ON THE SUM OF DIGITS OF PRIMES IN IMAGINARY QUADRATIC FIELDS

By
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1. Introduction. Let \( r \geq 2 \) be a fixed integer. Any positive integer \( n \) can be uniquely written in the form

\[
n = \sum_{j=1}^{k} a_j r^{k-j} = a_1 a_2 \cdots a_k,
\]

where each \( a_j \) is one of \( 0, 1, \ldots, r-1 \) and

\[
k = k(n) = \left\lceil \frac{\log n}{\log r} \right\rceil + 1,
\]

where \( \lceil u \rceil \) is the integral part of the real number \( u \). We put

\[
s(n) = \sum_{j=1}^{k} a_j.
\]

I. Kátaı [1] proved, assuming the validity of density hypothesis for the Riemann zeta function, that

\[
\sum_{p \leq x} s(p) = \frac{r-1}{2 \log r} x + O\left( \frac{x}{(\log \log x)^{1/3}} \right),
\]

where in the sum \( p \) runs through the prime numbers. The second-named author [6] proved, without any hypothesis, the result of Kátaı with an improved remainder term

\[
O\left( x \left( \frac{\log \log x}{\log x} \right)^{1/4} \right).
\]

His method is to appeal to a simple combinatorial inequality (see Lemma in § 4), and the deepest result on which he depends is the prime number theorem in a weak form

\[
\sum_{p \leq x} 1 = \frac{x}{\log x} + O\left( \frac{x}{(\log x)^2} \right).
\]

E. Heppner [2] independently proved a more general result by making use of a Chebyshev's inequality to the sum of independent random variables (cf. [5] p. 387, Theorem 2): Let \( B \) be a set of positive integers such that

\[ \log \frac{x}{B(x)} = o(\log x), \]

where

\[ B(x) = \sum_{n \in B} 1. \]

Then

\[ \sum_{n \in B} s(n) = \frac{r-1}{2} \log \frac{x}{\log r} B(x) \left( 1 + O \left( \frac{\log \log x + \log \frac{x}{B(x)}}{\log x} \right)^{1/2} \right). \]

This together with (4) implies (3).

In the present paper we shall show that the estimate (3) is also valid, in some sense, for primes in each imaginary quadratic field \( \mathbb{Q}(\sqrt{-m}) \), where \( m \) is any positive square free integer.

2. Representation of integers in \( \mathbb{Q}(\sqrt{-m}) \) in the scale of \( r \). Let \( \mathfrak{o} \) be the ring of all integers in \( \mathbb{Q}(\sqrt{-m}) \). Any \( \alpha \in \mathfrak{o} \) can be expressed in a unique way as

\[ \alpha = a + b\omega \quad (a, b \in \mathbb{Z}), \]

where

\[ \omega = \begin{cases} \sqrt{-m} & \text{if } -m \equiv 2, 3 \pmod{4}, \\ 1 + \sqrt{-m} / 2 & \text{if } -m \equiv 1 \pmod{4}, \end{cases} \]

and \( \mathbb{Z} \) denotes as usual the set of all rational integers. So by means of the expressions

\[ |a| = a_1 a_2 \cdots a_k \in \mathbb{Q}, \quad |b| = b_1 b_2 \cdots b_k \in \mathbb{Q} \]

given by (1), we can define coordinatewisely the representation of \( \alpha \in \mathfrak{o} \) in the scale of \( r \); i.e.

\[ \alpha = \sum_{j=1}^{k} \alpha_j r^{k-j} = \alpha_1 \alpha_2 \cdots \alpha_k, \]

where

\[ k = k(\alpha) = \max \{ k(|a|), k(|b|) \}, \quad k(0) = 1, \]

\[ \alpha_j = \text{sgn}(a_1) a_1 + \text{sgn}(b) b_j \omega, \]

and \( \text{sgn}(c) = c/|c| \) if \( c \neq 0 \), \( = 0 \) otherwise. We define

\[ s(\alpha) = \sum_{j=1}^{k} \alpha_j. \]

We write

\[ \mathcal{A}_1 = \{ a + b\omega | a, b \in \mathbb{Z}; a \geq 0, b \geq 0 \}. \]
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\[ \mathcal{A}_i = \{-a + bw | a + bw \in \mathcal{A}_i \}, \]

so that \( \mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4 \). We denote by \( \mathcal{B}_i \) the set of all 'digits' \( a_j \) needed for the expressions (5) of all \( \alpha \in \mathcal{A}_j \). Then

\[ \mathcal{B}_1 = \{ c + d \omega | c, d = 0, 1, \ldots, r - 1 \}, \]

\[ \mathcal{B}_2 = \{ c + d \omega | c + d \omega \in \mathcal{B}_1 \}, \]

\[ \mathcal{B}_3 = \{ -c - d \omega | c + d \omega \in \mathcal{B}_3 \}, \]

\[ \mathcal{B}_4 = \{ c - d \omega | c + d \omega \in \mathcal{B}_1 \}, \]

and \( \text{card } \mathcal{B}_i = r^2 \) (1 ≤ i ≤ 4). So we may say that the \( r \)-adic expression (5) of \( \alpha \in \mathcal{A} \) is a kind of representation in the scale of \( r^2 \). For any fixed \( \beta \in \mathcal{B}_i \), we denote by \( F(\alpha, \beta) \) the number of \( \beta \) appearing in the expression (5) of an integer \( \alpha \in \mathcal{A}_i \). By definition

\[ s(\alpha) = \sum_{\beta \in \mathcal{B}_i} F(\alpha, \beta) \quad (\alpha \in \mathcal{A}_i) \]

and

\[ F(a + bw, c + d\omega) = F(-a + bw, -c + d) = F(-a - bw, -c - d\omega) = F(a - bw, c - d\omega) \quad (a, b \in \mathcal{Z}). \]

The norm of \( \alpha = a + bw \in \mathcal{A} \) is a rational integer

\[ N(\alpha) = \begin{cases} a^2 + mb^2 & \text{if } -m \equiv 2, 3 \pmod{4} \\ a^2 + ab + \frac{m+1}{4} b^2 & \text{if } -m \equiv 1 \pmod{4} \end{cases} \]

so that for \( \alpha \neq 0 \)

\[ |k(\alpha) - \frac{\log N(\alpha)}{2 \log r}| \leq c_1, \]

where \( c_1 \) is a constant depending only on \( m \), since by definition

\[ |k(a) - \max \left( \frac{\log |a|, \log |b|}{\log r} \right)| \leq 1 \]

(we mean that max \( (\log 0, x) = x \) and

\[ |2 \max \{ \log |a|, \log |b| - \log N(\alpha) \} | \leq \begin{cases} \log (1+m) & \text{if } -m \equiv 2, 3 \pmod{4}, \\ \log \left(2 + \frac{m+1}{4}\right) & \text{if } -m \equiv 1 \pmod{4}. \end{cases} \]
3. A prime number theorem (A. Mitsui [3], [4]). An integer \( \alpha \in \mathfrak{O} \) is said to be prime if \( (\alpha) \) is an prime ideal in \( \mathcal{O}(\sqrt{-m}) \). Let \( \theta_1, \theta_2 \) be two real numbers such that \( 0 \leq \theta_1 < \theta_2 \leq 2\pi \). Then

\[
\sum_{\substack{\alpha \text{ prime} \\ \theta_1 \leq \arg \alpha \leq \theta_2}} \frac{(\theta_2 - \theta_1)w}{2\pi h} \int_{\theta_2}^{\theta_1} \frac{dt}{\log t} + O(x \exp (-c_4(\log x)^{3/6}(\log \log x)^{-1/5})),
\]

where \( h \) is the class number of \( \mathcal{O}(\sqrt{-m}) \) and

\[
w = \begin{cases} 
4 & \text{if } m = 1, \\
6 & \text{if } m = 3, \\
2 & \text{otherwise}
\end{cases}
\]

We note that a weaker estimate \( O(x/(\log x)^3) \) is sufficient for the proof of our theorem.

4. A combinatorial lemma (I. Shiokawa [6]). Let \( \beta_1, \ldots, \beta_\ell \) be given \( g \) symbols and let \( A^j \) be the set of all sequences of these symbols of length \( j \geq 1 \). Denote by \( F_j(\alpha, \beta) \) the number of any fixed symbol \( \beta \) appearing in a sequence \( \alpha \in A^j \). Then for any \( \varepsilon \) with \( 0 < \varepsilon < 1/2 \) there exist a positive integer \( j_0 \) independent of \( \varepsilon \) such that the number of sequences \( \alpha \in A^j \) satisfying

\[
\left| F_j(\alpha, \beta) - \frac{j}{g} \right| > j^{1/\varepsilon}
\]

is less that \( j g^j \exp (-c_5 j^2) \) for all \( j \geq j_0 \), where \( c_5 \) is an absolute constant.

5. Theorem. Let \( \varphi_1 = 0, \varphi_2 = 2\pi, \varphi_3 = \arg \omega, \varphi_4 = \pi, \) and \( \varphi_5 = \varphi_3 + \pi \). Then for any \( \theta_1, \theta_2 \) satisfying \( \varphi_3 \leq \theta_1 < \theta_2 \leq \varphi_3 + \pi \) for some \( j \) we have

\[
\sum_{\substack{\alpha \text{ prime} \\ \theta_1 \leq \arg \alpha \leq \theta_2}} s(\alpha) = \frac{(\theta_2 - \theta_1)w}{2\pi h} \frac{(r-1)}{4 \log r} \lambda_j x + O\left(x \left( \frac{\log \log x}{\log x} \right)^{1/5}\right),
\]

where

\[
\lambda_j = \begin{cases} 
1+\omega & \text{if } j=1, \\
-1+\omega & \text{if } j=2, \\
-1-\omega & \text{if } j=3, \\
1-\omega & \text{if } j=4,
\end{cases}
\]

and the \( O \)-constant depends at most on \( r \) and \( m \).
6. Proof of Theorem. By (7) and (8) we may assume \( j=1 \). We define for \( \alpha \in \mathcal{A}_1 \) and \( \beta \in \mathfrak{B}_1 \),

\[
D(\alpha, \beta) = \left| P(\alpha, \beta) - \frac{k(\alpha)}{r^2} \right|.
\]

Put for brevity

\[
C(x) = \{ \alpha \in \mathcal{A} : \text{prime}, \ N(\alpha) \leq x, \ \theta_1 \leq \arg \alpha \leq \theta_4 \}.
\]

Then by (7) and (12)

\[
\sum_{\alpha \in C(x)} s(\alpha) = \sum_{\beta \in \mathcal{B}_1} \sum_{\alpha \in C(x)} F(\alpha, \beta) = \frac{r-1}{2} \lambda_1 \sum_{\alpha \in C(x)} k(\alpha) + O\left( \sum_{\beta \in \mathcal{B}_1} \sum_{\alpha \in C(x)} D(\alpha, \beta) \right).
\]

By (9) and (10) we have

\[
\sum_{\alpha \in C(x)} k(\alpha) = \frac{(\theta_2 - \theta_4) x}{2\pi h} - \frac{x}{2 \log r} + O\left( \frac{x}{\log x} \right).
\]

Put \( D(\alpha) = D(\alpha, \beta_0) \), where \( \beta_0 \) is any fixed integer in \( \mathfrak{B}_1 \). We have from (9), (10), and (12)

\[
\sum_{x \in C(x)} D(\alpha) \leq \sum_{\alpha \in C(x)} k(\alpha)^{1/2 + \varepsilon} + \sum_{D(\alpha) > k(\alpha)^{1/2 + \varepsilon}} D(\alpha) = O\left( \sum_{\alpha \in C(x)} (\log N(\alpha))^{1/2 + \varepsilon} \right) = O\left( \sum_{D(\alpha) > k(\alpha)^{1/2 + \varepsilon}} D(\alpha) \right).\]

Besides, using (9),

\[
\sum_{\alpha \in C(x)} \frac{1}{k(\alpha)^{1/2 + \varepsilon}} \leq \sum_{x \in C(x)} \sum_{\alpha \in C(x)} \frac{1}{D(\alpha)^{1/2 + \varepsilon}} = \frac{1}{2 \log r} \left( \frac{\log x}{c_1} \right).
\]

where

\[
k(x) = \frac{\log x}{2 \log r} + c_1.
\]

Applying now the lemma in § 4 with \( g = r^2 \) and \( A^2 = \mathfrak{B}_1 \), we get

\[
\sum_{D(\alpha) > k(\alpha)^{1/2 + \varepsilon}} 1 < j r^{2j} \exp(-c_3 j^{2r})
\]

for all \( j \geq j_0 \), which leads to

\[
\sum_{D(\alpha) > k(\alpha)^{1/2 + \varepsilon}} 1 = O(1) + \sum_{j < j_0} j r^{2j} \exp(-c_3 j^{2r})\]
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\[ = O(1) + \sum_{j_0 < j \leq \frac{x}{2}} + \sum_{l(x) \leq j \leq l(x)} = O(x(\log x)^2 \exp \left( - \frac{c}{4} \left( \frac{\log x}{\log r} \right)^{\frac{1}{2}} \right)). \]

where the $O$-constant is uniform in $\varepsilon$.

If we take a constant $c_4 = c_4(r)$ large enough and choose $\varepsilon = \varepsilon(x, r)$ with $0 < \varepsilon < 1/2$ in such a way that

\[(\log x)^{\varepsilon} = c_4 \log \log x\]

we obtain from (15) and (16)

\[ \sum_{\sigma \in \mathcal{D}(x)} D(\alpha, \beta_0) = O \left( x \left( \frac{\log \log x}{\log x} \right)^{1/2} \right). \]

This together with (13) and (14) yealds the theorem.

References


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