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SPAN ZERO CONTINUA AND THE PSEUDO-ARC

Dedicated to Professor Ryosuke Nakagawa on his 60th birthday

By

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0. Introduction

A compact connected metric space is called a *continuum*. Let X be a continuum and d be a metric of X. A. Lelek [6], [7] defined the *span, semispan, surjective span* and *surjective semispan* by the following formulas (the map π_j denotes the projection map from $X\times X$ onto the i-th factor).

$\tau=\sigma,\ \sigma_0,\ \sigma^*,\ \sigma_0{}^*$.

$$\tau=\sup\left\{ c\geqq 0 \;\middle|\; \begin{array}{l} \text{there exists a continuum } Z\subset X\times X \text{ such that} \\ Z \text{ satisfies the condition } \tau) \text{ and} \\ d(x, y)\geqq c \quad \text{for each } (x, y)\in Z \end{array} \right\}.$$

Where the condition τ) is

$$\begin{array}{ll} \pi_1(Z)=\pi_2(Z) & \text{if } \tau=\sigma \\ \pi_1(Z)\supset\pi_2(Z) & \text{if } \tau=\sigma_0 \\ \pi_1(Z)=\pi_2(Z)=X & \text{if } \sigma=\sigma^* \\ \pi_1(Z)=X & \text{if } \tau=\sigma_0{}^* \end{array}$$

The property of having zero span (semispan, surjective span, surjective semispan resp.) does not depend on the choice of metrics of X.

A continuum is said to be *arc-like* if it is represented as the limit of an inverse sequence of arcs. It is known that each arc-like continuum has span zero. But it is not known whether the converse implication is true or not. A continuum X is said to be *hereditarily indecomposable* if each subcontinuum Y of X cannot be represented as the union of two proper subcontinua of Y. Hereditarily indecomposable arc-like continuum is topologically unique. It is called the *pseudo-arc* and denoted by P in this paper. It is known to be a homogeneous plane continuum and is also important in span theory. For example, each span zero continuum is a continuous image of the pseudo-arc ([11] and [2]).

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The purpose of this paper is to study some roles of the pseudo-arc in span theory. The paper is divided into three parts. In section 1, a uniformization theorem of maps from the pseudo-arc onto span zero continua is proved. As an application, we obtain a method of constructing maps from the pseudo-arc onto span zero continua. In section 2 and 3, we study the (weak) confluency of product maps. Using these results, we have an equivalent condition that a map preserves the property of having zero span in terms of (weak) confluency of product maps (cf. [10]). In section 4, we prove fixed point theorems for span zero continua, which are generalizations of [13].

To obtain these results, we use some techniques of Oversteegen [10] and Oversteegen-Tymchatyn [11].

Notations and definitions

Throughout this paper, $\boldsymbol{Q}$ denoted the Hilbert cube with a fixed metric. Let $f, g: X \to Y$ be maps and $\varepsilon>0$. We say that f and g are *ε-near* (denoted by $f \underset{\varepsilon}{=} g$) if $\sup\{d(f(x), g(x)) \mid x \in X\} < \varepsilon$. The map $f \triangle g: X \to Y \times X$ is defined by $f \triangle g\ (x)=(f(x), g(x))$.

A collection $\mathcal{W}=\{W_1, \cdots, W_n\}$ is called a *weak chain* if $W_i \cap W_{i+1} \neq \emptyset$ for each $1 \leqq i \leqq n-1$. Let $\mathcal{U}=\{U_1, \cdots, U_m\}$ be another weak chain and $f: \{1, \cdots, m\} \to \{1, \cdots, n\}$ be a pattern (i.e. $|f(i)-f(i+1)| \leqq 1$ for each i). Then $\mathcal{U}$ is said to *follow* f in $\mathcal{W}$ if $U_i \subset W_{f(i)}$ for each $1 \leqq i \leqq m$. A continuum W is called *weakly chainable* if there exists a sequence $(\mathcal{W}_n)$ of weak chain covers of W such that mesh $\mathcal{W}_n \to 0$ as $n \to \infty$, and for each n, $\mathcal{W}_{n+1}$ follows a pattern in $\mathcal{W}_n$.

A continuum is weakly chainable if and only if it is a continuous image of the pseudo-arc ([5]).

Let $f: X \to Y$ be an onto map between continua, The map f is called *confluent* (*weakly confluent* resp.) if for each subcontinuum K of Y, each (some resp.) component C of $f^{-1}(K)$ satisfies $f(C)=K$.

1. Uniformizations

The following proposition is proved by the same way as [11] Theorem 1 and [12] Lemma 6. We give an outline of the proof (cf. [10] Lemma 2).

PROPOSITION 1. *Let $X \subset \boldsymbol{Q}$ be a continum and suppose that $\sigma_0 X \leqq c$ $(c \geqq 0)$. Let Z be a subcontinuum of X.*

1) *For each $\varepsilon>0$, there exists a $\delta>0$ such that for each pair of maps $h, k: I \to \boldsymbol{Q}$ satisfying $d_H(h(I), Z)$, $d_H(k(I), Z)<\delta$, there exist onto maps $a, b: I \to I$ such*

that $h \circ a \underset{c+\varepsilon}{=} k \circ b$.

2) *Suppose that* X *is hereditarily indecomposable and* $z \in Z$. *If the maps* $h, k: I \to \boldsymbol{Q}$ *in* 1) *further satisfy* $d(h(0), z)$, $d(k(0), z) < \delta$, *then the maps* a *and* b *can be chosen so that* $a(0) = b(0) = 0$.

OUTLINE OF PROOF. We give an outline of the case 2). Give any subcontinuum Z and any $\varepsilon > 0$. For each pair of maps $h, k: I \to \boldsymbol{Q}$, we define

$$N(h, k; \varepsilon) = \{(x, y) \in I \times I \mid d(h(x), k(y)) < c + \varepsilon\}.$$

As in the proof of [11] Theorem 1 and [12] Lemma 6, we have

a) there exists an $\varepsilon > 0$ which satisfies the following condition:

Let $h, k: I \to \boldsymbol{Q}$ be any pair of maps satisfying

$$d_H(h(I), Z) < \delta, \qquad d_H(k(I), Z) < \delta$$

$$d(h(0), z) < \delta \quad \text{and} \quad d(k(0), z) < \delta.$$

Then each continuum $K \subset I \times I$ with $K \cap I \times 0 \neq \emptyset \neq K \cap 0 \times I$ intersects $N(h, k : \varepsilon)$.

This δ is the required number. To prove this, we take maps $h, k: I \to \boldsymbol{Q}$ as in the hypothesis. Then as in [12] Lemma 6 again,

b) there exists a component $C(\varepsilon)$ of $N(h, k; \varepsilon)$ such that each continuum $K \subset I \times I$ satisfying $K \cap I \times 0 \neq \emptyset \neq K \cap 0 \times I$ intersects $C(\varepsilon)$.

Let p_i be the projection map from $I \times I$ to the i-th factor. It is easy to see that $(0, 0) \in C(\varepsilon)$ and

$$p_1(C(\varepsilon)) = I \quad \text{or} \quad p_2(C(\varepsilon)) = I.$$

Assume that $p_1(C(\varepsilon)) = I$. By the similar argument of [11] Theorem 1, we see that there exists a component $D(\varepsilon)$ of $N(h, k; \varepsilon)$ such that $p_2(D(\varepsilon)) = I$. But clearly, $C(\varepsilon) \cap D(\varepsilon) \neq \emptyset$ so, $C(\varepsilon) = D(\varepsilon)$.

Take a graph $G \subset C(\varepsilon)$ such that $(0, 0) \in G$ and $p_i(G) = I$ $i = 1, 2$. Let $f: I \to G$ be an onto map such that $f(0) = (0, 0)$. Then $a = p_1 \circ f$ and $b = p_2 \circ f$ are the required.

Let X_i be continua and d_i be a metric of X_i $(i = 1, 2)$. In this paper, the metric of $X_1 \times X_2$ is defined by $d((x_1, x_2), (y_1, y_2)) = \max_{i=1,2} d_i(x_i, y_i)$.

Using Proposition 1.1 and the same way as [10] Theorem 3, we can prove the following.

PROPOSITION 1.2. *Let* X_i *be continua in* $\boldsymbol{Q}$ *such that* $\sigma_0{}^* X_i \leqq c$ $(c \leqq 0)$ $i = 1, 2$. *Then each pair of onto maps* $f_i: Y_i \to X_i$ $(i = 1$ '$2)$ *satisfies the following condition.*

For each subcontinuum $K\subset X\times X$ *satisfying* $\pi_i{}^X(K)=X_i$ $(i=1, 2)$, *there exists a continuum* $L\subset Y_1\times Y_2$ *such that* $\pi_i{}^Y(L)=Y_i$, $i=1, 2$ *and* $d_H((f_1\times f_2)(L), K)\leqq c$, *where, the map* $\pi_i{}^X$ *denotes the projection* $X_i\times X_2$ *to the i-th factor etc.*

REMARK. In the proof of [10] Theorem 3, the weak conluency of each factor of the product map is used. The map f_i in the above proposition need not be weakly confluent, but the same proof works in our situation.

THEOREM 1.3. *Let* $X\subset \boldsymbol{Q}$ *be a continuum such that* $\sigma_0{}^*X\leqq c$ $(c\geqq 0)$.

1) *For each pair of onto maps* $f, g: Y\rightarrow X$, *there exists a continuum* Z *and onto maps* $\alpha, \beta: Z\rightarrow Y$ *such that* $f\circ\alpha \underset{2c}{=} g\circ\beta$.

2) *In particular, if* $Y=P$, *then for each* $\varepsilon>0$, *there exists a homeomorphism* $h: P\rightarrow P$ *such that* $f\underset{2c+\varepsilon}{=} g\circ h$.

PROOF. 1) Consider the map $f\times g: Y\times Y\rightarrow X\times X$ and the diagonal set ΔX of X. By Proposition 1.2, there exists a continuum $Z\subset Y\times Y$ such that $\pi_1(Z)=\pi_2(Z)=Y$ and $d_H(f\times g(Z), X)\leqq c$. Let $\alpha=\pi_1|Z$ and $\beta=\pi_2|Z: Z\rightarrow Y$, then α and β are onto maps. For each $(x, y)\in Z$, there exists a point $(p, p)\in\Delta X$ such that $d(f(x), p)$, $d(g(y), p)\leqq c$. Hence $d(f(x), g(y))\leqq 2c$. This means $f\circ\alpha \underset{2c}{=} g\circ\beta$.

2) Give any $\varepsilon>0$. There exists a $\delta>0$ such that
 for each $x, y\in P$ with $d(x, y)<\delta$, $d(f(x), f(y))<\varepsilon/2$
 and $d(g(x), g(y))<\varepsilon/2$.

Consider the continuum Z as in 1). By [14], there exists a homeomorphism $h: P\rightarrow P$ such that $d_H(G(h), Z)<\delta/2$, where $G(h)=\{x, h(x))|x\in P\}$, the *graph* of h.

For each $p\in P$, there exists a point $(x, y)\in Z$ such that $d(x, p)$, $d(h(p), y)<\delta$. Since $f(x)\underset{2c}{=} g(y)$, we have that

$$d(f(p), g\circ h(p))\leqq d(f(p), f(x))+d(f(x), g(y))+d(g(y), g\circ h(p))$$
$$<\varepsilon/2+2c+\varepsilon/2<2c+\varepsilon\,.$$

This completes the proof.

As an application of Theorem 1.3, we obtain a characterization of span zero continua as follows.

THEOREM 1.4. *Let* $X\subset\boldsymbol{Q}$ *be a tree-like continuum in* $\boldsymbol{Q}$. *Then the following are equivalent.*

1) $\sigma X=0$.

2) *For each subcontinuum Z of X and for each $\varepsilon>0$, there exists a $\delta>0$ such that*

> *for each pair of maps $f, g: P\to \boldsymbol{Q}$ satisfying $f(P)\supset g(P)$ and $d_H(f(P), Z)<\delta$, there exists a subcontinuum $P_1\subset P$ and an (onto) homeomorphism $h: P_1\to P$ such that $g\circ h \underset{\varepsilon}{=} f\,|\,P_1$.*

We need the following lemma for the proof.

LEMMA 1.5. *Let $f: P\to X$ be a map from the pseudo-arc into a weakly chainable continuum X. Then there exists an arc-like continuum $P^*\supset P$ and an extension $F: P^*\to X$ of f such that $F(P)=X$.*

PROOF. Take a point p of P and let $x=f(p)$. Take another pseudo-arc P' and an onto map $g: P'\to X$. Fix a point $p'\in g^{-1}(x)$ and let P^* be the one point union of P and P' identified at p and p'. Define $F: P^*\to X$ by $F|P=f$ and $F|P'=g$. For each $\varepsilon>0$, there exist a chain cover $\mathcal{C}$ ($\mathcal{C}'$ resp.) of P (P' resp.) such that mesh $\mathcal{C}$ (mesh $\mathcal{C}'$ resp.)$<\varepsilon$ and p (p' resp.) is contained in the first link of $\mathcal{C}$ ($\mathcal{C}'$ resp.). Using this fact, it is easy to see that P^* is arc-like.

PROOF OF THEOREM 1.4.

1)$\to$2). Notice that $\sigma_0 X=0$ by [2]. Fix any subcontinum Z and give any $\varepsilon>0$. As $\sigma_0 Z=0$, there exists a $\delta>0$ such that

> each continuum $K\subset\boldsymbol{Q}$ with $d_H(K, Z)<\delta$, satisfies $\sigma_0 K<\varepsilon/4$.

To prove that this δ is the required number, take any pair of maps $f, g: P\to\boldsymbol{Q}$ as in the hypothesis. Then $\sigma_0 f(P)<\varepsilon/4$ by the choice of δ. By Lemma 1.5, there exist an arc-like continuum $P^*\supset P$ and a surjective extension $G: P^*\to f(P)$ of g. Fix an onto map $k: P\to P^*$. Applying Theorem 1.3 to f and $G\circ k: P\to f(P)$, there exists a homeomorphism $h^*: P\to P$ such that $f \underset{\varepsilon/2}{=} G\circ k\circ h^*$.

Since P^* is arc-like, it is in class W (i. e. each map onto P^* is weakly confluent). Hence there exists a continuum $P_1\subset P$ such that $k\circ h^*(P_1)=P$. Define $h'=k\circ h^*|P_1: P_1\to P$. Each onto map from P_1 onto P is a near-homeomorphism by [14]. A homeomorphism $h: P_1\to P$ which is sufficiently close to h' satisfies the required condition.

2)$\to$1). Suppose that $\sigma X=c>0$. There exist maps $\alpha, \beta: C\to X$ from a continuum C such that $\alpha(C)=\beta(C)$ and $d(\alpha(p), \beta(p))\geqq c$ for each $p\in C$. We assume that $C\subset\boldsymbol{Q}$ and let $Z=\alpha(C)=\beta(C)$ and $0<\varepsilon<c/4$. Take δ for ε as in 2). Let $X=\varprojlim X_n$ be the inverse limit description of X by an inverse sequence of trees.

We may assume that $X\cup\cup X_n\subset \boldsymbol{Q}$ and the projection map $p_n: X\to X_n$ is $1/2^n$-translation in $\boldsymbol{Q}$. Take sufficiently large n, so that $1/2^n<\delta$ and let $T=p_n(Z)$. Since T is a tree, $p_n\circ\alpha$ and $p_n\circ\beta$ has extensions $A, B: \boldsymbol{Q}\to T$ respectively. There exists an $\eta>0$ such that

for each $x, y\in \boldsymbol{Q}$ with $d(x, y)<\eta$, $d(A(x), A(y))<\varepsilon/2$
and $d(B(x), B(y))<\varepsilon/2$.

Let E be the set of all end points of T. For each $p\in E$, take $x_p\in(p_n\circ\alpha)^{-1}(p)$. It is easy to find a pseudo-arc $P\subset \boldsymbol{Q}$ such that $d_H(P, C)<\eta$ and $\{x_p \mid p\in E\}\subset P$. Then $A(P)=T$.

Applying 2) to $A|P$ and $B|P: P\to T$, we can find a subcontinuum $P_1\subset P$ and a homeomorphism $h: P_1\to P$ such that $B\circ h\underset{\varepsilon}{=}A|P_1$. There exists a point $p\in P_1$ such that $h(p)=p$. As $d_H(C, P)<\eta$, we can find a point $x\in C$ such that $d(p, x)<\eta$. But then,

$$\begin{aligned} d(\alpha(x), \beta(x)) &= d(A(x), B(x)) \\ &\leqq d(A(x), A(p))+d(A(p), B\circ h(p))+d(B(p), B(x)) \\ &< \varepsilon/2+\varepsilon+\varepsilon/2=2\varepsilon<c/2\,, \end{aligned}$$

which is a contradiction.

This completes the proof.

The following theorem gives a method of constructing maps from P onto span zero continua.

THEOREM 1.6. *Let X be a continuum which is the limit of an inverse sequence $(X_n, p_{n\,n+1}: X_{n+1}\to X_n)$. If $\sigma X=0$, then X has the following property.*

For each sequence $(a_n: P\to X_n)$ of onto maps, there exists a subsequence (m_n) and a sequence of homeorphism $(h_{n\,n+1}: P\to P)$ such that the following diagram is $1/2^{i-1}$-commutative.

$$\begin{tikzcd} & P \arrow[d, "a_{n_i}"'] & P \arrow[l, "h_{ij}"'] \arrow[d, "a_{n_j}"] \\ X_{n_k} & X_{n_i} \arrow[l, "p_{n_k n_i}"'] & X_{n_j} \arrow[l, "p_{n_i n_j}"'] \end{tikzcd} \qquad k\leqq i\leqq j\,.$$

Where, h_{ij} denotes $h_{i\,i+1}\circ h_{i+1\,i+2}\circ, \cdots, \circ h_{j-1\,j}$, etc.

Hence an onto map $a: P\to X$ is induced [9].

Again, we can assume that $X\cup\cup X_n\subset \boldsymbol{Q}$ and the projection $p_n: X\to X_n$ is an $1/2^n$-translation in $\boldsymbol{Q}$. For the proof, we need the following lemma.

LEMMA 1.7. *Under the above notation, the following condition holds.*
For each $i\geqq 1$ *and for each* $\varepsilon>0$, *there exist an integer* $N>0$ *and a* $\delta>0$ *such that*

for each $n\geqq N$ *and for any points* $x, y\in X_n$ *with* $d(x, y)<\delta$,
$d(p_{in}(x), p_{in}(y))<\varepsilon$.

PROOF. Define $\pi: X\cup\bigcup_{n\geqq i} X_n\to X_i$ by $\pi|X=p_i$ and $\pi|X_n=p_{in}$. Then π is continuous. Hence for each $\varepsilon>0$, there exists a $\delta>0$ such that for any points $x, y\in X\cup\bigcup_{n\geqq i} X_n$ with $d(x, y)<3\delta$, $d(\pi(x), \pi(y))<\varepsilon/2$. Take sufficiently large N such that for each $n\geqq N$, p_n is a δ-translation in $\boldsymbol{Q}$. It is easy to see that N and δ are the required numbers.

PROOF OF THEOREM 1.6. Inductively we will construct the desired diagram. Since $\lim \sigma_0 X_n=\sigma_0 X=0$ by [8] ((3.1), (3.2)), [4] and [2], taking a subsequence if necessary, we may assume that $\sigma_0 X_n<1/2^n$.

$i=1$; Let $n_1=1$, $a_{n_1}=a_1$, and $\delta_1=1/2$. Choose an $\varepsilon_1>0$ so that $2(\sigma_0 X_{n_1})+\varepsilon_1<\delta_1$.

$i=2$; Applying Lemma 1.5 to $i=1$ and $\varepsilon=1/2^2$, we have an integer $N_2>0$ such that $\delta_2<1/2^2$ and

for each $n\geqq N_2$ and for each $x, y\in X_n$ with $d(x, y)<\delta_2$,
$d(p_{1n}(x), p_{1n}(y))<1/2^2$.

Take an $n_2>n_1$, N_2 such that $\sigma_0 X_{n_2}<\delta_2/2$ and choose $\varepsilon_2>0$ such that $2(\sigma_0 X_{n_2})+\varepsilon_2<\delta_2$. Applying Theorem 1.3 to ε_1, a_{n_1}, and $p_{n_1n_2}\circ a_{n_2}$, then we have a homeomorphism $h_{12}: P\to P$ such that $a_{n_1}\circ h_{12}\underset{1/2}{=} p_{n_1n_2}\circ a_{n_2}$.

$i=3$; Applying Lemma 1.5 to n_1 and $1/2^3$, take $N_3^1>0$ and $\delta_3^1>0$. Applying Lemma 1.5 again to n_2 and $1/2^3$, take $N_3^2>0$ and $\delta_3^2>0$.

Let $N_3>\max(N_3^1, N_3^2)$ and $0<\delta_3<\min(\delta_3^1, \delta_3^2)$, and take $n_3>n_2$, N_3 such that $\sigma_0 X_{n_3}<\delta_3/2$. Choose an $\varepsilon_3>0$ such that $2(\sigma_0 X_{n_3})+\varepsilon_3<\delta_3$. Apply Theorem 1.3 to ε_2, a_{n_3} and $p_{n_2n_3}\circ a_{n_2}$. Then, there exists a homeomorphism $h_{23}: P\to P$ such that $a_{n_2}\circ h_{23}\underset{\delta_2}{=} p_{n_2n_3}\circ a_{n_3}$. Since $2(\sigma_0 X)+\varepsilon_2<\delta_2<1/2^2$, we have

$$a_{n_2}\circ h_{23}\underset{1/2^2}{=} p_{n_2n_3}\circ a_{n_3} \quad\text{and}$$

$$p_{n_1n_2}\circ a_{n_2}\circ h_{23}\underset{1/2^2}{=} p_{n_1n_2}\circ p_{n_2n_3}\circ a_{n_3}\,.$$

Continuing these steps, we have a subsequence (n_i) and a sequence of homeomorphisms $(h_{i\,i+1}: P\to P)$ such that

$$\text{for each } k\leqq i\leqq j,\ p_{n_k n_i}\circ a_{n_i}\circ h_{ij} \underset{1/2^{i-1}}{=} p_{n_k}p_i\circ a_{n_i n_j}\circ a_{n_j}.$$

This completes the proof.

2. (Weak) Confluency of product maps

PROPOSITION 2.1 (cf. [10] Theorem 3) *Let Y be a continuum such that $\sigma Y=0$.*

1) *For each map $f: X\to Y$ and for each continuum Z, $f\times id_Z$ is weakly confluent.*

2) *In particular, if Y is hereditarily indecomposable, then $f\times id_Z$ is confluent.*

PROOF. The proof uses the method of [10] Theorem 3. We prove only the case 2). Let $X=\lim(X_n, p_{n\,n+1}: X_{n+1}\to X_n)$, $Y=\lim(Y_n, q_{n\,n+1}: Y_{n+1}\to Y_n)$ and $Z=\lim(Z_n, r_{n\,n+1}: Z_{n+1}\to Z_n)$ be inverse limit descriptions of X, Y and Z respectively. Taking a subsequence if necessary, we may assume that f is induced by the following diagram.

$$\begin{tikzcd}
 & X_m \arrow[d, "f_m"'] \arrow[dr, phantom, "\circlearrowright\varepsilon_m"] & X_n \arrow[l] \arrow[d, "f_n"'] \arrow[dr, phantom, "\circlearrowright\varepsilon_n"] & X \arrow[l] \arrow[d, "f"] \\
Y_l & Y_m \arrow[l] & Y_n \arrow[l] & Y. \arrow[l]
\end{tikzcd}$$

Where $\varepsilon_n\to 0$ as $n\to\infty$.

Further we assume that $X\cup\cup X_n$, $Y\cup\cup Y_n$ and $Z\cup\cup Z_n\subset \boldsymbol{Q}$ and projection maps $p_n: X\to X_n$, $q_n: Y\to Y_n$ and $r_n: Z\to Z_n$ are $1/2^n$-translations in $\boldsymbol{Q}$. The map $F: X\cup\cup X_n\to Y\cup\cup Y_n$ defined by $F|X=f$, $F|X_n=f_n$ is continuous.

To prove that $f\times id_Z$ is confluent, we take any continuum $K\subset Y\times Z$ and choose a point $(x, z)\in(f\times id_Z)^{-1}(K)$. It suffices to construct a continuum $C\subset X\times Z$ such that $f\times id_Z(C)=K$ and $(x, z)\in C$. By an induction, we take a suitable subsequence (m_n) and a sequence (C_n) of continua such that

a) $C_n\subset X_{m_n}\times Z_{m_n}$ b) $d_H(f_{m_n}\times id_{Z_{m_n}}(C_n), K)<1/n$.

c) $d((x, z), C_n)<1/n$.

Let π_Y and π_Z be the projection from $Y\times Z$ to Y and Z respectively. Define $K^Y=\pi_Y(K)$, $K^Z=\pi_Z(K)$ and $(y, z)=f\times id_Z(x, z)$.

Let $m_0=0$ and $C_0=X\times Z$ and assume that m_{n-1} and C_{n-1} have been defined. Since Y is hereditarily indecomposable and $\sigma Y=0$, by Proposition 1.1, there exists a $\delta>0$ such that $0<\delta<1/2n$ and

d) for each pair of maps $h, k: I\to \boldsymbol{Q}$ which satisfy $d_H(h(I), K^Y)<\delta$ and $d_H(k(I), K^Y)<\delta$, there exist maps $a, b: I\to \boldsymbol{Q}$ such that $h\circ a \underset{1/2n}{=} k\circ b$ and $a(0)=b(0)=0$.

Since f is a confluent map, there exists a continuum C of X such that

e) $x\in C$ and $f(C)=K^Y$.

We use the following notation;

f) $K_m=q_m\times r_m(K)$, $K_m{}^Y=q_m(K^Y)$, $K_m{}^Z=r_m(K^Z)$,
$C_m{}^X=p_m(C)$, $C_m{}^Z=K_m{}^Z$.

Take sufficiently large m such that

g) $m>m_{n-1}$, $d_H(K_m, K)<\delta/3$, $d_H(f_m(C_m{}^X), K_m{}^Y)<\delta/3$ and $\varepsilon_m<\delta/3$.

Now we define maps $\alpha_1: I\to Y_m$, $\beta_1: I\to X_m$, α_2, $\beta_2: I\to Z_m$ as follows;

h) $d(\alpha_1(0), y)<\delta$ and $d_H(\alpha_1(I), K_m{}^Y)<\delta/3$.

i) $d(\beta_1(0), x)<1/n$, $d(f_m\beta_1(0), y)<\delta$ and $d_H(f_m\beta_1(I), K_m{}^Y)<\delta/3$.

j) $d(\alpha_2(0), z)<\delta$ and $d_H(\alpha_2(I), K_m{}^Z)<\delta/3$.

k) The map $\alpha=\alpha_1\Delta\alpha_2: I\to Y_m\times Z_m$ satisfies $d_H(\alpha(I), K_m)<1/2n$.

l) $\beta_2=\alpha_2$.

Then by h), i) and d), there exist maps $a_1, b_1: I\to I$ such that $\alpha_1\circ a_1 \underset{1/2n}{=} f_m\circ\beta_1\circ b_1$ and $a_1(0)=b_1(0)=0$. Let $\omega=\beta_1\circ b_1\Delta\alpha_2\circ a_1: I\to X_m\times Z_m$. Then we have

m) $d(\omega(0), (x, z))<1/n$.

n) $d(f_m\times id_{Z_m}(\omega(t)), \alpha(a_1(t)))<1/n$.

Let $m_n=m$. As a_1 is an onto map, we see that $C_n=\omega(I)$ is the required continuum.

$$
\begin{tikzcd}
& I \arrow[r, two heads, "\beta_1"] & X_m \arrow[d, "f_m"] \\
I \arrow[ur, "b_1"] \arrow[dr, "a_1"'] & & Y_m \\
& I \arrow[ur, "\alpha_1"] \arrow[r, "\alpha"] \arrow[dr, "\alpha_2"'] & Y_m \times Z_m \arrow[u] \arrow[d] \\
& & Z_m
\end{tikzcd}
$$

We may assume that $C_n)$ converges to a continuum $C \subset X \times Z$. Then $(x, z) \in C$ and $f \times id_Z(C) = K$.

THEOREM 2.2. *Let $f: Y \rightarrow Y$ be an onto map between continua. The following are equivalent respectively.*

1) *The map $f \times id_P: X \times P \rightarrow Y \times P$ is weakly confluent (confluent resp.).*
2) *For each continuum Z with $\sigma Z = 0$ (for each hereditarily indecomposable continuum Z with $\sigma Z = 0$ resp.), $f \times id_Z: X \times Z \rightarrow Y \times Z$ is weakly confluent (confluent resp.).*
3) *There exists a hereditarily indecomposable continum Z such that $f \times id_Z$ is weakly confluent (confluent resp.).*

PROOF. We prove the confluent case. Another case is similarly proved.

1)→2). Since Z is weakly chainable, there exists an onto map $\varphi: P \rightarrow Z$. Clearly,

$$
\begin{aligned}
f \times \varphi &= (f \times id_Z) \circ (id_X \times \varphi) \\
&= (id_Y \times \varphi) \circ (f \times id_P) .
\end{aligned}
$$

By Theorem 2.1, $id_Y \times \varphi$ is confluent and by the assumption, $f \times id_P$ is confluent, so $f \times \varphi$ is confluent. Hence $f \times id_Z$ is confluent.

2)→1)→3). These are trivial.

3)→1). By [1], there exists an onto map $\psi: Z \rightarrow P$. Then $f \times \psi = (f \times id_P) \circ (id_X \times \psi) = (id_Y \times \psi) \circ (f \times id_Z)$. The similar argument as above implies the conclusion.

3. The preservation of the property of having zero span

LEMMA 3.1. *Let $f: X \rightarrow Y$ be an irreducible map (i.e. no proper subcontinuum of X can be mapped onto Y). If $f \times id_P: X \times P \rightarrow Y \times P$ is weakly confluent, then*

f has the following property;

(*) *for each onto map $\alpha: P \to Y$, there exists a continuum $Z \subset X \times P$ such that $\pi_X(Z)=X$, $\pi_P(Z)=P$, and $f \circ \pi_X | Z = \alpha \circ \pi_P | Z$.*

Where π_X and π_P is the projections from $X \times P$ to X and P respectively.

PROOF. Let $H_\alpha = \{(\alpha(p), p) | p \in P\}$. Then $\pi_P(H_\alpha)=P$ and $\pi_Y(H_\alpha)=Y$. Since $f \times id_P$ is weakly confluent, there exists a continuum $Z \subset X \times P$ such that $f \times id_P(Z) = H_\alpha$. Then $f(\pi_Y(Z))=\pi_Y(H_\alpha)=Y$, so by the irreducibility of f, $\pi_X(Z)=X$. It is easy to see that Z satisfies the other conditions which are required.

THEOREM 3.2. *Let $f: X \to Y$ be a map which satisfies the following conditions.* 1) *f satisfies (*)* 2) *$f \times f: X \times X \to Y \times Y$ is weakly confluent. If $\sigma X=0$, then $\sigma^* Y=0$.*

PROOF. We first show that

a) for each pair of onto maps $\alpha, \beta: P \to Y$ from the pseudo-arc, there exists a point $p \in P$ such that $\alpha(p)=\beta(p)$.

To prove a), we apply the property (*) to α and β respectively. There exist continua Z_α and Z_β such that $f \circ \pi_X{}^\alpha = \alpha \circ \pi_P{}^\alpha$ and $f \circ \pi_X{}^\beta = \beta \circ \pi_P{}^\beta$, where $\pi_X{}^\alpha = \pi_X | Z_\alpha$ etc. By Theorem 1.3, there exist a continuum W and onto maps $f_\alpha: W \to Z_\alpha$ and $f_\beta: W \to Z_\beta$ such that $\pi_P{}^\alpha \circ f_\alpha = \pi_P{}^\beta \circ f_\beta$. Since $\pi_X{}^\alpha \circ f_\alpha$ and $\pi_X{}^\beta \circ f_\beta: W \to X$ are onto maps and $\sigma X=0$, there exists a point $w \in W$ such that $\pi_X{}^\alpha \circ f_\alpha(w) = \pi_X{}^\beta \circ f_\beta(w)$. Then we can see that $\alpha \circ \pi_P{}^\alpha \circ f_\alpha(w) = \beta \circ \pi_P{}^\beta \circ f_\beta(w)$. So $p = \pi_P{}^\alpha \circ f_\alpha(w) = \pi_P{}^\beta \circ f_\beta(w)$ satisfies the conclusion of a).

$$\begin{tikzcd}
& Z_\alpha \arrow[r, two heads] \arrow[d, two heads] & X \arrow[d, two heads, "f"] \\
W \arrow[ur, "f_\alpha"] \arrow[dr, "f_\beta"'] & P \arrow[r, shift left, two heads, "\alpha"] \arrow[r, shift right, two heads, "\beta"'] & Y \\
& Z_\beta \arrow[r, two heads] \arrow[u, two heads] & X . \arrow[u, two heads, "f"']
\end{tikzcd}$$

Using a), it is easy to see that

b) for each pair of onto maps $\alpha, \beta: W \to Y$ from any weakly chainable continuum W onto X, there exists a point $w \in W$ such that $\alpha(w)=\beta(w)$.

Next we prove that

c) for each subcontinuum $Z \subset Y \times Y$, there exists a sequence (W_n) of weakly chainable continua such that

$W_n \subset Y\times Y$, $\mathrm{Lim}\, W_n = Z$ and $p_i(W_n)=p_i(Z)$,
where p_i denotes projection from $Y\times Y$ to the i-th factor.

To see this, we note that $\sigma X=0$ and hence X is weakly chainable. Take an onto map $\varphi: P\to X$, then $\varphi\times\varphi: P\times P\to X\times X$ is weakly confluent ([10], Theorem 3). From this fact and condition 2), there exists a continuum $C\subset P\times P$ so that $f\varphi\times f\varphi(C)=Z$. Let $P_i=\pi_P{}^i(C)$ $i=1, 2$, where each $\pi_P{}^i$ denotes projection from $P\times P$ to the i-th factor. By [14], there exist a sequence of homeomorphism $(h_n: P_1\to P_2)_{n\geq 0}$ such that $G(h_n)$'s, the graphs of h_n's ($\subset P\times P$), converges to C. Define W_n by $W_n=f\varphi\times f\varphi(G(h_n))$, which is clearly weakly chainable. Moreover, $W_n\to f\varphi\times f\varphi(C)=Z$, and for $i=1, 2$,

$$\begin{aligned} p_i(W_n) &= f\varphi(\pi_P{}^i(G(h_n))) \\ &= f\varphi(P_i)=p_i(f\varphi\times f\varphi)(C)=p_i(Z). \end{aligned}$$

This prove c).

Now we prove that $\sigma^*Y=0$. Take any continuum $Z\subset Y\times Y$ satisfying $p_i(Z)=Y$ $i=1, 2$. By c), there exists a sequence (W_n) of weakly chainable continua such that $p_i(w_n)=Y$ and $W_n\to Z$. By b), $W_n\cap\Delta Y\neq\emptyset$ for each n. So we have $Z\cap\Delta Y\neq\emptyset$. This completes the proof.

Using Theorem 3.2, we have

THEOREM 3.3 (cf. [10] Theorem 7). *Let $f: X\to Y$ be an onto map between continua and suppose that $\sigma X=0$.*

1) *The following are equivalent.*

a) $\sigma Y=0$.

b) *For each subcontinuum K of X.*

$$(f\,|\,K)\times id_P: K\times P \longrightarrow f(K)\times P \quad \text{and} \quad (f\,|\,K)\times id_Y: K\times Y \longrightarrow f(K)\times Y$$

are weakly confluent.

2) *Suppose that X is hereditarily indecomposable and f is confluent. Then the following are equivalent.*

a) $\sigma Y=0$.

b) *$f\times id_Y: X\times Y\to X\times Y$ is confluent.*

c) *$f\times f: X\times X\to Y\times Y$ is confluent.*

PROOF. 1) a)$\to$b). This follows for [10] Theorem 3.

b)$\to$a). Take any subcontinuum Z in Y. There exists a contituum $K\subset X$

such that $f|K: K\to Z$ is an irreducible map. By the assumption and Theorem 2.2, we see that $(f|K)\times id_X$ is, and hence $(f|K)\times(f|K)$ is weakly confluent. Hence by Theorem 3.2 and Lemma 3.1, we have $\sigma^*Z=0$. So $\sigma Y=0$.

2) a)$\to$b). This follows from [10] Theorem 3.

b)$\to$c). Since Y is hereditarily indecomposable (Notice that confluent maps preserve hereditary indecomposability), it follows that $f\times id_X$ is confluent by Theorem 2.2. Then $f\times f=(id_Y\times f)\circ(f\times id_X)$ is confluent.

c)$\to$a). This follows from [10] Theorem 7.

4. Fixed points for multi-valued map on span zero continua

We prove some fixed point theorem for multi-valued map of span zero continua, which generalize some results of Rosen [14]. Also in this section, [10] Theorem 3 is used.

Let X be a continuum. The space of all nonempty compact subsets of X (the space of all nonempty subcontinua of X resp.) with the Hausdorff metric is denoted by 2^X ($C(X)$ resp.). Let $f: X\to 2^Y$ be a (not necessarily continuous) function. The set $G(f)=\bigcup_{x\in X}\{x\}\times f(x)\subset X\times Y$ is called the *graph* of f. The *image* of f, denoted by $f(X)$, is defined by $\bigcup_{x\in X} f(x)$. A function f is *uppersemi-* (*lowersemi-* resp.) *continuous.* abbreviated u. s. c. (l. s. c. resp.), if for each open set U of Y, $\{x\in X|f(x)\subset U\}$ ($\{x\in X|f(x)\cap U\neq\emptyset\}$ resp.) is open. A function $f: X\to 2^Y$ is continuous if and only if f is both upper- and lower- semi- continuous. We say that f is *onto* if $f(X)=X$,

THEOREM 4.1 (cf. [13] Theorem 1). *Let $f, g: X\to 2^Y$ be u. s. c. functions. Suppose that*

1) $\sigma X=\sigma Y=0$ 2) *$G(f)$ and $G(g)$ are connected and*

3) *f is onto.*

The there exists a point $x\in X$ such that $f(x)\cap g(x)\neq\emptyset$.

PROOF. Since X and Y are weakly chainable by 1), there exist irreducible onto maps $a: P\to X$ and $b: P\to Y$. By the uppersemicontinuity and 2), $G(f)$, $G(g)\subset X\times Y$ are continua. By [10] Theorem 3, there exist subcontinua K and L of $P\times P$ such that $a\times b(K)=G(f)$ and $a\times b(L)=G(g)$. Let p_i's (π_i's resp.) denote the projection maps from $P\times P$ ($X\times Y$ resp.) to the i-th factor, $i=1, 2$. Then $a(p_1(K))=\pi_1(G(f))=X$, and by the irreducibility of a, $p_1(K)=P$. Similarly, $p_1(L)=P$, $p_2(K)=P$.

Since P is arc-like, it is easy to see that $K\cap L\neq\emptyset$, hence $G(f)\cap G(g)\neq\emptyset$.

Take $(x, y)\in G(f)\cap G(g)$. The point x satisfies the conclusion.

COROLLARY 4.2. *Let $f, g: X\to 2^{Y}$ be u.s.c. functions and suppose that*

1) $\sigma X=\sigma Y=0$
2) *f is onto and $G(f)$ is connected, and*
3) *g is continuous.*

Then there exists a point $x\in X$ such that $f(x)\cap g(x)\neq\emptyset$.

PROOF. By [13] Lemma 1, there exists an u.s.c. function $h: X\to 2^{Y}$ such that $h(x)\subset g(x)$ for each $x\in X$ and $G(h)$ is connected.

THEOREM 4.3 (cf. [13] Theorem 2). *Let $f, g: X\to C(Y)$ be u.s.c. functions. Suppose that*

2) $\sigma Y=0$ *and* 2) *f is onto.*

Then there exists a point $x\in X$ such that $f(x)\cap g(x)\neq\emptyset$.

PROOF. Define a subset $G(f, g)$ of $Y\times Y$ by $\bigcup_{x\in X} f(x)\times g(x)$. Since $f(x)$ and $g(x)$ are continua for each $x\in X$, and f and g are uppersemicontinuous, $G(f, g)$ is a subcontinuum of $Y\times Y$, and $\pi_1(G(f, g))=Y$ (π_1 is the projection to the first factor). By [2], $\sigma_0 Y=0$, so $G(f, g)\cap\Delta Y\neq\emptyset$. This means the conclusion.

Let $f: X\to 2^{X}$ be a function. A point $x\in X$ is called a *fixed point* of f if $x\in f(x)$.

COROLLARY 4.4. *Let X be a continuum with $\sigma X=0$. Then X has the fixed point property for the following classes of multi-valued functions.*

1) $\{f: X\to 2^{X} \mid f$ *is u.s.c. and* $G(f)$ *is connected*$\}$.
2) $\{f: X\to 2^{X} \mid f$ *is continuous*$\}$.
3) $\{f: X\to C(X) \mid f$ *is u.s.c.*$\}$.

References

[1] Bellamy, D., Mapping hereditarily indecomposable continua, Lecture Notes in Math. Springer, Berlin 375, Topology Conference, Virginia Polytechique Institute and State University, March 1973, 22-24.
[2] Davis, J.F., The equivalence of zero span and zero semispan, Proc. A.M.S. **96** (1984), 135-138.
[3] Ingram, W.T., Positive span and inverse limits, Topology Proc. **9** (1984), 313-317.
[4] Kawamura, K., On span and inverse limits, Tsukuba J. of Math. **12** (1988), 333-340.
[5] Lelek, A., On weakly chainable continua, Fund. Math. **51** (1962-3), 271-282.
[6] ———, Disjoint mappings and the span of spaces, Fund. Math. **55** (1964), 199-

214.
[7] ———, On surjective span and semispan of connected metric spaces, Colloq. Math. **37** (1977), 35-45.
[8] ———, The span and the width of continua, Fund. Math. **98** (1978), 181-199.
[9] Mioduszewski, J., Mappings of inverse limits, Colloq. Math. **10** (1964), 39-44.
[10] Oversteegen, L.G., On products of confluent and weakly confluent mappings related to span, Houston J. of Math. **12** (1986), 109-116.
[11] Oversteegen, L.G. and Tymchatyn, E.D., On span and weakly chainable continua, Fund. Math. **122** (1984), 159-174.
[12] ——— and ———, On span and chainable continua, Fund. Math. **123** (1984), 137-149.
[13] Rosen, R., Fixed points for multi-valued functions on snake-like continua, Proc. A.M.S. **10** (1959), 167-173.
[14] Smith, M., Concerning the homeomorphisms of the pseudoarc X as a subspace of $C(X \times X)$, Houston J. Math. **12** (1986), 431-440.

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