TWO MOORE SPACES ON WHICH EVERY CONTINUOUS REAL-VALUED FUNCTION IS CONSTANT

By

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Abstract We construct two Moore spaces on which every continuous real-valued function is constant. The first is Moore, screenable and the second, Moore separable. As corollaries we obtain two more Moore spaces on which every continuous real-valued function is constant (a Moore separable and a Moore, screenable) and having a dispersion point.

Key words: Moore, metacompact, screenable, separable, dispersion point.

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§ 1. Introduction.

Moore spaces on which every continuous real-valued function is constant are given in [1], [2], [7], [8]. The space by J.N. Younglove [8] is, in addition locally connected, complete and separable and the space in [2], by H. Brandenburg and A. Mysior, metacompact.

We construct two Moores spaces on which every continuous real-valued function is constant. The first is Moore, screenable (hence metacompact, since every developable screenable space is metacompact [4]) and the second, Moore separable. As corollaries we obtain two more Moore spaces on which every continuous real-valued function is constant (a Moore separable and a Moore, screenable) and having a dispersion point.

In order to construct these spaces, we first consider two auxiliary spaces (a Moore, screenable for the first space and a Moore separable for the second) containing two points not separated by a continuous real-valued function. Then we construct an appropriate Moore space (which is screenable in the first case or separable in the second) on which, with the help of a sequence of functions, we define a decomposition. Finally, on the quotient set we define a topology and we prove that this, in each case, is the required space.

A space \( X \) is called (1) developable, if it has a development, i.e. a sequence \( F_1, F_2, \ldots, F_n, \ldots \) of open coverings such that if \( K \) is a closed subset of \( X \) and \( x \not\in K \), then there exists a covering \( F_n \) such that \( St(x, F_n) \cap K = \emptyset \), where \( St(x, F_n) \) is the union of all sets in \( F_n \) containing \( x \) (2) metacompact, if every open covering of \( X \) has a point-finite open refinement and (3) screenable, if for every open covering \( F \) of \( X \) there exists a sequence \( F_1, F_2, \ldots, F_n, \ldots \) of collections of pairwise disjoint open sets such that \( \bigcup_{n=1}^{\infty} F_n \) covers \( X \) and refines \( F \).

A regular developable space is called a Moore space.

A point \( p \) of a connected space \( X \) is called a dispersion point if the space \( X \setminus \{ p \} \) is totally disconnected.

\[ \text{§ 2. The space } X. \]

The following space \( K \) is a slight modification of the Heath's space \([4]\). The idea of “splitting” the neighbourhoods is due to A. Mysior.

We consider the set

\[ K = \left[ (-1, \infty) \times [0, 1] \setminus \{(x, y) : -1 < x < 0, |x| > y \} \right] \cup \{p\}. \]

Let \( L_1 \) (resp. \( M_1 \)) be the set of rationals (resp. irrationals) of the intervals \([n, n+1)\), \( n=0, 2, 4, \ldots \), and \( L_2 \) (resp. \( M_2 \)) be the set of rationals (resp. irrationals) of the intervals \([n, n+1)\), \( n=1, 3, 5, \ldots \).

On the set \( K \) we define the following topology: Every point \((x, y) \in K \setminus \{p\}, y > 0\), is isolated.

For every \((q, 0) \in L_1 \) (resp. \((s, 0) \in M_1\)) a basis of open neighbourhoods are the sets

\[ U_n(q, 0) = \{(q, 0)\} \cup \{(q-y, y) : 0 < y < \frac{1}{n}\} \]

\[ \cup \{(q+1-y, y) : 0 < y < \frac{1}{n}\}. \]

\[ \left( \text{resp. } U_n(s, 0) = \{(s, 0)\} \cup \{(s+y, y) : 0 < y < \frac{1}{n}\} \right) \]

\[ \cup \{(s+1+y, y) : 0 < y < \frac{1}{n}\}, \]

\( n=1, 2, \ldots \).

For every \((r, 0) \in L_2 \) (resp. \((t, 0) \in M_2\)) a basis of open neighbourhoods are the sets
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\[ U_n(r, 0) = \{(r, 0)\} \cup \{(r+y, y) : 0 < y < \frac{1}{n}\} \]

\[ \cup \{(r+1+y, y) : 0 < y < \frac{1}{n}\}, \]

(resp. \( U_n(t, 0) = \{(t, 0)\} \cup \{(t-y, y) : 0 < y < \frac{1}{n}\} \]

\[ \cup \{(t+1-y, y) : 0 < y < \frac{1}{n}\}, \]

\( n=1, 2, \ldots \).

For the point \( p \) a basis of open neighbourhoods are the sets

\[ U_n(p) = \{p\} \cup \{(x, y) : x > n\}, \quad n=1, 2, \ldots \]

It can be easily proved that \( K \) is Moore, screenable not completely regular.

Let \( K^+, K^- \) be two disjoint copies of \( K \) and let \( [0, 1)^+, [0, 1)^- \) be the copies of the interval \( [0, 1] \) in \( K^+, K^- \), respectively. We attach \( K^+ \) to \( K^- \) identifying each point of \( [0, 1)^+ \) with its corresponding point of \( [0, 1)^- \). We set \( [0, 1)^+ = [0, 1)^- = [0, 1] \) and we consider the space

\[ X = (K^+[0, 1)^+) \cup [0, 1) \cup (K\setminus[0, 1)^-). \]

It is easy to prove that \( X \) is regular, first countable, containing two points \( a, b \) (the copies of \( p \) in \( K^+, K^- \), respectively) not separated by a continuous real-valued function of \( X \).

Let \( x \in X \) and \( U_n(x), n=1, 2, \ldots \), be a countable local basis of \( x \). It is obvious that the collection \( F_n = \{U_n(x) : x \in X\}, n=1, 2, \ldots \), is a development for \( X \) and hence \( X \) is a Moore space.

Let \( L_1^+, L_1^- \) (resp. \( M_1^+, M_1^- \)) and \( L_2^+, L_2^- \) (resp. \( M_2^+, M_2^- \)) be the copies of \( L_1 \) (resp. \( M_1 \)) and \( L_2 \) (resp. \( M_2 \)) in \( K^+, K^- \), respectively.

We set

\[ P = (L_1^\setminus\{0^+\}) \cup \{0\} \cup (L_1^-\{0^-\}) \]

\[ R = M_1^\uparrow \cup M_1^- \]

\[ Q = L_2^\uparrow \cup L_2^- \]

\[ T = M_2^\uparrow \cup M_2^- \]

and we observe that \( P, R, Q, T \) are pairwise disjoint sets and that if \( p, p' \) (resp. \( r, r', q, q' \) and \( t, t' \)) are distinct points of \( F \) (resp. of \( R, Q \) and \( T \)) then for every \( n, m, U_n(p) \cap U_m(p') = \emptyset \) (resp. \( U_n(r) \cap U_m(r') = \emptyset \), \( U_n(q) \cap U_m(q') = \emptyset \) and \( U_n(t) \cap U_m(t') = \emptyset \)). Based on this, it is easy to prove that \( X \) is screenable.
§ 3. The space \((Z, \tau)\).

The set of isolated points of \(X\) has cardinality \(c\). Let \(I\) be an index set having the same cardinality and let \(X^{(i)}, i \in I\) be disjoint copies of \(X\) and \(a^{(i)}, b^{(i)} \in X^{(i)}\) be points corresponding to \(a, b \in X\), respectively. Let \(Y\) be the disjoint union (i.e. topological sum) of \(X^{(i)}, i \in I\) and let \(D\) be the dense subset of isolated points of \(Y\). Obviously, \(|D| = c\).

Set \(A = \{a^{(i)} : i \in I\}\) and on the quotient set \(Z = Y / A\) we define a topology \(\tau\) as follows: For every point \(x^{(i)} \in X^{(i)}, x^{(i)} \neq a^{(i)}\), a basis of open neighbourhoods is \(B(x^{(i)}), B(x)\) is the basis of \(x\) in \(X\). For the point \(A \in Z\) a basis of open neighbourhoods are the sets

\[O_n(A) = \{A\} \cup \bigcup V_n(a^{(i)}), \quad n = 1, 2, \ldots\]

where \(V_n(a^{(i)})\) is the copy of \(U_n(a) \setminus \{a\}\) in \(X^{(i)}\).

Observe that this topology is regular, first countable, strictly weaker than the quotient topology on \(Z\) and that the subspace \((X^{(i)} \setminus \{a^{(i)}\}) \cup \{A\}\) is homeomorphic to \(X^{(i)}\), for every \(i \in I\).

Obviously \((Z, \tau)\) is Moore screenable.

§ 4. The space \((S_\infty L, \tau^*)\).

We consider a copy \(Z_0\) of \(Z\) and let \(A_0, B_0\) be the copies of the point \(A\) and of the set \(B = \{b^{(i)} : i \in I\}\), in \(Z_0\), respectively.

Let \(Y_k, k = 1, 2, \ldots\), be disjoint copies of \(Y\) and let \(A_k, B_k\) be the copies of \(A, B\) in \(Y_k\), respectively.

We attach the space \(Y_1\) to \(Z_0\) replacing each point \(b_0^{(i)}\) of \(B_0\) by its corresponding point \(a_1^{(i)}\) of \(A_1\).

We set \(S_1 = (Z_0 \setminus B_0) \cup Y_1\).

By induction (replacing each point \(b_k^{(i)}\) of \(B_{k-1}\) by its corresponding point \(a_k^{(i)}\) of \(A_k\)) we construct the space \(S_k = (S_{k-1} \setminus B_{k-1}) \cup Y_k, k = 2, 3, \ldots\).

Finally, we consider the space

\[S_\infty = \bigcup_{k=1}^{\infty} S_k.\]

It can be easily proved that \(S_\infty\) is Moore, screenable and that every continuous real-valued function of \(S_\infty\) is constant on \(\{A, a_k^{(i)} : k = 1, 2, \ldots, i \in I\}\).

Observe that the basis of open neighbourhoods of each point \(a_k^{(i)} \in A_k, k = 1, 2, \ldots\), has the form

\[O_n(a_k^{(i)}) = V_n(a_k^{(i)}) \cup U_n(a_k^{(i)}), \quad n = 1, 2, \ldots,\]
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where $V_n(a^k)$ is the deleted neighbourhood of $b^k$ in $S_{k-1}$ and $U_n(a^k)$ is the neighbourhood of $a^k$ in $Y_k$.

Let $D_0, D_1, D_2, \ldots, D_k, \ldots$, be the sets of isolated points of $Z_0, Y_1, Y_2, \ldots, Y_k, \ldots$, respectively.

Since the sets $A_k, D_k, k=2, 3, \ldots$ have the same cardinality there exists an one-to-one function $f_k$ of $A_k$ onto $D_k$.

Let $L$ be the decomposition of $S_m$ consisting of the points $A_k, a^i, i \in I$, the pairs $(a^i, f_k(a^i))$, $k=2, 3, \ldots$, and the points of the sets

\[ P_k = \{ p^k : p \in P, k=0, 1, 2, \ldots, i \in I \} \]
\[ R_k = \{ r^k : r \in R, k=0, 1, 2, \ldots, i \in I \} \]
\[ Q_k = \{ q^k : q \in Q, k=0, 1, 2, \ldots, i \in I \} \]
\[ T_k = \{ t^k : t \in T, k=0, 1, 2, \ldots, i \in I \} \]

where again $P_0, R_0, Q_0, T_0$ are the corresponding copies for $k=0$, in $Z_0$.

On the quotient set $S_m/L$ we define a topology $\tau_*$ as follows:

If $s \in S_m/L$ and $s=(a^k, f_k(a^k))$ we set
\[ E_0^m(s) = \{ f_k(a^k) \} \cup V_n(a^k) \cup U_n(a^k) \]
and we consider the set
\[ E^m_1(s) = E_0^m(s) \cup M^{k+1}_m(s) \cup N^{k+2}_m(s), \]
where,
\[ M^{k+1}_m(s) = \bigcup \{ O_n(a^m_{k+1}) : f_{k+1}(a^m_{k+1}) \in V_n(a^k) \} \]
and
\[ N^{k+2}_m(s) = \bigcup \{ O_n(a^m_{k+2}) : f_{k+2}(a^m_{k+2}) \in U_n(a^k) \}. \]

By induction, we consider the set
\[ E^{m+1}_n(s) = E^m_n(s) \cup \bigcup \{ O_n(a^m_{k+1}) : f_{k+1}(a^m_{k+1}) \in M^{k+1}_m \} \]
\[ \quad \cup \bigcup \{ O_n(a^m_{k+2}) : f_{k+2}(a^m_{k+2}) \in N^{k+2}_m \}, \]
and we set $E_n(s) = \bigcup_{m=0}^\infty E^m_n(s)$.

A basis of open neighbourhoods for the point $s=(a^k, f(a^k))$ are the sets $E_n(s)$, $n=1, 2, \ldots$.

Similarly, we define the open bases $E_n(s)$, $n=1, 2, \ldots$, if $s=A_0$, whence we set $E_0^0(A_0) = \{ A_0 \} \cup V_n(a^d)$ or, if $s=a^i$, $i \in I$, whence we set $E_0^0(a^i) = V_n(a^i) \cup U_n(a^i)$, or if $s \in P_k \cup R_k \cup Q_k \cup T_k$, $k=0, 1, 2, \ldots$, whence we set $E_0^0(s) = U_n(s)$, where $U_n(s)$, $n=1, 2, \ldots$, is the basis of $s$ in $S_m$.

It can be easily proved that the space $(S_m/L, \tau_*)$ is regular, first countable
and that the topology $\tau^*$ is strictly weaker than the quotient topology on $S_\omega/L$.

**Proposition.** The space $(S_\omega/L, \tau^*)$ is Moore, screenable, on which every continuous real-valued function is constant.

**Proof.** Since the collection $F_n=\{E_n(s): s\in S_\omega/L\}$, $n=1, 2, \ldots$, is a development, it follows that $S_\omega/L$ is a Moore space.

To prove that $S_\omega/L$ is screenable observe that for every $k=0, 1, 2, \ldots$, the sets $P_k, Q_k$ and $T_k$ are pairwise disjoint and that if $p_k^{(i)}$, $q_k^{(i)}$, $q_k^{(i)}$ and $t_k^{(i)}$, $t_k^{(i)}$ are distinct points of $P_k$ (resp. $R_k$, $Q_k$ and $T_k$) then for every $n, m$, $E_n(p_k^{(i)})\cap E_m(p_k^{(i)})=\emptyset$ (resp. $E_n(r_k^{(i)})\cap E_m(r_k^{(i)})=\emptyset$, $E_n(q_k^{(i)})\cap E_m(q_k^{(i)})=\emptyset$ and $E_n(t_k^{(i)})\cap E_m(t_k^{(i)})=\emptyset$). Based on this it is easy to prove that $S_\omega/L$ is screenable.

Finally, since every continuous real-valued function of $S_\omega/L$ is constant on the dense subset $\{(a_k^{(i)}, f_k(a_k^{(i)})): k=1, 2, \ldots, i\in I\}$, it follows that every continuous real-valued function of $S_\omega/L$ is constant.

**Remark 1.** Based on the above we can easily construct a Moore separable space on which every continuous real-valued function is constant (see, also, [8]): Let $K$ be the set

$$\{(x, y): x, y\in Q, x, y>0\} \cup \{(r, 0): r\geq 0, r\in R\} \cup \{p\}$$

($Q, R$ denote the rationals and the reals, respectively). On $K$ we define the following topology: Every point $(x, y)$, $x, y\in Q$, $y>0$ is isolated. For every point $(r, 0)$, $r\geq 0$ a basis of open neighbourhoods are the sets

$$U_n(r, 0) = \{(r, 0)\} \cup \{(t, s)\in K: t>r, (t-r)^2+\left(s-\frac{1}{n}\right)^2<\frac{1}{n^2}\}$$

$$\cup \{(t, s)\in K: t<r+1, (t-r-1)^2+\left(s-\frac{1}{n}\right)^2<\frac{1}{n^2}\}$$

$n=1, 2, \ldots$. For the point $p$, a basis of open neighbourhoods are the sets

$$U_n(p) = \{p\} \cup \{(t, s)\in K: t>n\}, \quad n=1, 2, \ldots .$$

The space $K$ (which is called splitted Niemytzki's space) is Moore, separable not completely regular and it is due to A. Mysior.

Then the corresponding space $X$ (see §2) is Moore separable (since its subset of isolated points is countable and dense) containing two points $a, b$ (the copies of $p$ in $K^*, K^-$, respectively) not separated by a continuous real-valued function of $X$. Hence, if $X^{(n)}$, $n=1, 2, \ldots$, are disjoint copies of $X$, then the corresponding spaces $Y, Z, S_\omega$ (see §3 and 4) are Moore separable and there-
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fore \( S_\infty / L \) is Moore separable on which every continuous real-valued function is constant.

**COROLLARY 1.** There exists a Moore separable space on which every continuous real-valued function is constant and having a dispersion point.

**Proof.** Let \( Z \) be the Moore separable space corresponding to the space \( X \) of Remark 1. Let \( f \) be a one-to-one function of \( B=\{b^{(k)}: k=1, 2, \ldots \} \) onto the countable dense subset \( D \) of isolated points of \( Z \). If \( L \) is the decomposition of \( Z \) consisting of the points of \( Z\setminus \text{BU}D \) and the pairs \( \langle b^{(k)}, f(b^{(k)}) \rangle \), \( k=1, 2, \ldots \), and if on the set \( Z/L \) we define a topology \( \tau^* \) in the same manner as on the set \( S_\infty / L \), then the space \( \langle Z/L, \tau^* \rangle \) is, obviously, Moore separable on which every continuous real-valued function is constant (hence, is connected) and having the point \( A \) as a dispersion point, (since \( X \) is totally disconnected; see the remark in [3]).

**COROLLARY 2.** There exists a Moore screenable space on which every continuous real-valued function is constant and having a dispersion point.

**Proof.** Let \( Z_k, k=1, 2, \ldots \), be disjoint copies of the space \( Z \) of \( \S \) 3 and let \( A_k \) be the copy of the point \( A \) in \( Z_k \). Let \( Y_\infty \) be the disjoint union of \( Z_k \). We set \( A_\infty = \{ A_k : k=1, 2, \ldots \} \) and on the quotient set \( Z_\infty = Y_\infty / A_\infty \) we define a topology as on the set \( Z=Y/A \) of \( \S \) 3. Let \( D_k, k=1, 2, \ldots \), be the (dense) subset of isolated points of \( Z_k \). We set \( B_k = \{ b^{(i)} : i \in I \} \) and we consider a sequence of one-to-one functions \( f_k, k=1, 2, \ldots \), from \( B_{k+1} \) onto \( D_k \). We set \( B_\infty = \bigcup_{k=1}^\infty B_k \), \( D_\infty = \bigcup_{k=1}^\infty D_k \) and let \( L \) be the decomposition of \( Z_\infty \) consisting of the points of \( Z_\infty \setminus B_\infty \cup D_\infty \) and the pairs \( \langle b^{(i)}, f_k(b^{(i)}) \rangle \), \( k=2, 3, \ldots \), \( i \in I \).

Then, defining on the quotient set \( Z_\infty / L \) a topology \( \tau^* \) as on the set \( S_\infty / L \) (in \( \S \) 4), it can be proved, in a similar manner as for the space \( S_\infty / L \), that \( Z_\infty / L \) is Moore, screenable on which every continuous real-valued function is constant. That \( A_\infty \) is a dispersion point, is proved as in Corollary 1.

**Remarks.** A direct application of the van Douwen's method [3] on the space \( X \) either if it is the Moore, screenable of \( \S \) 2, or it is the Moore separable of Remark 1, leads to a regular, not separable and nowhere first countable space. The quotient topology on \( S_\infty / L \) if \( X \) is the Moore, screenable (resp. if it is the Moore separable) gives a regular, nowhere first countable, metacompact, screenable (resp. a regular, nowhere first countable, separable) space.
The quotient topology on $Z/L$ of Corollary 1 if $X$ is the Moore separable space of Remark 1 gives a regular, separable, nowhere first countable with a dispersion point. The quotient topology on $Z^\omega/L$ gives a regular, nowhere first countable, metacompact screenable space with a dispersion point. On each of these spaces, every continuous real-valued function is constant.

References


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