alpha^{2}$. We compare the norm of the $\gamma$-Jacobi field $V_{t}$ with the norm of $\gamma$-Jacobi fields for uniform magnetic fields on a hyperbolic space. Since we have

$$\nabla_{s} V_{t}(0) = \frac{\partial}{\partial s} \frac{\partial}{\partial t} \sigma(t, s) \bigg|_{s=0} = \frac{\partial}{\partial t} \Omega_{0}(\gamma(t)) = -f(\gamma(t))\dot{\gamma}(t),$$

we get the following estimate by the Rauch's comparison theorem;

$$\| V_{t}(s) \| \geq \cosh \alpha s - \frac{1}{\alpha} f(\gamma(t)) \sinh \alpha s .$$

This guarantees that if $|f(\gamma(t))| \leq \alpha$ then $V_{t}$ does not vanish anywhere and

$$\liminf_{s \to \pm \infty} \exp(-\alpha s) \| V_{t}(s) \| \geq \frac{1}{2} (1 - |f(\gamma(t))|/\alpha)$$

for every $t$. Since $M$ is diffeomorphic to an Euclidean plane, we find that the geodesic $\sigma(t_{1}, \cdot)$ and $\sigma(t_{2}, \cdot)$ do not intersect each other if $t_{1} \neq t_{2}$.

Let $S_{r}(p)$ denote the geodesic circle $\{ x \in M \, | \, d(x, p) = r \}$ of radius $r$ centered at $p$. If we suppose $\gamma \big|_{[0, \infty)}$ is tangent to a geodesic circle $S_{r}(\gamma(0))$ at $\gamma(t_{0})$, then $\sigma(t_{0}, \cdot)$ passes $\gamma(0)$, which is a contradiction. We therefore have

PROPOSITION. The trajectory rays $\gamma \big|_{[0, \infty)}$ and $\gamma \big|_{(-\infty, 0]}$ cross only once to every geodesic circle $S_{r}(\gamma(0))$.

This proposition leads us to Theorem 1. In order to see Theorem 2, we denote by $u_{i}$ for $i \neq 0$ the unit tangent vector at $p = \gamma(0)$ such that the geodesic emanating from $p$ with the initial speed $u_{i}$ joins $p$ and $\gamma(t)$. We set $u_{0} = \dot{\gamma}(0)$. Since $\gamma$ is unbounded in both directions, we may treat the case that $f$ is nonpositive (or nonnegative) on $M$. We then find the smooth curve $(u_{i})_{i \in [0, \infty)}$ on $U_{p}M = S^{1}$ rotates counterclockwisely if $f \geq 0$ and rotates clockwise if $f \leq 0$. If we suppose $u_{0} = \pm \Omega_{0}(u_{0})$ for some $t_{0}$, then $\sigma(0, \cdot)$ passes $\gamma(t_{0})$. Hence we find that $\{ u_{i} \}_{i} \subset U_{p}M \setminus \{ \pm \Omega_{0}(u_{0}) \}$ and the limit $u_{\infty} = \lim_{i \to \infty} u_{i}$ exists. Similarly, we find that the limit $u_{-\infty} = \lim_{i \to -\infty} u_{i}$ exists. We therefore get that $\gamma$ has points at infinity;

$$\gamma(\infty) = \rho_{u_{\infty}}(\infty) \quad \text{and} \quad \gamma(-\infty) = \rho_{u_{-\infty}}(-\infty),$$

where $\rho_{v}$ denote the geodesic with $\dot{\rho}(0) = v$. Now we suppose that $\gamma$ has a single point at infinity: $\gamma(\infty) = \gamma(-\infty)$. This means $u_{\infty} = u_{-\infty}$, hence $\gamma(\infty) = \sigma(t, \infty)$ for every $t$. This can not occur when $f < \alpha$. We get the conclusion of Theorem 2.

In view of our proof, we can conclude the following.
REMARK. Consider a magnetic field $B = f \cdot \text{Vol}_M |f| \leq \alpha$, on a Hadamard surface $M$ of curvature $\text{Riem}_M \leq -\alpha^2 < 0$.

(1) A trajectory $\gamma$ for $B$ has a single point at infinity $\gamma(\infty) = \gamma(-\infty)$ if and only if all the geodesic $\sigma(t, \cdot)$ converges to that point $\sigma(t, \infty) = \gamma(\infty)$.

(2) If a trajectory $\gamma$ has a single point at infinity, then the magnetic angle at that point is $\pi/2$. Here the magnetic angle means the angle between the outer tangent vector of $\gamma$ and the outer tangent vector of geodesics $\rho$ with $\rho(\infty) = \gamma(\infty)$ (c.f. [2]).

REMARK. Let $B = k \cdot \text{Vol}_M, |k| < \alpha$ be a uniform magnetic field on a Hadamard surface $M$ of bounded negative curvature $-\beta^2 \leq \text{Riem}_M \leq -\alpha^2 < 0$. We have a positive $\varepsilon$ such that the angle $\angle (\dot{\gamma}(0), \dot{\rho}(0))$ between a trajectory $\gamma$ for $B$ and a geodesic $\rho$ with $\gamma(0) = \rho(0)$ and $\gamma(\infty) = \rho(\infty)$ is always not greater than $\pi - \varepsilon$.

References


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