0. Introduction

A compact connected metric space is called a continuum. Let $X$ be a continuum and $d$ be a metric of $X$. A. Lelek [6], [7] defined the span, semispan, surjective span and surjective semispan by the following formulas (the map $\pi_i$ denotes the projection map from $X \times X$ onto the $i$-th factor).

$$
\tau = \sigma, \sigma_0, \sigma^*, \sigma_0^*.
$$

$$
\tau = \sup \left\{ \alpha \geq 0 \left| \begin{array}{l}
\text{there exists a continuum } Z \subseteq X \times X \text{ such that} \\
Z \text{ satisfies the condition } \tau \text{ and} \\
d(x, y) \geq \alpha \text{ for each } (x, y) \in Z
\end{array} \right. \right\}.
$$

Where the condition \( \tau \) is

$$
\pi_1(Z) = \pi_1(Z) \quad \text{if } \tau = \sigma
$$
$$
\pi_1(Z) \supseteq \pi_1(Z) \quad \text{if } \tau = \sigma_0
$$
$$
\pi_1(Z) = \pi_1(Z) = X \quad \text{if } \sigma = \sigma^*
$$
$$
\pi_1(Z) = X \quad \text{if } \tau = \sigma_0^*
$$

The property of having zero span (semispan, surjective span, surjective semispan resp.) does not depend on the choice of metrics of $X$.

A continuum is said to be arc-like if it is represented as the limit of an inverse sequence of arcs. It is known that each arc-like continuum has span zero. But it is not known whether the converse implication is true or not. A continuum $X$ is said to be hereditarily indecomposable if each subcontinuum $Y$ of $X$ cannot be represented as the union of two proper subcontinua of $Y$. Hereditarily indecomposable arc-like continuum is topologically unique. It is called the pseudo-arc and denoted by $P$ in this paper. It is known to be a homogeneous plane continuum and is also important in span theory. For example, each span zero continuum is a continuous image of the pseudo-arc ([11] and [2]).

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The purpose of this paper is to study some roles of the pseudo-arc in span theory. The paper is divided into three parts. In section 1, a uniformization theorem of maps from the pseudo-arc onto span zero continua is proved. As an application, we obtain a method of constructing maps from the pseudo-arc onto span zero continua. In section 2 and 3, we study the (weak) confluency of product maps. Using these results, we have an equivalent condition that a map preserves the property of having zero span in terms of (weak) confluency of product maps (cf. [10]). In section 4, we prove fixed point theorems for span zero continua, which are generalizations of [13].

To obtain these results, we use some techniques of Oversteegen [10] and Oversteegen-Tymchatyn [11].

Notations and definitions

Throughout this paper, \( Q \) denoted the Hilbert cube with a fixed metric. Let \( f, g : X \to Y \) be maps and \( \varepsilon > 0 \). We say that \( f \) and \( g \) are \( \varepsilon \)-near (denoted by \( f \sim \varepsilon g \)) if \( \sup \{ d(f(x), g(x)) | x \in X \} < \varepsilon \). The map \( f \triangle g : X \to Y \times X \) is defined by \( f \triangle g (x) = (f(x), g(x)) \).

A collection \( \mathcal{W} = \{ W_1, \ldots, W_n \} \) is called a weak chain if \( W_i \cap W_{i+1} \neq \emptyset \) for each \( 1 \leq i \leq n-1 \). Let \( \mathcal{U} = \{ U_1, \ldots, U_m \} \) be another weak chain and \( f : \{ 1, \ldots, m \} \to \{ 1, \ldots, n \} \) be a pattern (i.e. \( |f(i) - f(i+1)| \leq 1 \) for each \( i \)). Then \( \mathcal{U} \) is said to follow \( f \) in \( \mathcal{W} \) if \( U_i \subseteq W_{f(i)} \) for each \( 1 \leq i \leq m \). A continuum \( W \) is called weakly chainable if there exists a sequence \( (\mathcal{W}_n) \) of weak chain covers of \( W \) such that mesh \( \mathcal{W}_n \to 0 \) as \( n \to \infty \), and for each \( n \), \( \mathcal{W}_{n+1} \) follows a pattern in \( \mathcal{W}_n \).

A continuum is weakly chainable if and only if it is a continuous image of the pseudo-arc \([5]\).

Let \( f : X \to Y \) be an onto map between continua. The map \( f \) is called confluent (weakly confluent resp.) if for each subcontinuum \( K \) of \( Y \), each (some resp.) component \( C \) of \( f^{-1}(K) \) satisfies \( f(C) = K \).

1. Uniformizations

The following proposition is proved by the same way as [11] Theorem 1 and [12] Lemma 6. We give an outline of the proof (cf. [10] Lemma 2).

Proposition 1. Let \( X \subseteq Q \) be a continuum and suppose that \( \sigma X \subseteq c \) (\( c \geq 0 \)). Let \( Z \) be a subcontinuum of \( X \).

1) For each \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that for each pair of maps \( h, k : I \to Q \) satisfying \( d_H(h(I), Z), d_H(k(I), Z) < \delta \), there exist onto maps \( a, b : I \to I \) such
that \( h \circ a = k \circ b \).

2) Suppose that \( X \) is hereditarily indecomposable and \( z \in Z \). If the maps \( h, k : I \to \mathbb{Q} \) in 1) further satisfy \( d(h(0), z), d(k(0), z) < \delta \), then the maps \( a \) and \( b \) can be chosen so that \( a(0) = b(0) = 0 \).

**Outline of proof.** We give an outline of the case 2). Give any subcontinuum \( Z \) and any \( \varepsilon > 0 \). For each pair of maps \( h, k : I \to \mathbb{Q} \), we define

\[
N(h, k; \varepsilon) = \{(x, y) \in I \times I \mid d(h(x), k(y)) < \varepsilon \}.
\]

As in the proof of [11] Theorem 1 and [12] Lemma 6, we have

a) there exists an \( \varepsilon > 0 \) which satisfies the following condition:

Let \( h, k : I \to \mathbb{Q} \) be any pair of maps satisfying

\[
\begin{align*}
\delta &< d_H(I, Z) < 2\delta, \\
d_H(0, I, Z) < \delta &< d_H(I, Z) < 2\delta.
\end{align*}
\]

Then each continuum \( K \subset I \times I \) with \( K \cap I \times 0 \not\subset K \cap 0 \times I \) intersects \( N(h, k; \varepsilon) \).

This \( \delta \) is the required number. To prove this, we take maps \( h, k : I \to \mathbb{Q} \) as in the hypothesis. Then as in [12] Lemma 6 again,

b) there exists a component \( C(\varepsilon) \) of \( N(h, k; \varepsilon) \) such that each continuum \( K \subset I \times I \) satisfying \( K \cap I \times 0 \not\subset K \cap 0 \times I \) intersects \( C(\varepsilon) \).

Let \( p_t \) be the projection map from \( I \times I \) to the \( i \)-th factor. It is easy to see that \( (0, 0) \in C(\varepsilon) \) and

\[
\begin{align*}
p_1(C(\varepsilon)) &\supseteq I \quad \text{or} \quad p_2(C(\varepsilon)) = I.
\end{align*}
\]

Assume that \( p_i(C(\varepsilon)) = I \). By the similar argument of [11] Theorem 1, we see that there exists a component \( D(\varepsilon) \) of \( N(h, k; \varepsilon) \) such that \( p_2(D(\varepsilon)) = I \). But clearly, \( C(\varepsilon) \cap D(\varepsilon) \not\subset \emptyset \) so, \( C(\varepsilon) = D(\varepsilon) \).

Take a graph \( G \subset C(\varepsilon) \) such that \( (0, 0) \in G \) and \( p_i(G) = I \) \( i = 1, 2 \). Let \( f : I \to G \) be an onto map such that \( f(0) = (0, 0) \). Then \( a = p_1 \circ f \) and \( b = p_2 \circ f \) are the required.

Let \( X_i \) be continua and \( d_i \) be a metric of \( X_i \) \( (i = 1, 2) \). In this paper, the metric of \( X_i \times X_i \) is defined by \( d((x_1, x_2), (y_1, y_2)) = \max_{i=1,2} d_i(x_i, y_i) \).

Using Proposition 1.1 and the same way as [10] Theorem 3, we can prove the following.

**Proposition 1.2.** Let \( X_i \) be continua in \( \mathbb{Q} \) such that \( \sigma_i X_i \leq c \) \( (c \leq 0) \) \( i = 1, 2 \). Then each pair of onto maps \( f_i : Y_i \to X_i \) \( (i = 1, 2) \) satisfies the following condition.
For each subcontinuum $K \subset X \times X$ satisfying $\pi_i^X(K) = X_i$ ($i = 1, 2$), there exists a continuum $L \subset Y_1 \times Y_2$ such that $\pi_i^L(L) = Y_i$, $i = 1, 2$ and $d_H(f_i \times f_2)(L), K) \leq c$, where, the map $\pi_i^X$ denotes the projection $X_i \times X_2$ to the $i$-th factor etc.

REMARK. In the proof of [10] Theorem 3, the weak confluency of each factor of the product map is used. The map $f_i$ in the above proposition need not be weakly confluent, but the same proof works in our situation.

Theorem 1.3. Let $X \subset Q$ be a continuum such that $\sigma^*_s X \leq c$ ($c \geq 0$).

1) For each pair of onto maps $f, g : Y \to X$, there exists a continuum $Z$ and onto maps $\alpha, \beta : Z \to Y$ such that $f \circ \alpha = g \circ \beta$.

2) In particular, if $Y = P$, then for each $\epsilon > 0$, there exists a homeomorphism $h : P \to P$ such that $f = g \circ h$.

Proof. 1) Consider the map $f \times g : Y \times Y \to X \times X$ and the diagonal set $\Delta X$ of $X$. By Proposition 1.2, there exists a continuum $Z \subset Y \times Y$ such that $\pi_i(Z) = \pi_i(Z) = Y$ and $d_H(f \times g(Z), X) \leq c$. Let $\alpha = \pi_1 | Z$ and $\beta = \pi_2 | Z : Z \to Y$, then $\alpha$ and $\beta$ are onto maps. For each $(x, y) \in Z$, there exists a point $(p, p) \in \Delta X$ such that $d(f(x), p), d(g(y), p) \leq c$. Hence $d(f(x), g(y)) \leq 2c$. This means $f \circ \alpha = g \circ \beta$.

2) Give any $\epsilon > 0$. There exists a $\delta > 0$ such that for each $x, y \in P$ with $d(x, y) < \delta$, $d(f(x), f(y)) < \epsilon/2$ and $d(g(x), g(y)) < \epsilon/2$.

Consider the continuum $Z$ as in 1). By [14], there exists a homeomorphism $h : P \to P$ such that $d_H(G(h), Z) < \delta/2$, where $G(h) = \{x, h(x) | x \in P\}$, the graph of $h$.

For each $p \in P$, there exists a point $(x, y) \in Z$ such that $d(x, p), d(h(p), y) < \delta$. Since $f(x) = g(y)$, we have that

\[d(f(p), g \circ h(p)) \leq d(f(p), f(x)) + d(f(x), g(y)) + d(g(y), g \circ h(p)) < \epsilon/2 + 2c + \epsilon/2 < 2c + \epsilon.\]

This completes the proof.

As an application of Theorem 1.3, we obtain a characterization of span zero continua as follows.

Theorem 1.4. Let $X \subset Q$ be a tree-like continuum in $Q$. Then the following are equivalent.

1) $\sigma X = 0$. 

2) For each subcontinuum $Z$ of $X$ and for each $\epsilon>0$, there exists a $\delta>0$ such that

for each pair of maps $f, g: P \to Q$ satisfying $f(P) \supseteq g(P)$ and

$\delta(f(P), Z) < \delta$, there exists a subcontinuum $P_1 \subseteq P$ and an (onto)
homeomorphism $h: P_1 \to P$ such that $g \circ h = f | P_1$.

We need the following lemma for the proof.

**Lemma 1.5.** Let $f: P \to X$ be a map from the pseudo-arc into a weakly chain-able continuum $X$. Then there exists an arc-like continuum $P^* \supseteq P$ and an extension $F: P^* \to X$ of $f$ such that $F(P) = X$.

**Proof.** Take a point $p$ of $P$ and let $x = f(p)$. Take another pseudo-arc $P'$ and an onto map $g: P' \to X$. Fix a point $p' \in g^{-1}(x)$ and let $P^*$ be the one point union of $P$ and $P'$ identified at $p$ and $p'$. Define $F: P^* \to X$ by $F|_P = f$ and $F|_{P'} = g$. For each $\epsilon > 0$, there exist a chain cover $C$ (or resp.) of $P$ ($P'$ resp.) such that mesh $C$ (mesh $C'$ resp.) < $\epsilon$ and $p$ ($p'$ resp.) is contained in the first link of $C$ (or resp.). Using this fact, it is easy to see that $P^*$ is arc-like.

**Proof of Theorem 1.4.**

1) $\to$ 2). Notice that $\sigma vX = 0$ by [2]. Fix any subcontinuum $Z$ and give any $\epsilon > 0$. As $\sigma vZ = 0$, there exists a $\delta > 0$ such that

each continuum $K \subseteq Q$ with $\delta(K, Z) < \delta$, satisfies $\sigma vK < \epsilon / 4$.

To prove that this $\delta$ is the required number, take any pair of maps $f, g: P \to Q$ as in the hypothesis. Then $\sigma v(f(P)) < \epsilon / 4$ by the choice of $\delta$. By Lemma 1.5, there exist an arc-like continuum $P^* \supseteq P$ and a surjective extension $G: P^* \to f(P)$ of $g$. Fix an onto map $k: P \to P^*$. Applying Theorem 1.3 to $f$ and $G \circ k: P \to f(P)$, there exists a homeomorphism $h^*: P \to P$ such that $h = G \circ k \circ h^*$.

Since $P^*$ is arc-like, it is in class $W$ (i.e. each map onto $P^*$ is weakly confluent). Hence there exists a continuum $P_1 \subseteq P$ such that $k \circ h^*(P_1) = P$. Define $h' = k \circ h^* | P_1: P_1 \to P$. Each onto map from $P_1$ onto $P$ is a near-homeomorphism by [14]. A homeomorphism $h: P_1 \to P$ which is sufficiently close to $h'$ satisfies the required condition.

2) $\to$ 1). Suppose that $\sigma vX = c > 0$. There exist maps $\alpha, \beta: C \to X$ from a continuum $C$ such that $\alpha(C) = \beta(C)$ and $d(\alpha(p), \beta(p)) \geq c$ for each $p \in C$. We assume that $C \subseteq Q$ and let $Z = \alpha(C) = \beta(C)$ and $0 < \epsilon < c / 4$. Take $\delta$ for $\epsilon$ as in 2). Let $X = \lim X_n$ be the inverse limit description of $X$ by an inverse sequence of trees.
We may assume that \( X \cup \bigcup X_n \subset \mathcal{Q} \) and the projection map \( p_n : X \to X_n \) is \( 1/2^n \)-translation in \( \mathcal{Q} \). Take sufficiently large \( n \), so that \( 1/2^n < \delta \) and let \( T = p_n(Z) \). Since \( T \) is a tree, \( p_n \alpha \) and \( p_n \beta \) has extensions \( A, B : \mathcal{Q} \to T \) respectively. There exists an \( \eta > 0 \) such that

for each \( x, y \in \mathcal{Q} \) with \( d(x, y) < \eta \), \( d(A(x), A(y)) < \varepsilon/2 \)
and \( d(B(x), B(y)) < \varepsilon/2 \).

Let \( E \) be the set of all end points of \( T \). For each \( p \in E \), take \( x_p \in (p_n \alpha)^{-1}(p) \).
It is easy to find a pseudo-arc \( P \subset \mathcal{Q} \) such that \( d_H(P, C) < \eta \) and \( \{ x_p \mid p \in E \} \subset P \).
Then \( A(P) = T \).

Applying 2) to \( A|P \) and \( B|P : P \to T \), we can find a subcontinuum \( P_i \subset P \) and a homeomorphism \( h : P_i \to P \) such that \( B \circ h = A \circ P_i \). There exists a point \( p \in P_i \) such that \( h(p) = p \). As \( d_H(C, P) < \eta \), we can find a point \( x \in C \) such that \( d(p, x) < \eta \). But then,

\[
d(a(x), \beta(x)) = d(A(x), B(x)) \
\leq d(A(x), A(p)) + d(A(p), B \circ h(p)) + d(B(p), B(x)) \
< \varepsilon/2 + \varepsilon + \varepsilon/2 = 2\varepsilon < \varepsilon/2,
\]

which is a contradiction.

This completes the proof.

The following theorem gives a method of constructing maps from \( P \) onto span zero continua.

**Theorem 1.6.** Let \( X \) be a continuum which is the limit of an inverse sequence \((X_n, p_{n+1} : X_{n+1} \to X_n)\). If \( \sigma X = 0 \), then \( X \) has the following property.

For each sequence \((a_n : P \to X_n)\) of onto maps, there exists a subsequence \((m_n)\) and a sequence of homeomorphism \((h_n : P \to P)\) such that the following diagram is \( 1/2^{k-1} \)-commutative.

\[
\begin{array}{ccc}
P & \xleftarrow{h_{ij}} & P \\
\downarrow{a_{i}} & & \downarrow{a_{j}} \\
X_{n_i} & \xleftarrow{p_{n_i}n_{i}} & X_{n_j} \\
\end{array}
\]

Where, \( h_{ij} \) denotes \( h_{i+1}\circ h_{i+1}\circ \cdots \circ h_{j-1} \), etc.

Hence an onto map \( a : P \to X \) is induced \([9]\).
Again, we can assume that $X \cup \cup X_n \subset \mathbb{Q}$ and the projection $p_n : X \rightarrow X_n$ is an $1/2^n$-translation in $\mathbb{Q}$. For the proof, we need the following lemma.

**Lemma 1.7.** Under the above notation, the following condition holds.
For each $i \geq 1$ and each $\epsilon > 0$, there exist an integer $N > 0$ and a $\delta > 0$ such that

for each $n \geq N$ and any points $x, y \in X_n$ with $d(x, y) < \delta$,

$d(p_{i_n}(x), p_{i_n}(y)) < \epsilon$.

**Proof.** Define $\pi : X \cup \cup X_n \rightarrow X_i$ by $\pi|X = p_i$ and $\pi|X_n = p_{i_n}$. Then $\pi$ is continuous. Hence for each $\epsilon > 0$, there exists a $\delta > 0$ such that for any points $x, y \in X \cup \cup X_n$ with $d(x, y) < 3\delta$, $d(\pi(x), \pi(y)) < \epsilon/2$. Take sufficiently large $N$ such that for each $n \geq N$, $p_n$ is a $\delta$-translation in $\mathbb{Q}$. It is easy to see that $N$ and $\delta$ are the required numbers.

**Proof of Theorem 1.6.** Inductively we will construct the desired diagram. Since $\lim_{s_0} \sigma_s X_n = \sigma_0 X = 0$ by [8] (3.1), (3.2), [4] and [2], taking a subsequence if necessary, we may assume that $\sigma_s X_n < 1/2^n$.

$i = 1$; Let $a_{n_{i-1}} = a_1$, and $\delta_i = 1/2$. Choose an $\epsilon_1 > 0$ so that $2(\sigma_0 X_n) + \epsilon_1 < \delta_i$.

$i = 2$; Applying Lemma 1.5 to $i = 1$ and $\epsilon = 1/2^2$, we have an integer $N_1 > 0$ such that $\delta_i < 1/2^2$ and

for each $n \geq N_1$ and for each $x, y \in X_n$ with $d(x, y) < \delta_2$,

$d(p_{i_n}(x), p_{i_n}(y)) < 1/2^2$.

Take an $a_{n_{i-2}} > N_1$, $N_2$ such that $\sigma_s X_n < \delta_i/2$ and choose $\epsilon_i > 0$ such that $2(\sigma_0 X_n) + \epsilon_i < \delta_i$. Applying Theorem 1.3 to $\epsilon_i$, $a_{n_{i-2}}$, and $p_{n_{i-2}} a_{n_{i-2}}$, then we have a homeomorphism $h_{i_2} : P \rightarrow P$ such that $a_{n_{i-2}} h_{i_2} = p_{n_{i-2}} a_{n_{i-2}}$.

$i = 3$; Applying Lemma 1.5 to $i = 1$ and $1/2^3$, take $N_1^3 > 0$ and $\delta_i^3 > 0$. Applying Lemma 1.5 again to $n_2$ and $1/2^3$, take $N_2^3 > 0$ and $\delta_i^3 > 0$.

Let $N_1 > \max(N_3, N_3^3)$ and $0 < \delta_i^3 < \min(\delta_i^3, \delta_3^3)$, and take $n_{i-3} > n_3$, $N_3$ such that $\sigma_s X_n < \delta_i^3/2$. Choose an $\epsilon_2 > 0$ such that $2(\sigma_0 X_n) + \epsilon_2 < \delta_3$. Apply Theorem 1.3 to $\epsilon_2$, $a_{n_3}$ and $p_{n_3} a_{n_3}$. Then, there exists a homeomorphism $h_{i_3} : P \rightarrow P$ such that $a_{n_3} h_{i_3} = p_{n_3} a_{n_3}$. Since $2(\sigma_0 X) + \epsilon_2 < \delta_3 < 1/2^3$, we have

$$a_{n_3} h_{i_3} = p_{n_3} a_{n_3}$$

and

$$p_{n_3} a_{n_3} h_{i_3} = p_{n_3} a_{n_3}.$$


Continuing these steps, we have a subsequence \((n_i)\) and a sequence of homeomorphisms \((h_{i+1}: P \to P)\) such that

\[
\text{for each } k \leq i \leq j, \quad p_{n_k} a_{n_i} a_{n_j} h_{i+1} = p_{n_k} p_{i} a_{n_i} a_{n_j}.
\]

This completes the proof.

2. (Weak) Confluency of product maps

**Proposition 2.1** (cf. [10] Theorem 3) Let \(Y\) be a continuum such that \(\sigma Y = 0\).

1) For each map \(f: X \to Y\) and for each continuum \(Z\), \(f \times \text{id}_Z\) is weakly confluent.

2) In particular, if \(Y\) is hereditarily indecomposable, then \(f \times \text{id}_Z\) is confluent.

**Proof.** The proof uses the method of [10] Theorem 3. We prove only the case 2). Let \(X = \lim (X_n, p_{n+1}: X_{n+1} \to X_n)\), \(Y = \lim (Y_n, q_{n+1}: Y_{n+1} \to Y_n)\) and \(Z = \lim (Z_n, r_{n+1}: Z_{n+1} \to Z_n)\) be inverse limit descriptions of \(X\), \(Y\) and \(Z\) respectively. Taking a subsequence if necessary, we may assume that \(f\) is induced by the following diagram.

\[
\begin{array}{ccc}
X_m & \leftarrow & X_n & \leftarrow & X \\
\downarrow f_m & & \downarrow f_n & & \downarrow f \\
Y_m & \leftarrow & Y_n & \leftarrow & Y \\
\end{array}
\]

Where \(\varepsilon_n \to 0\) as \(n \to \infty\).

Further we assume that \(X \cup X_n\), \(Y \cup Y_n\) and \(Z \cup Z_n \subset Q\) and projection maps \(p_n: X \to X_n\), \(q_n: Y \to Y_n\) and \(r_n: Z \to Z_n\) are \(1/2^n\)-translations in \(Q\). The map \(F: X \cup X_n \to Y \cup Y_n\) defined by \(F|X = f\), \(F|X_n = f_n\) is continuous.

To prove that \(f \times \text{id}_Z\) is confluent, we take any continuum \(K \subset Y \times Z\) and choose a point \((x, z) \in (f \times \text{id}_Z)^{-1}(K)\). It suffices to construct a continuum \(C \subset X \times Z\) such that \(f \times \text{id}_Z(C) = K\) and \((x, z) \subset C\). By an induction, we take a suitable subsequence \((m_n)\) and a sequence \((C_n)\) of continua such that

a) \(C_n \subset X_{m_n} \times Z_{m_n}\)

b) \(d_H(f_{m_n} \times \text{id}_{Z_{m_n}}(C_n), K) < 1/n\).

c) \(d((x, z), C_n) < 1/n\).

Let \(\pi_Y\) and \(\pi_Z\) be the projection from \(Y \times Z\) to \(Y\) and \(Z\) respectively. Define \(K_Y = \pi_Y(K)\), \(K_Z = \pi_Z(K)\) and \((y, z) = f \times \text{id}_Z(x, z)\).
Let $m_0=0$ and $C_0=X \times Z$ and assume that $m_{n-1}$ and $C_{n-1}$ have been defined.

Since $Y$ is hereditarily indecomposable and $\sigma Y=0$, by Proposition 1.1, there exists a $\delta >0$ such that $0<\delta<1/2n$ and

**d)** for each pair of maps $h$, $k : I \rightarrow Q$ which satisfy $d_H(h(I), K^y) < \delta$ and $d_H(k(I), K^y) < \delta$, there exist maps $a$, $b : I \rightarrow Q$ such that $h \ast a = k \ast b$ and $a(0)=b(0)=0$.

Since $f$ is a confluent map, there exists a continuum $C$ of $X$ such that

**e)** $x \in C$ and $f(C)=K^y$.

We use the following notation;

**f)** $K_m=q_m \times r_m(K)$, $K_m^y=q_m(K^y)$, $K_m^z=r_m(K^z)$, $C_m^x=p_m(C)$, $C_m^z=K_m^z$.

Take sufficiently large $m$ such that

**g)** $m> m_{n-1}$, $d_H(K_m, K)< \delta/3$, $d_H(f_m(C_m^x), K_m^y)< \delta/3$

and $\varepsilon_m< \delta/3$.

Now we define maps $\alpha_1 : I \rightarrow Y_m$, $\beta_1 : I \rightarrow X_m$, $\alpha_2$, $\beta_2 : I \rightarrow Z_m$ as follows;

**h)** $d(\alpha_1(0), y)< \delta$ and $d_H(\alpha_1(I), K_m^y)< \delta/3$.

**i)** $d(\beta_1(0), x)<1/n$, $d(f_m \beta_1(0), y)< \delta$ and $d_H(f_m \beta_1(I), K_m^y)< \delta/3$.

**j)** $d(\alpha_2(0), z)< \delta$ and $d_H(\alpha_2(I), K_m^z)< \delta/3$.

**k)** The map $\alpha=\alpha_1 \Delta \alpha_2 : I \rightarrow Y_m \times Z_m$ satisfies $d_H(\alpha(I), K_m)<1/2n$.

**l)** $\beta_2 = \alpha_2$.

Then by h), i) and d), there exist maps $a_1$, $b_1 : I \rightarrow I$ such that $\alpha_1 \ast a_1 = f_m \beta_1 \ast b_1$

and $a_1(0)=b_1(0)=0$. Let $\omega=\beta_1 \ast b_1 \Delta \alpha_2 \ast a_1 : I \rightarrow X_m \times Z_m$. Then we have

**m)** $d(\omega(0), (x, z))<1/n$.

**n)** $d(f_m \times id_{Z_m}(\omega(t)), \alpha(a_i(t)))<1/n$.

Let $m_2=m$. As $a_1$ is an onto map, we see that $C_n=\omega(I)$ is the required continuum.
We may assume that $C_n$ converges to a continuum $C \subset X \times Z$. Then $(x, z) \in C$ and $f \times \text{id}_Z(C) = K$.

**Theorem 2.2.** Let $f : Y \rightarrow Y$ be an onto map between continua. The following are equivalent respectively.

1) The map $f \times \text{id}_Z : X \times P \rightarrow Y \times P$ is weakly confluent (confluent resp.).

2) For each continuum $Z$ with $\sigma Z = 0$ (for each hereditarily indecomposable continuum $Z$ with $\sigma Z = 0$ resp.), $f \times \text{id}_Z : X \times Z \rightarrow Y \times Z$ is weakly confluent (confluent resp.).

3) There exists a hereditarily indecomposable continuum $Z$ such that $f \times \text{id}_Z$ is weakly confluent (confluent resp.).

**Proof.** We prove the confluent case. Another case is similarly proved.

1)→2). Since $Z$ is weakly chainable, there exists an onto map $\varphi : P \rightarrow Z$. Clearly,

$$f \times \varphi = (f \times \text{id}_Z) \times (\text{id}_X \times \varphi) = (\text{id}_Y \times \varphi) \times (f \times \text{id}_P).$$

By Theorem 2.1, $\text{id}_Y \times \varphi$ is confluent and by the assumption, $f \times \text{id}_P$ is confluent, so $f \times \varphi$ is confluent. Hence $f \times \text{id}_Z$ is confluent.

2)→1). These are trivial.

3)→1). By [1], there exists an onto map $\phi : Z \rightarrow P$. Then $f \times \phi = (f \times \text{id}_P) \times (\text{id}_X \times \phi) = (\text{id}_Y \times \phi) \times (f \times \text{id}_Z)$. The similar argument as above implies the conclusion.

3. The preservation of the property of having zero span

**Lemma 3.1.** Let $f : X \rightarrow Y$ be an irreducible map (i.e., no proper subcontinuum of $X$ can be mapped onto $Y$). If $f \times \text{id}_P : X \times P \rightarrow Y \times P$ is weakly confluent, then
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$f$ has the following property:

(*) for each onto map $\alpha: P \rightarrow Y$, there exists a continuum $Z \subset X \times P$ such that $\pi_X(Z)=X$, $\pi_P(Z)=P$, and $f \cdot \pi_X|Z=\alpha \cdot \pi_P|Z$.

Where $\pi_X$ and $\pi_P$ are the projections from $X \times P$ to $X$ and $P$ respectively.

**Proof.** Let $H_a=\{(\alpha(p), p)| p \in P\}$. Then $\pi_P(H_a)=P$ and $\pi_Y(H_a)=Y$. Since $f \times id_P$ is weakly confluent, there exists a continuum $Z \subset X \times P$ such that $f \times id_P(Z)=H_a$. Then $f(\pi_Y(Z))=\pi_Y(H_a)=Y$, so by the irreducibility of $f$, $\pi_X(Z)=X$. It is easy to see that $Z$ satisfies the other conditions which are required.

**Theorem 3.2.** Let $f: X \rightarrow Y$ be a map which satisfies the following conditions.

1) $f$ satisfies $(*)$ 2) $f \times f: X \times X \rightarrow Y \times Y$ is weakly confluent. If $\sigma X=0$, then $\sigma Y=0$.

**Proof.** We first show that

a) for each pair of onto maps $\alpha, \beta: P \rightarrow Y$ from the pseudo-arc, there exists a point $p \in P$ such that $\alpha(p)=\beta(p)$.

To prove a), we apply the property $(*)$ to $\alpha$ and $\beta$ respectively. There exist continua $Z_\alpha$ and $Z_\beta$ such that $f \cdot \pi_X|Z_\alpha=\alpha \cdot \pi_P|Z_\alpha$ and $f \cdot \pi_X|Z_\beta=\beta \cdot \pi_P|Z_\beta$, where $\pi_X=\pi_X|Z_\alpha$ etc. By Theorem 1.3, there exist a continuum $W$ and onto maps $f_\alpha: W \rightarrow Z_\alpha$ and $f_\beta: W \rightarrow Z_\beta$ such that $\pi_P \cdot f_\alpha=\pi_P \cdot f_\beta$. Since $\pi_X \cdot f_\alpha$ and $\pi_X \cdot f_\beta: W \rightarrow X$ are onto maps and $\sigma X=0$, there exists a point $w \in W$ such that $\pi_X \cdot f_\alpha(w)=\pi_X \cdot f_\beta(w)$. Then we can see that $\alpha \cdot \pi_P \cdot f_\alpha(w)=\beta \cdot \pi_P \cdot f_\beta(w)$. So $p=\pi_P \cdot f_\alpha(w)=\pi_P \cdot f_\beta(w)$ satisfies the conclusion of a).

![Diagram](https://via.placeholder.com/150)

Using a), it is easy to see that

b) for each pair of onto maps $\alpha, \beta: W \rightarrow Y$ from any weakly chainable continuum $W$ onto $X$, there exists a point $w \in W$ such that $\alpha(w)=\beta(w)$.

Next we prove that

c) for each subcontinuum $Z \subset Y \times Y$, there exists a sequence $(W_n)$ of weakly
chainable continua such that
\[ W_n \subset Y \times Y, \ \text{Lim} \ W_n = Z \text{ and } p_i(W_n) = p_i(Z), \]
where \( p_i \) denotes projection from \( Y \times Y \) to the \( i \)-th factor.

To see this, we note that \( \sigma X = 0 \) and hence \( X \) is weakly chainable. Take an onto map \( \varphi : P \to X \), then \( \varphi \times \varphi : P \times P \to X \times X \) is weakly confluent ([10], Theorem 3). From this fact and condition 2), there exists a continuum \( C \subset P \times P \) so that \( f \varphi \times f \varphi(C) = Z \). Let \( P_i = \pi_{P'}(C) \ i = 1, 2 \), where each \( \pi_{P'} \) denotes projection from \( P \times P \) to the \( i \)-th factor. By [14], there exist a sequence of homeomorphism \( (h_n : P_1 \to P_2)_{n \in \mathbb{N}} \) such that \( G(h_n)'s \), the graphs of \( h_n \)'s \( (\subset P \times P) \), converges to \( C \). Define \( W_n \) by \( W_n = f \varphi \times f \varphi(G(h_n)) \), which is clearly weakly chainable. Moreover, \( W_n \to f \varphi \times f \varphi(C) = Z \), and for \( i = 1, 2 \),
\[ p_i(W_n) = f \varphi(\pi_{P'}(G(h_n))) = f \varphi(P_i) = p_i(f \varphi \times f \varphi(C)) = p_i(Z). \]

This prove c).

Now we prove that \( \sigma^* Y = 0 \). Take any continuum \( Z \subset Y \times Y \) satisfying \( p_i(Z) = Y \ i = 1, 2 \). By c), there exists a sequence \( (W_n) \) of weakly chainable continua such that \( p_i(W_n) = Y \) and \( W_n \to Z \). By b), \( W_n \cap \Delta Y \neq \emptyset \) for each \( n \). So we have \( Z \cap \Delta Y \neq \emptyset \). This completes the proof.

Using Theorem 3.2, we have

**Theorem 3.3** (cf. [10] Theorem 7). Let \( f : X \to Y \) be an onto map between continua and suppose that \( \sigma X = 0 \).

1) The following are equivalent.
   a) \( \sigma Y = 0 \).
   b) For each subcontinuum \( K \) of \( X \).
\[ (f \mid K) \times p : K \times P \to f(K) \times P \]
\[ (f \mid K) \times id_Y : K \times Y \to f(K) \times Y \]
are weakly confluent.

2) Suppose that \( X \) is hereditarily indecomposable and \( f \) is confluent. Then the following are equivalent.
   a) \( \sigma Y = 0 \).
   b) \( f \times id_Y : X \times Y \to X \times Y \) is confluent.
   c) \( f \times f : X \times X \to Y \times Y \) is confluent.

**Proof.** 1) a) \( \to b) \). This follows for [10] Theorem 3.

b) \( \to a) \). Take any subcontinuum \( Z \) in \( Y \). There exists a continuum \( K \subset X \)
such that \( f|K: K \to Z \) is an irreducible map. By the assumption and Theorem 2.2, we see that \( (f|K) \times id_X \) is, and hence \( (f|K) \times (f|K) \) is weakly confluent. Hence by Theorem 3.2 and Lemma 3.1, we have \( \sigma^*Z = 0 \). So \( \sigma Y = 0 \).

2) a)\( \rightarrow \)b). This follows from [10] Theorem 3.

b)\( \rightarrow \)c). Since \( Y \) is hereditarily indecomposable (Notice that confluent maps preserve hereditary indecomposability), it follows that \( f \times id_X \) is confluent by Theorem 2.2. Then \( f \times f = (id_Y \times f)(f \times id_X) \) is confluent.

c)\( \rightarrow \)a). This follows from [10] Theorem 7.

4. Fixed points for multi-valued map on span zero continua

We prove some fixed point theorem for multi-valued map of span zero continua, which generalize some results of Rosen [14]. Also in this section, [10] Theorem 3 is used.

Let \( X \) be a continuum. The space of all nonempty compact subsets of \( X \) (the space of all nonempty subcontinua of \( X \) resp.) with the Hausdorff metric is denoted by \( 2^X \) (\( C(X) \) resp.). Let \( f: X \to 2^Y \) be a (not necessarily continuous) function. The set \( G(f) = \bigcup_{x \in X} \{ \{ x \} \times f(x) \subset X \times Y \} \) is called the graph of \( f \). The image of \( f \), denoted by \( f(X) \), is defined by \( \bigcup_{x \in X} f(x) \). A function \( f \) is uppersemi- (lowersemi- resp.) continuous, abbreviated u.s.c. (l.s.c. resp.), if for each open set \( U \) of \( Y \), \( \{ x \in X \mid f(x) \subset U \} \) \( \{ x \in X \mid f(x) \cap U \neq \emptyset \} \) (resp.) is open. A function \( f: X \to 2^Y \) is continuous if and only if \( f \) is both upper- and lower-semi-continuous.

We say that \( f \) is onto if \( f(X) = X \).

**Theorem 4.1** (cf. [13] Theorem 1). Let \( f, g: X \to 2^Y \) be u.s.c. functions. Suppose that

1) \( \sigma X = \sigma Y = 0 \)

2) \( G(f) \) and \( G(g) \) are connected and

3) \( f \) is onto.

The there exists a point \( x \in X \) such that \( f(x) \cap g(x) \neq \emptyset \).

**Proof.** Since \( X \) and \( Y \) are weakly chainable by 1), there exist irreducible onto maps \( a: P \to X \) and \( b: P \to Y \). By the uppersemicontinuity and 2), \( G(f) \), \( G(g) \subset X \times Y \) are continua. By [10] Theorem 3, there exist subcontinua \( K \) and \( L \) of \( P \times P \) such that \( a \times b(K) = G(f) \) and \( a \times b(L) = G(g) \). Let \( p_i \)'s (\( \pi_i \)'s resp.) denote the projection maps from \( P \times P \) \( X \times Y \) resp.) to the \( i \)-th factor, \( i = 1, 2 \). Then \( a(p_i(K)) = \pi_i(G(f)) = X \), and by the irreducibility of \( a \), \( p_i(K) = P \). Similarly, \( p_i(L) = P \), \( p_i(K) = P \).

Since \( P \) is arc-like, it is easy to see that \( K \cap L \neq \emptyset \), hence \( G(f) \cap G(g) \neq \emptyset \).
Take \((x, y)\in G(f)\cap G(g)\). The point \(x\) satisfies the conclusion.

**Corollary 4.2.** Let \(f, g : X\to 2^Y\) be u.s.c. functions and suppose that
1) \(\sigma X = \sigma Y = 0\)
2) \(f\) is onto and \(G(f)\) is connected, and
3) \(g\) is continuous.
Then there exists a point \(x\in X\) such that \(f(x)\cap g(x) \neq \emptyset\).

**Proof.** By [13] Lemma 1, there exists an u.s.c. function \(h : X\to 2^Y\) such that \(h(x)\subseteq g(x)\) for each \(x\in X\) and \(G(h)\) is connected.

**Theorem 4.3** (cf. [13] Theorem 2). Let \(f, g : X\to C(Y)\) be u.s.c. functions. Suppose that
2) \(\sigma Y = 0\) and 2) \(f\) is onto.
Then there exists a point \(x\in X\) such that \(f(x)\cap g(x) \neq \emptyset\).

**Proof.** Define a subset \(G(f, g)\) of \(Y\times Y\) by \(\bigcup_{x\in X} f(x)\times g(x)\). Since \(f(x)\) and \(g(x)\) are continua for each \(x\in X\), and \(f\) and \(g\) are uppersemicontinuous, \(G(f, g)\) is a subcontinuum of \(Y\times Y\), and \(\pi_1(G(f, g)) = Y\) (\(\pi_1\) is the projection to the first factor). By [2], \(\sigma_{\emptyset} = 0\), so \(G(f, g)\cap \Delta Y \neq \emptyset\). This means the conclusion.

Let \(f : X\to 2^X\) be a function. A point \(x\in X\) is called a *fixed point* of \(f\) if \(x\in f(x)\).

**Corollary 4.4.** Let \(X\) be a continuum with \(\sigma X = 0\). Then \(X\) has the fixed point property for the following classes of multi-valued functions.
1) \(\{f : X\to 2^X | f\) is u.s.c. and \(G(f)\) is connected\}.
2) \(\{f : X\to 2^X | f\) is continuous\}.
3) \(\{f : X\to C(X) | f\) is u.s.c.\}.

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