REALIZATIONS OF INVOLUTIVE AUTOMORPHISMS
$\sigma$ AND $G^\sigma$ OF EXCEPTIONAL LINEAR LIE GROUPS $G$, PART II, $G=E_7$

By
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M. Berger [1] classified involutive automorphisms $\sigma$ of simple Lie algebras $g$ and determined the type of the subalgebras $g^\sigma$ of fixed points. In the preceding paper [Y], we found involutive automorphisms $\sigma$ and realized the subgroups $G^\sigma$ of fixed points explicitly for the connected exceptional universal linear Lie groups $G$ of type $G_2$, $F_4$ and $E_6$. In this paper we consider the case of type $E_7$. Our results are as follows.

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<thead>
<tr>
<th>$G$</th>
<th>$G^\sigma$</th>
<th>$\sigma$</th>
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<tbody>
<tr>
<td>$E_7^C$</td>
<td>$(C^* \times E_6^C)/\mathbb{Z}_3$</td>
<td>$\iota$</td>
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<tr>
<td></td>
<td>$SL(8, C)/\mathbb{Z}_2$</td>
<td>$\lambda \gamma$</td>
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<td>$(SL(2, C) \times Spin(12, C))/\mathbb{Z}_3$</td>
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<td>$E_7^C$</td>
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<td>$\tau \lambda$</td>
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<td>$E_7$</td>
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<td>$E_7^C$</td>
<td>$E_{7(1)}$</td>
<td>$\tau \gamma$ $\tau \gamma \sigma$ $\tau \lambda \gamma$ $\tau \lambda \gamma \gamma$ $\tau \lambda \gamma \gamma \gamma$ $\tau \lambda \gamma \rho$</td>
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<td>$E_{7(2)}$</td>
<td>$(\mathbb{R}^+ \times E_{6(1)}) \times 2$</td>
<td>$\iota$</td>
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<tr>
<td></td>
<td>$(U(1) \times E_{6(1)})/\mathbb{Z}_3$</td>
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<td>$SU(8)/\mathbb{Z}_3$</td>
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<td>$SU(4, 4)/\mathbb{Z}_3 \times 2$</td>
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<td>$SU^*(8)/\mathbb{Z}_3 \times 2$</td>
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<td>$SL(8, \mathbb{R})/\mathbb{Z}_3 \times 2$</td>
<td>$\lambda \gamma$</td>
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<td>$(SL(2, \mathbb{R}) \times Spin(6, 6))/\mathbb{Z}_2 \times 2$</td>
<td>$\sigma$</td>
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<td>$E_{7(-1)}$</td>
<td>$\tau \lambda \gamma$ $\tau \lambda \sigma$ $\tau \lambda \sigma'$ $\tau \lambda \gamma \rho$</td>
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<td>$E_{7(-5)}$</td>
<td>$(U(1) \times E_{6(-5)})/\mathbb{Z}_3$</td>
<td>$\iota$</td>
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<td>$\iota$</td>
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This paper is a continuation of [Y] and we use the same notations as [Y]. So the numbering of sections and theorems starts from 4.1 and 4.1.1, respectively.

**Group $E_7$**

4.1. The Freudenthal vector space and the complex Lie group $E_7^c$

We define a $C$-vector space $\mathfrak{F}^c$, called the Freudenthal $C$-vector space, by

$$\mathfrak{F}^c = \mathfrak{F}^c \oplus \mathfrak{F}^c \oplus C \oplus C.$$  

An element $\begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix}$ of $\mathfrak{F}^c$ is often denoted by $(X, Y, \xi, \eta)$, sometimes $X + Y + \xi + \eta$.

In $\mathfrak{F}^c$, the inner products $(P, Q)$ and elements $\{P, Q\}$ are defined by

$$(P, Q) = (X, Z) + (Y, W) + \xi \omega + \eta \bar{\omega},$$

$$\{P, Q\} = \{X, W\} - \{Y, Z\} + \xi \omega - \eta \bar{\omega},$$

respectively, where $P = (X, Y, \xi, \eta)$, $Q = (Z, W, \zeta, \omega) \in \mathfrak{F}^c$.

For $\phi \in e^c$, $A, B \in \mathfrak{F}^c$, $\nu \in C$, we define a $C$-linear transformation $\Phi(\phi, A, B, \nu)$ of $\mathfrak{F}^c$ by

$$\Phi(\phi, A, B, \nu) \begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \phi - \frac{1}{3} \nu & 2B & 0 & A \\ 2A & -i \phi + \frac{1}{3} \nu & B & 0 \\ 0 & A & \nu & 0 \\ B & 0 & 0 & -\nu \end{pmatrix} \begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix}.$$
Realizations of involutive automorphisms

For $P=(X, Y, \xi, \eta), Q=(Z, W, \zeta, \omega) \in \mathfrak{B}^c$, we define a $C$-linear transformation $P \times Q$ of $\mathfrak{B}^c$ by

$$P \times Q = \Phi(\phi, A, B, \nu),$$

where $X \lor Y \in C^c$, $X, Y \in \mathfrak{Y}^c$, is defined by

$$(X \lor Y)Z = \frac{1}{2}(Y, Z)X + \frac{1}{6}(X, Y)Z - 2Y \times (X \times Z), \quad Z \in \mathfrak{Y}^c.$$
Proposition 4.1.3 ([9]).

\[ e^c_e = \{ \Phi(\phi, A, B, \nu) \in \text{Hom}_c(\mathfrak{g}^c, \mathfrak{g}^c) | \phi \in e^c_e, A, B \in \mathfrak{g}^c, \nu \in C \}. \]

The Lie bracket \([\Phi_1, \Phi_2] \) in \( e^c_e \) is given by

\[
[\Phi(\phi_1, A_1, B_1, \nu_1), \Phi(\phi_2, A_2, B_2, \nu_2)] = \Phi(\phi, A, B, \nu),
\]

\[
\begin{align*}
\phi &= [\phi_1, \phi_2] + 2A_1 \vee B_2 - 2A_2 \vee B_1, \\
A &= (\phi_1 + \frac{2}{3} \nu_1)A_1 - (\phi_2 + \frac{2}{3} \nu_2)A_1, \\
B &= -(i\phi_1 + \frac{2}{3} \nu_1)B_2 + (i\phi_2 + \frac{2}{3} \nu_2)B_1, \\
\nu &= (A_1, B_2) - (B_1, A_2).
\end{align*}
\]

4.2. Involutions of Lie group \( E_7^c \)

We arrange here main involutions used in this chapter \( E_7 \). We define \( C \)-linear transformations \( \gamma, \sigma, \iota, \lambda_J, \tau \) of \( \mathfrak{g}^c \) by

\[
\gamma(X, Y, \xi, \eta) = (\gamma X, \gamma Y, \xi, \eta),
\]

\[
\sigma(X, Y, \xi, \eta) = (\sigma X, \sigma Y, \xi, \eta),
\]

respectively, where \( \gamma, \sigma \) of the right sides are the same ones as \( \gamma \in G_2^c \subset F_4^c \subset E_6^c \), \( \sigma \in F_4^c \subset E_6^c \),

\[
\iota(X, Y, \xi, \eta) = (-iX, iY, -i\xi, i\eta),
\]

\[
\lambda_J(X, Y, \xi, \eta) = (Y, -X, \eta, -\xi).
\]

Then \( \gamma, \sigma, \iota, \lambda_J \in E_7^c \) and \( \gamma^2 = \sigma^2 = 1, \iota^2 = \lambda_J^2 = -1 \). The complex conjugation in \( \mathfrak{g}^c \) is denoted by \( \tau \):

\[
\tau(X, Y, \xi, \eta) = (\tau X, \tau Y, \tau \xi, \tau \eta).
\]

These linear transformations \( \gamma, \sigma, \iota, \lambda_J, \tau \) of \( \mathfrak{g}^c \) induce involutive automorphisms \( \tilde{\gamma}, \tilde{\sigma}, \tilde{\iota}, \tilde{\lambda}_J, \tilde{\tau} \) of \( E_7^c \):

\[
\tilde{\gamma}(\alpha) = \gamma^i \alpha^i, \quad \tilde{\sigma}(\alpha) = \sigma \alpha \sigma, \quad \tilde{\iota}(\alpha) = \iota \alpha \iota^{-1}, \quad \tilde{\lambda}_J(\alpha) = \lambda_J \alpha \lambda_J^{-1}, \quad \tilde{\tau}(\alpha) = \tau \alpha \tau.
\]

We define one more involutive automorphism \( \lambda \) of \( E_7^c \) by

\[
\lambda(\alpha) = ^t \alpha^{-1}, \quad \alpha \in E_7^c
\]

where \( ^t \alpha \) is the transpose of \( \alpha \) with respect to the inner product \( (P, Q) \):

\[
(\gamma^t \alpha P, Q) = (P, \gamma^t \alpha Q).
\]

\( \lambda \) is surely an automorphism of \( E_7^c \) (see Proposition 4.2.1).
PROPOSITION 4.2.1. \( \lambda(\alpha) = \lambda_j \alpha \lambda_j^{-1}, \quad \alpha \in E_7^c. \)

PROOF. The inner products \( (P, Q), \{P, Q\} \) in \( \mathfrak{P}^c \) are related with
\[
\{P, Q\} = (P, \lambda_j Q) = -(\lambda_j P, Q).
\]
Now \( (P, \lambda_j Q) = \{P, Q\} = \{\alpha P, \alpha Q\} \) (Lemma 4.1.2) = \( (\alpha P, \lambda_j \alpha Q) \) for \( P, Q \in \mathfrak{P}^c \). Hence \( \lambda_j = \alpha \lambda_j \alpha \), that is, \( \alpha^{-1} = \lambda_j \alpha \lambda_j^{-1} \).

REMARK. The group \( E_7^c \) has a subgroup \( E_8^c \) (see Proposition 4.4.1) and the restriction of \( \lambda \) to \( E_8^c \) is the outer automorphism \( \lambda \) of \( E_8^c \) (Theorem 3.3.1.1). Since \( E_7^c \) has no outer automorphism, \( \lambda \) should be inner. Proposition 4.2.1 shows that \( \lambda \) is realized by \( \lambda_j : \lambda = \lambda_j \). After this, we denote \( \lambda \) by \( \lambda \) in the sense of Proposition 4.2.1:
\[
\lambda = \lambda_j.
\]

LEMMA 4.2.2. The involutive automorphisms of \( e_8^c \) induced by \( \gamma, \sigma, \tau, \lambda, \tau \) are, respectively, as follows.
\[
\begin{align*}
\gamma \Phi(\phi, A, B, \nu)\gamma &= \Phi(\gamma \phi, \gamma A, \gamma B, \nu), \\
\sigma \Phi(\phi, A, B, \nu)\sigma &= \Phi(\sigma \phi, \sigma A, \sigma B, \nu), \\
\tau \Phi(\phi, A, B, \nu)\tau^{-1} &= \Phi(-\phi, -A, -B, \nu), \\
\lambda \Phi(\phi, A, B, \nu)\lambda^{-1} &= \Phi(-\phi, -A, -B, -\nu), \\
\tau \Phi(\phi, A, B, \nu)\tau &= \Phi(\tau \phi, \tau A, \tau B, \tau \nu).
\end{align*}
\]

4.3. Lie groups of type \( E_7 \)

We define \( R \)-vector spaces \( \mathfrak{P}, \mathfrak{P}' \), called the Freudenthal \( R \)-vector spaces, by
\[
\begin{align*}
\mathfrak{P} &= \mathfrak{K}(3, \mathbb{C}) \oplus \mathfrak{K}(3, \mathbb{C}) \oplus \mathbb{R}, \\
\mathfrak{P}' &= \mathfrak{K}(3, \mathbb{C}') \oplus \mathfrak{K}(3, \mathbb{C}') \oplus \mathbb{R}.
\end{align*}
\]

The universal linear connected Lie groups of type \( E_7 \) are obtained as
\[
\begin{align*}
E_7^c &= \{ \alpha \in \text{Iso}_c(\mathfrak{P}') \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q \}, \\
E_7 &= \{ \alpha \in \text{Iso}_c(\mathfrak{P}) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \langle \alpha P, \alpha Q \rangle = \langle P, Q \rangle \}, \\
E_7^{(r)} &= \{ \alpha \in \text{Iso}_c(\mathfrak{P}^r) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q \}, \\
E_7^{(r \gamma)} &= \{ \alpha \in \text{Iso}_c(\mathfrak{P}') \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \langle \alpha P, \alpha Q \rangle \gamma = \langle P, Q \rangle \gamma \}, \\
E_7^{(r \sigma)} &= \{ \alpha \in \text{Iso}_c(\mathfrak{P}) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q \}.
\end{align*}
\]
where \( \langle P, Q \rangle = (\tau P, Q) = -\langle \tau \lambda P, Q \rangle \), \( \langle P, Q \rangle = (\tau P, Q) = -\langle \tau \lambda P, Q \rangle \), \( P, Q \in \mathbb{P}^c \). 

\( E_7^c \), \( E_7 \) are simply connected (see Appendix). Note that each group of them has the center \( \{1, -1\} \).

**Lemma 4.3.1.** \( (\mathbb{P}^c)_{\tau} = \mathbb{P} \), \( (\mathbb{P}^c)_{\tau} = \mathbb{P}^c \).

**Theorem 4.3.2.** \( (E_7^c)^{\tau} = E_7 \), \( (E_7^c)^{\tau} = E_7^{(\tau)} \), \( (E_7^c)^{\tau} = E_7^{(\tau)} \), \( (E_7^c)^{\tau} = E_7^{(-\tau)} \).

**Proof.** As for \( E_7^{(\tau)} \), \( E_7^{(-\tau)} \), these are direct results of Lemma 4.3.1. \( E_7^{(\tau)} \), \( E_7^{(-\tau)} \) are nothing but their definitions (Lemm 4.1.2).

Remark that \( \gamma, \sigma, \iota, \lambda \in E_7 \). The Lie algebras of these groups are given as follows.

**Proposition 4.3.3.**

1. \( e_7 = \{ \Phi \in \mathfrak{e}_7^c \mid \tau \lambda \Phi = \Phi \tau \lambda \} = \{ \Phi(\phi, A, -\tau A, \nu) \in \mathfrak{e}_7^c \mid \phi \in (\mathfrak{e}_7^c)^{\tau} \} \), \( A \in \mathfrak{g}, \nu = -\tau \nu \).

2. \( e_{7(\gamma)} = \{ \Phi \in \mathfrak{e}_7^c \mid \tau \gamma \Phi = \Phi \tau \gamma \} = \{ \Phi(\phi, A, B, \nu) \in \mathfrak{e}_7^c \mid \phi \in (\mathfrak{e}_7^c)^{\tau} \} \), \( A \in \mathfrak{g}(3, \mathbb{C}) \), \( \nu \in \mathbb{R} \).

3. \( e_{7(-\gamma)} = \{ \Phi \in \mathfrak{e}_7^c \mid \tau \gamma \Phi = \Phi \tau \gamma \} = \{ \Phi(\phi, A, -\tau A, \nu) \in \mathfrak{e}_7^c \mid \phi \in (\mathfrak{e}_7^c)^{\tau} \} \), \( A \in \mathfrak{g}, \nu = -\tau \nu \).

4. \( e_{7(-\gamma)} = \{ \Phi \in \mathfrak{e}_7^c \mid \tau \Phi = \Phi \tau \} = \{ \Phi(\phi, A, B, \nu) \in \mathfrak{e}_7^c \mid \phi \in (\mathfrak{e}_7^c)^{\tau} \} \), \( A, B \in \mathfrak{g}(3, \mathbb{C}) \), \( \nu \in \mathbb{R} \).

**Proof.** These follow from Lemma 4.2.2.

**Lemma 4.3.4.** For \( 0 \neq a \in \mathfrak{g} \), the mapping \( \alpha_i(a) : \mathbb{P}^c \to \mathbb{P}^c, i = 1, 2, 3 \),

\[
\alpha_i(a) = \begin{pmatrix}
1 + (\cos |a| - 1)\rho_i & -2\tau a \sin |a| \frac{|a|}{|a|} E_i & 0 & a \sin |a| \frac{|a|}{|a|} E_i \\
2a \sin |a| \frac{|a|}{|a|} E_i & 1 + (\cos |a| - 1)\rho_i & -\tau a \sin |a| \frac{|a|}{|a|} E_i & 0 \\
0 & a \sin |a| \frac{|a|}{|a|} E_i & \cos |a| & 0 \\
-\tau a \sin |a| \frac{|a|}{|a|} E_i & 0 & 0 & \cos |a| \\
\end{pmatrix}
\]

belongs to \( E_i \), where \( |a| = \sqrt{\tau a} a \) and \( \rho_i : \mathbb{P}^c \to \mathbb{P}^c \) is defined by \( \rho_i(X) = (X, a \sin |a| \frac{|a|}{|a|} E_i) \).
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\( E_7 \subset E_6 \times (E_6 \times X) \). \( \alpha_i(a), \alpha_i(b), \alpha_i(c) \ (a, b, c \in C) \) commute mutually.

**Proof.** For \( \Phi_i(a) = \Phi(0, a E_i, -a E_i, 0) \in \mathfrak{c} \), we have \( \alpha_i(a) = \exp \Phi_i(a) \). Hence \( \alpha_i(a) \in E_7 \). Since \( [\Phi_i(a), \Phi_i(b)] = 0 \), \( \alpha_i(a) \) and \( \alpha_i(b) \) are commutative.

**Proposition 4.3.5.** (1) \( \iota \) and \( \lambda \) are conjugate in \( E_7 \): \( \delta \iota = \lambda \delta \), moreover under \( \delta \in E_7 \), such that \( \delta \iota = \gamma \delta \), \( \delta \sigma = \sigma \delta \).

(2) \( \iota \) and \( -\iota \sigma \) are conjugate in \( E_7 \): \( \delta \iota = -\iota \sigma \delta \), moreover under \( \delta \in E_7 \) such that \( \delta \lambda = \lambda \delta \), \( \delta \sigma = \sigma \delta \).

(3) \( \gamma \) and \( -\sigma \) are conjugate in \( E_7 \): \( \delta \gamma = -\sigma \delta \), \( \delta \in E_7 \).

**Proof.** (1) \( \delta = \exp \Phi(0, \frac{i \pi}{4} E, -\frac{i \pi}{4} E, 0) \) is the required one \( \delta = \alpha_i(\frac{\pi}{4}) \).

\( \alpha_i(\frac{\pi}{4}) \alpha_i(\frac{i \pi}{4}) \) (Lemma 4.3.4). The explicit form of \( \delta \) is

\[
\delta = \begin{pmatrix}
X \\
Y \\
\xi \\
\eta
\end{pmatrix} = \frac{1}{\sqrt{8}} \begin{pmatrix}
-(\text{tr}(X)E-2X)+i(\text{tr}(Y)E-2Y)-\xi E+i\eta E \\
\text{tr}(X)E-2X)-i(\text{tr}(Y)E-2Y)+i\xi E-i\eta E \\
-\text{tr}(X)+i\text{tr}(Y)+\xi-i\eta \\
i\text{tr}(X)-\text{tr}(Y)-i\xi+\eta
\end{pmatrix}
\]

(2) \( \delta = \exp \Phi(0, \frac{\pi}{2} E, -\frac{\pi}{2} E, 0) \) is the required one \( \delta = \alpha_i(\frac{\pi}{2}) \) (Lemma 4.3.4).

(3) The proof will be given in 4.5.6.

**4.4. Subgroups of type \( C \oplus E_6 \) of Lie groups of type \( E_7 \)**

We consider a subgroup \( (E_7^c)_{1,1} = \{ \sigma \in E_7^c \mid \sigma i = i, \sigma a = a \} \) of \( E_7^c \).

**Proposition 4.4.1.** \( (E_7^c)_{1,1} \cong E_6^c \).

**Proof.** ([10]). For \( \beta \in E_6^c \), we correspond \( \alpha \in E_7^c \),

\[
\alpha = \begin{pmatrix}
\beta & \varepsilon & 0 & 0 \\
\delta & \beta' & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \beta, \beta', \delta, \varepsilon \in \text{Hom}_c(\mathfrak{h}^c, \mathfrak{h}^c).
\]

In fact, the fact that the left bottom parts are 0 follows from \( \{ \alpha X, \hat{i} \} = \{ \alpha \hat{i} X, \alpha \hat{i} \} = \{ \hat{i} X, \hat{i} \} = 0 \), \( \{ \alpha X, \hat{1} \} = \{ \alpha \hat{i} X, \alpha \hat{1} \} = \{ \hat{i} X, \hat{1} \} = 0 \) and \( \{ \alpha Y, \hat{1} \} = \{ \alpha Y, \hat{1} \} = 0 \) for all
\[ X, Y \in \mathfrak{F}. \] To prove \( \delta = \varepsilon = 0 \), define a space \( \mathfrak{V}^c \) by

\[
\mathfrak{V}^c = \{ P \in \mathfrak{V}^c \mid P \times P = 0, P \neq 0 \}
\]

\[
= \left\{ P = (X, Y, \xi, \eta), P \neq 0 \middle| X \vee Y = 0, X \times X = \xi Y, Y \times Y = \eta X, (X, Y) = 3 \xi \eta \right\}.
\]

Obviously the group \( \mathbb{E}^c \) acts on \( \mathfrak{V}^c \). Since \( \left( X, \frac{1}{\eta} X \times X, \frac{1}{\eta} \det X, \eta \right) \in \mathfrak{V}^c \),

\[
\left( \beta X + \frac{1}{\eta} \varepsilon (X \times X), \delta X + \frac{1}{\eta} \beta' (X \times X), \frac{1}{\eta} \det X, \eta \right) \in \mathfrak{V}^c.
\]

Hence by the second condition in \( \mathfrak{V}^c \),

\[
\left( \beta X + \frac{1}{\eta} \varepsilon (X \times X) \right) \times \left( \beta X + \frac{1}{\eta} \beta' (X \times X) \right) = \eta \left( \delta X + \frac{1}{\eta} \beta' (X \times X) \right)
\]

holds for all \( \neq C \). Compare the coefficients of \( \eta \), then we have \( \delta = 0 \).

Similarly, by \( \left( \frac{1}{\xi} (Y \times Y), Y, \xi, \frac{1}{\xi} \det Y \right) \in \mathfrak{V}^c \), we have \( \varepsilon = 0 \). Next, by the condition \( \alpha (X, X \times X, \det X, 1) = (\beta X, \beta' (X \times X), \det X, 1) \in \mathfrak{V}^c \),

\[
\beta X \times \beta X = \beta' (X \times X), \quad (\beta X, \beta' (X \times X)) = 3 \det X.
\]

Hence \( 3 \det \beta X = (\beta X, \beta X \times \beta X) = (\beta X, \beta' (X \times X)) = 3 \det X \). Therefore \( \beta \in \mathbb{E}^c \). Furthermore, in \( \beta'(X \times X) = \beta X \times \beta X = \beta^{-1} (X \times X) \), put \( X \times X \) instead of \( X \), then \( (\det X) \beta' X = (\det X) \beta^{-1} X \), hence we have \( \beta' X = \beta^{-1} X, X \in \mathfrak{F} \) (even if \( \det X = 0 \), because \( \{ X \in \mathfrak{F} \mid \det X \neq 0 \} \) is dense in \( \mathfrak{F} \)). Therefore \( \beta' = \beta^{-1} \). Thus the proof of Proposition 4.4.1 is completed.

**PROPOSITION 4.4.2.** \((E^c)'^c \) has a subgroup \( \phi(C^*) = \{ \phi(\theta) \mid \theta \in C^* \} \) which is isomorphic to the group \( C^* = \mathbb{C} - \{ 0 \} \). Where \( \phi(\theta), \theta \in C^* \), is the \( C \)-linear transformation of \( \mathfrak{V}^c \) defined by

\[
\phi(\theta)(X, Y, \xi, \eta) = (\theta^{-1} X, \theta Y, \theta^2 \xi, \theta^{-1} \eta).
\]

**PROOF.** It is easy to verify that \( \phi(\theta) \in (E^c)' \).

**LEMMA 4.4.3.** \( \phi : C^* \rightarrow (E^c)' \) of Proposition 4.4.2 satisfies

\[
\tau \phi(\theta) \tau = \phi(\tau \theta), \quad \lambda \phi(\theta) \lambda^{-1} = \phi(\theta^{-1}), \quad \gamma \phi(\theta) \gamma = \sigma \phi(\theta) \sigma = \phi(\theta).
\]

**THEOREM 4.4.4.** \( (E^c)' \cong (C^* \times E^c) / \mathbb{Z}_2, \mathbb{Z}_2 = \{ (1, 1), (\phi(\omega), \phi(\omega^2)), (\phi(\omega^3), \phi(\omega)) \}, \omega \in C, \omega^2 = 1, \omega \neq 1. \)

**PROOF.** We define a mapping \( \phi : C^* \times E^c \rightarrow (E^c)' \) by

\[
\phi(\theta, \beta) = \phi(\theta) \beta.
\]
Obviously $\phi(\theta, \beta) \in (E_2)'. \phi(\theta) \in \phi(C^*)$ and $\beta \in E_6 C$ are commutative, $\phi$ is a homomorphism. $\ker \phi = \{(1, 1), (\phi(\omega), \phi(\omega^\beta)), (\phi(\omega^\beta), \phi(\omega^\beta))\}$ is easily obtained. $\langle E_2 \rangle'$ is connected (Lemma 0.7) and $\dim_{C}(e^* \otimes e^* C) = 1 + 78 = \dim_{C}(e^* C)'$ (because $(e^* C) = \{\Phi(\phi, 0, 0, \nu) | \phi \in e^* C, \nu \in C\}$ (Lemma 4.2.2)), hence $\phi$ is onto. Thus we have the required isomorphism.

**THEOREM 4.4.5.** (1) $\langle E_2 \rangle' \cong (U(1) \times E_6)/Z_3 \cong (\tau \lambda \gamma)' \sim (E_{7-(25)})'.

(2) $\langle E_{7-(25)} \rangle' \cong (U(1) \times E_{6(3)})/Z_3 \cong (\tau \lambda \gamma)' \sim (E_{7(7)})'.

(3) $\langle E_{7-(19)} \rangle' \sim (\tau \lambda \sigma)' \cong (U(1) \times E_{6(-14)})/Z_3 \cong (\tau \lambda \sigma)' \sim (E_{7-(25)})'.

**PROOF.** (1) Let $\alpha \in (E_{2})' = ((E_2)')^{'(\tau \lambda \gamma)} = (\tau \lambda \gamma)'$. By Theorem 4.4.4, there exist $\theta \in C^*, \beta \in E_6 C$ such that $\alpha = \phi(\theta) \beta$. From the condition $\tau \lambda \alpha = \alpha \tau \lambda$, we have $\phi(\theta) \beta = \alpha = \tau \lambda \alpha \lambda^{-1} \tau = \tau \lambda \phi(\theta) \lambda^{-1} \tau \lambda \beta \lambda^{-1} \tau = \phi(\tau \theta^{-1}) \tau \lambda \beta \lambda^{-1} \tau$ (Lemma 4.4.3). Hence

$$\{ \phi(\tau \theta^{-1}) = \phi(\theta) \} \quad \{ \phi(\tau \theta^{-1}) = \phi(\theta) \phi(\omega^\beta) \} \quad \{ \phi(\tau \theta^{-1}) = \phi(\theta) \phi(\omega^\beta) \}$$

$$\tau \lambda \beta \lambda^{-1} \tau = \beta, \quad \tau \lambda \beta \lambda^{-1} \tau = \phi(\theta) \beta.$$

The second and the third cases are impossible, because $(\tau \theta) \theta = \omega^\beta, \omega$ are false. In the first case, $(\tau \theta) \theta = 1$, that is, $\theta \in U(1) = \{ \theta \in C | (\tau \theta) \theta = 1\}$ and $\beta \in (E_6 C)'^{1, 2} = E_6$ (Theorem 3.2.2). Thus $(E_2)' = \phi(U(1) \times E_6) \cong (U(1) \times E_6)/Z_3$.

$E_{7-(25)} = (E_2)' \cong (E_2)'^{1, 2}.$

In fact, since $\tau \sim \lambda$ under $\delta \in E_7 : \delta \tau = \lambda \delta, \delta \tau \lambda = \tau \lambda \delta$ (Proposition 4.3.5), $(E_2)' \cong \alpha \rightarrow \delta^{-1} a \delta \in (E_2)'^{1, 2}$ gives an isomorphism. Now $\langle E_{7-(25)} \rangle' \sim (\tau \lambda \gamma)' = (\tau \lambda \gamma)'$.

(2) Let $\alpha \in (E_{7-(25)})' = (\tau \lambda \gamma)'$, $\alpha = \phi(\theta) \beta, \theta \in C^*, \beta \in E_6 C$. As similar to (1), $\theta \in U(1), \beta \in (E_6 C)'^{1, 2} = E_{6(3)}$ (Theorem 3.2.2). Thus $(E_{7-(25)})' \cong (U(1) \times E_{6(3)})/Z_3$.

$E_{7-(25)} = (E_2)' \cong (E_2)^{'1, 2} \cong (E_2)^{1, 2 \sigma}.$

In fact, since $\tau \sim \alpha$ under $\delta \in E_7 : \delta \tau = \alpha \delta, \delta \tau \alpha = \tau \lambda \delta$ (Proposition 4.3.5), $(E_2)' \cong \alpha \rightarrow \delta^{-1} a \delta \in (E_2)'^{1, 2 \sigma}$ gives an isomorphism. Now $(E_{7-(25)})' \sim (\tau \lambda \gamma)' = (\tau \lambda \gamma)'$.

(3) $E_{7-(25)} = (E_2)' \sim (E_2)^{1, 2 \sigma}$

because $\gamma \sim -\sigma$ under $\delta \in E_7 : \delta \gamma = -\sigma \delta, \delta \tau \lambda = \tau \lambda \delta$ (Proposition 4.3.5). Let $\alpha \in (E_2)' = (\tau \lambda \sigma)'$, $\alpha = \phi(\theta) \beta, \theta \in C^*, \beta \in E_6 C$. As similar to (1), $\theta \in U(1), \beta \in (E_6 C)'^{1, 2} = E_{6(-11)}$ (Theorem 3.2.2). Thus $(E_{7-(25)})' \sim (\tau \lambda \sigma)' \cong (U(1) \times E_{6(-14)})/Z_3$.

$E_{7-(25)} \cong (E_2)'^{1, 2 \sigma}$ (result of (1))$\cong (E_2)'^{1, 2 \sigma}$

because $\tau \sim -\sigma$ under $\delta \in E_7 : \delta \tau = -\sigma \delta, \delta \tau \lambda = \tau \lambda \delta$ (Proposition 4.3.5). Now $(E_{7-(25)})' \sim (\tau \lambda \sigma)' = (\tau \lambda \sigma)'$.

**THEOREM 4.4.6.** (1) $(E_{7(7)})' \cong (R^* \times E_{6(3)})' \times 2.$
(2) \((E_{17-25})\)'\(\cong (R^+ \times E_{6-26}) \times 2\).

**Proof.** (1) Let \(a \in (E_{17})' = (\tau \gamma)'\), \(a = \phi(\theta)\beta\), \(\theta \in C^*\), \(\beta \in E_6^c\) (Theorem 4.4.4). From \(\tau \gamma a = \alpha \tau \gamma\), we have \(\phi(\tau \gamma)\tau \gamma \beta \gamma \tau = \phi(\theta)\beta\) (Lemma 4.4.3). Hence

\[
\begin{align*}
\phi(\tau \gamma) &= \phi(\theta), \\
\tau \gamma \beta \gamma \tau &= \beta;
\end{align*}
\]

or

\[
\begin{align*}
\phi(\tau \gamma) &= \phi(\theta)\phi(\omega), \\
\tau \gamma \beta \gamma \tau &= \phi(\omega)\beta.
\end{align*}
\]

In the first case \(\tau \gamma = \theta\), that is, \(\theta \in R\) and \(\beta \in (E_6^c)' = E_{6\pm}\) (Theorem 3.2.2). In the second case, we can put \(\theta = \theta' \omega\), \(\theta' \in R\), \(\beta = \phi(\omega)\beta'\), \(\beta' \in (E_6^c)'\). Hence \(\phi(\theta)\beta = \phi(\theta')\beta' = \phi(R^+ \times E_{6\pm})\). The third case is similar to the second case. Thus \((E_{17})' = \phi(R^+ \times E_{6\pm})\). The kernel of the restriction \(\phi\) to \(R^+ \times E_{6\pm}\) is \(\{1\}\). Thus \((E_{17})' = R^+ \times E_{6\pm} \cup (-1)R^+ \times E_{6\pm}\) (exactly\(\times 2\) (which is element of the center of \(E_{17}\)) exists in \(E_{17}\)).

(2) Since we know \((E_6^c)' = E_{6-26}\) (Theorem 3.2.2), as similar to (1), \((E_{17-25})' = (\tau \gamma)' = (\tau \gamma) \cong R^+ \times E_{6-26} = (R^+ \times E_{6-26}) \times 2\).

### 4.5. Subgroups of type \(A_7\) of Lie groups of type \(E_7\)

**Lemma 4.5.1.** Any element \(D \in \mathfrak{su}(8, C)\) is uniquely expressed by

\[
D = k(S) + i k(T), \quad S \in \mathfrak{sp}(4, H^c), \quad T \in \mathfrak{sl}(4, H^c).
\]

**Proof.** For \(D \in \mathfrak{su}(8, C)\), \(S = \frac{1}{2} \mathrm{k}^{-1}(D - J\bar{D} J), \quad T = \frac{1}{2} \mathrm{k}^{-1}(D + J\bar{D} J)\) are the required ones.

Recall the \(C\)-linear isomorphism \(g : \mathfrak{so}(3, H^c) \oplus (H^c)^* = \mathfrak{so}(3, \mathbb{C}) \oplus \mathfrak{so}(3, \mathbb{C})\), \(g(M + a) =

\[
\begin{pmatrix}
\frac{1}{2} \mathrm{tr}(M) & ia \\
-ia^* & \frac{1}{2} \mathrm{tr}(M) E
\end{pmatrix}
\]

which is used to define the homomorphism \(\phi : \mathfrak{sp}(4, H^c) \rightarrow (e_c^c)^{2T}, \quad \phi(A)\mathfrak{X} = g^{-1}(A(\mathfrak{X})^*)\mathfrak{X}, \quad \mathfrak{X} \in \mathfrak{so}(\mathfrak{so}) \) (Theorem 3.4.2). The differential of \(\phi\) is denoted by \(\phi_* : \mathfrak{sp}(4, H^c) \rightarrow (\mathfrak{e}_c^c)^{2T}, \quad \phi_*(S)\mathfrak{X} = g^{-1}(S(\mathfrak{X}) + (\mathfrak{X})S^*), \quad \mathfrak{X} \in \mathfrak{so}(\mathfrak{so})\).

**Proposition 4.5.2.** \(\mathfrak{e}_{c}^{c} = \{ \Phi \in e_{c}^{c} \mid \lambda \gamma \Phi = \Phi \lambda \gamma \} = \{ \Phi(\Phi, A, -\gamma A, 0) \in e_{c}^{c} \mid \phi(\Phi(\Phi, A, -\gamma A, 0)) \}

\[
= \{ \Phi(\Phi, A, -\gamma A, 0) \in e_{c}^{c} \mid \phi(\Phi(\Phi, A, -\gamma A, 0)) \}
\]

\[
\times \{ A \in \mathfrak{so}(\mathfrak{so}) \}
\]

\[
\times \{ A(\mathfrak{X})^* \mathfrak{X}, \quad \mathfrak{X} \in \mathfrak{so}(\mathfrak{so}) \}
\]

\[
\times \{ A(\mathfrak{X})^* \mathfrak{X}, \quad \mathfrak{X} \in \mathfrak{so}(\mathfrak{so}) \}
\]

\[
\times \{ A(\mathfrak{X})^* \mathfrak{X}, \quad \mathfrak{X} \in \mathfrak{so}(\mathfrak{so}) \}
\]

We define a \(C\)-vector space \(\mathfrak{e}(8, C)\) by

\[
\mathfrak{e}(8, C) = \{ Q \in M(8, C) \mid \mathfrak{J} Q = -Q \}
\]
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...and consider its complexification $\mathcal{Z}(8, C)^c$. Now define a $C$-linear isomorphism $\chi: \mathcal{P}^c \to \mathcal{Z}(8, C)^c$ by

$$\chi(X, Y, \xi, \eta) = \left(k(gX - \frac{\xi}{2} E)\right)J + ik\left(k(g(\gamma Y) - \frac{\eta}{2} E)\right)J.$$

**Theorem 4.5.3** $(E, C)^{27} \cong SL(8, C)/\mathbb{Z}_2, \mathbb{Z}_2 = \{E, -E\}$.

**Proof.** We define $\phi: SU(8, C^c) \to (E, C)^{27}$ by

$$\phi(A)P = \chi^{-1}(A(\chi P)^t A), \quad P \in \mathcal{P}^c.$$  

First we have to prove $\phi(A) \in (E, C)^{27}$. To prove this, for the differential $d\phi: \mathfrak{g}\mathfrak{u}(8, C^c) \to (E, C)^{27}$ of $\phi$, $d\phi(D)P = \chi^{-1}(D(\chi P)^t D), P \in \mathcal{P}^c$, it suffices to show $d\phi(D) \in (E, C)^{27}$ (Lemma 0.6).

(1) For $D = k(S), S \in \mathfrak{g}\mathfrak{u}(4, H^c)$, $(P = (X, Y, \xi, \eta) \in \mathcal{P}^c)$

$$\gamma(d\phi(k(S)))P = k(S)(\chi P) + (\chi P)^t k(S)$$

$$= k\left(S\left(gX - \frac{\xi}{2} E\right)\right)J + ik\left(S\left(g(\gamma Y) - \frac{\eta}{2} E\right)\right)J + k\left((gX - \frac{\xi}{2} E)S^*\right)J$$

$$+ ik\left((g(\gamma Y) - \frac{\eta}{2} E)S^*\right)J$$

$$= k(S(gX) + (gX)S^*)J + ik(S(g(\gamma Y)) + (g(\gamma Y))S^*)J$$

$$= k(g(\phi_*(S)X))J + ik(g(\phi_*(S)(\gamma Y)))J = \chi(\phi_*(S)X, \gamma \phi_*(S)Y, 0, 0)$$

$$= \chi(\phi_*(S), 0, 0, 0)(X, Y, \xi, \eta).$$

Hence $d\phi(S) = \Phi(\phi_*(S), 0, 0, 0) \in (E, C)^{27}$.

(2) For $D = ik(T), T \in \mathfrak{g}\mathfrak{u}(4, H^c)$, $(P = (X, Y, \xi, \eta) \in \mathcal{P}^c)$

$$\chi(d\phi(ik(T)))P = ik(T(\chi P) + (\chi P)^t k(T)$$

$$= ik\left(T\left(gX - \frac{\xi}{2} E\right)\right)J - k\left(T\left(g(\gamma Y) - \frac{\eta}{2} E\right)\right)J + ik\left((gX - \frac{\xi}{2} E)T\right)J$$

$$- k\left((g(\gamma Y) - \frac{\eta}{2} E)T\right)J$$

$$= k(-T(g(\gamma Y)) - (g(\gamma Y))T + \eta T)J + ik(T(gX) + (gX)T - \xi T)J$$

$$= k(-2gA + g(\gamma Y) + \eta gA)J + ik(2gA + gX - \xi gA)J$$

$$= k(-2g(\gamma A X + Y) - \frac{1}{2}(A, Y)E + \eta gA) + ik(2g(\gamma A X + Y) + \frac{1}{2}(\gamma A, Y)E$$

$$- \xi gA)J$$ (Lemma 3.4.1)

$$= \chi(-2\gamma A X + Y + \eta A, 2A X - \xi A, 0, (A, Y), (-\gamma A, X))$$
= Z(Φ(0, A, −γ A, 0))(X, Y, ξ, η).

Hence \( d\psi(hk(T)) = \Phi(0, A, −γ A, 0) \in (\mathfrak{e}_\mathfrak{c})^{2\mathfrak{r}} \).

Thus we see that the homomorphism \( \phi : SU(8, C^C) \rightarrow (E_7^C)^{2\mathfrak{r}} \) is well-defined. Since \( (E_7^C)^{2\mathfrak{r}} \) is connected (Lemma 0.7) and \( \dim_c(\mathfrak{su}(8, C^C)) = 36 + 27 \) (Proposition 4.5.2) = 63 = \( \dim_c(gu(8, C^C)) \), \( \phi \) is onto. \( \text{Ker}\phi = \{ E, −E \} = Z_2 \). Thus we have the isomorphism \( (E_7^C)^{2\mathfrak{r}} \cong SU(8, C^C)/Z_2 \cong SL(8, C)/Z_2 \).

**Lemma 4.5.4.** \( \phi : SU(8, C^C) \rightarrow E_7^C \) satisfies

1. \( \tau = \phi(I_2), \quad \tau_c = \phi(J), \quad \tau_u = \phi(iI), \quad \sigma = \phi(I_0), \quad −\sigma = \phi(iI_0) \).
2. \( \tau \gamma(\alpha) \tau = \phi(\tau A), \quad \gamma(\alpha) \gamma = 2\phi(\tau A)^{−1} = \phi(I_1 A I_2), \quad \sigma(\alpha) \sigma = \phi(I_0 A I_3), \quad \tau \sigma(\alpha) \gamma = \phi(\tau A), \quad \alpha(\tau)^{−1} = \phi(\tau A) \).

**Proof.** We shall give the proof only the last formula of (2). Since \( k(x) = −J[x, x], x \in H \), we have \( \chi(tP) = iJ\chi(P)J, P \in \mathbb{R}^C \). Now \( \chi(\phi(\alpha)^{−1}P) = iJ\chi(\phi(\alpha)^{−1})J = iJ\chi(\phi(\alpha))J^\dagger AJ = −iJ\chi(\phi(\alpha))J^\dagger AJ \).

**Theorem 4.5.5.**

(1) \( (E_7^C)^{2\mathfrak{r}} \cong SU(8)/Z_2 \cong (E_7^C)^{2\mathfrak{r}} \).

(2) \( (E_7^C)^{2\mathfrak{r}} \cong SU(2, 6)/Z_2 \cong (E_7^C)^{2\mathfrak{r}} \).

**Proof.** (1) Let \( \alpha \in (E_7^C)^{2\mathfrak{r}} = (\tau \alpha)^{2\mathfrak{r}} \), \( \alpha = \phi(A), A \in SU(8, C^C) \) (Theorem 4.5.3). From \( \tau \gamma \alpha = \alpha \tau \gamma \), we have \( \phi(\tau A) = \phi(A) \) (Lemma 4.5.4). Hence \( \tau A = A \) or \( \tau A = −A \). The latter case is impossible. In fact, put \( A = iB \), then \( B^*B = −E, B \in M(8, C) \), a contradiction. Therefore \( A \in SU(8) \). Thus \( (E_7^C)^{2\mathfrak{r}} = SU(8)/Z_2 \).

(2) Define \( \phi : SU(2, 6, C^C) \rightarrow (E_7^C)^{2\mathfrak{r}} \) by \( \phi(A) = \phi(T^\mathfrak{r}_s A T^\mathfrak{r}_s^{−1}) \). Let \( \alpha \in (E_7^C)^{2\mathfrak{r}} = (\tau \alpha)^{2\mathfrak{r}} \), \( \alpha = \phi(A), A \in SU(2, 6, C^C) \). From \( \tau \alpha = \alpha \tau \), we have \( \phi(\tau A) = \phi(A) \). Thus \( (E_7^C)^{2\mathfrak{r}} = SU(2, 6)/Z_2 \) (cf. Theorem 3.4.5.3). \( (E_7^C)^{2\mathfrak{r}} = (\tau \alpha)^{2\mathfrak{r}} = (\tau \gamma \alpha)^{2\mathfrak{r}} \).

4.5.6. **Proposition 4.3.5.** (3). \( \gamma \sim −\sigma \).

**Proof.** Since \( J \sim iI \) in \( SU(8) \), \( \gamma_c = \phi(J) \sim \phi(iI) = −\sigma \) (Lemma 4.5.4) in \( \phi(SU(8)) \) \( = (E_7^C)^{2\mathfrak{r}} \) (Theorem 4.5.5.1) \( \in E_7^C \). Furthermore \( \gamma \sim \gamma_c \) in \( G_3 \) (Proposition 1.2.3) \( \subset F_4 \subset E_6 \subset E_7 \). Consequently \( \gamma \sim −\sigma \) in \( E_7^C \).

**Theorem 4.5.7.** \( (E_7^C)^{2\mathfrak{r}} \sim (\tau \gamma(\sigma))^{2\mathfrak{r}} = SU(4, 4)/Z_2 \times 2 \cong (\tau \lambda \sigma)^{2\mathfrak{r}} \sim (E_7^C)^{2\mathfrak{r}} \).

**Proof.**

\[ E_7^{2\mathfrak{r}} = (E_7^C)^{2\mathfrak{r}} = (E_7^C)^{2\mathfrak{r}} \]

because \( \gamma \sim \gamma \sigma \) under \( \delta \in F_4 \subset E_6 \subset E_7 ; \delta = \gamma \sigma \delta, \delta \tau = \tau \delta \) (Proposition 2.2.3). Define \( \phi : SU(4, 4, C^C) \rightarrow (E_7^C)^{2\mathfrak{r}} \) by \( \phi(A) = \phi(T^\mathfrak{r}_s A T^\mathfrak{r}_s^{−1}) \). From \( \tau \gamma \sigma \alpha = \alpha \gamma \sigma \), we have
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\[ (E_{1\ell})^{27} \sim (\tau^\alpha)^{27} = (SU(4, 4) \cup i k(0, 1^t, 0) SU(4, 4))/Z_2 = SU(4, 4)/Z_2 \times 2 \] (cf. Theorem 3.4.5.4). \((\phi(ik(0, 1^t, 0))) = \rho_\varepsilon \in E_6 \subset E_7 \) (Theorem 3.4.5.4)). \((E_{1\ell})^{27} = (\tau^\alpha)^{27} \) (Theorem 4.4.5.3). \((E_{1\ell})^{27} = (\tau^\alpha)^{27} \).

**Theorem 4.5.8.**

1. \((E_{1\ell})^{27} \sim (\tau^\alpha)^{27} \cong SU^*(8)/Z_2 \times 2 \equiv (\tau^\alpha)^{27} \).
2. \((E_{1\ell})^{27} \sim (\tau^\alpha)^{27} \cong SL(8, R)/Z_2 \times 2 \).

**Proof.**

1. \(h : SU^*(8, C^0) \to SU(8, C^0), h(A) = \varepsilon A - \varepsilon J^t A^{-1} J \) where \(\varepsilon = \frac{1}{2}(1 + i)\), is an isomorphism which satisfies \(h(\tau A) = -\overline{h(A)} h\). Define \(\phi : SU^*(8, C^0) \to (E_7^\circ)^{27} \) by \(\phi(h(A)) = h(A)\). Now \((E_{1\ell})^{27} = (\tau^\alpha)^{27} \) (Theorem 4.4.5.1). Let \(\alpha \in (\tau^\alpha)^{27}, \alpha = \phi(h(A)), A \in SU^*(8, C^0)\). From \(\tau^\alpha \alpha = \alpha \tau^\alpha\), we have \(\phi(\tau A) = \phi(A)\). Thus \((E_{1\ell})^{27} \sim (\tau^\alpha)^{27} = (SU^*(8) \cup (-i I)SU^*(8))/Z_2 = SU^*(8)/Z_2 \times 2\).

2. \((E_{1\ell})^{27} \equiv (E_7^\circ)^{27} \equiv (E_7^\circ)^{27}\) (Theorem 4.4.5.2) because \(\gamma \sim \gamma_c\) under \(\delta = G_3 \subset E_6 \subset E_7\) : \(\delta = \gamma_c \delta, \delta t = \varepsilon \delta, \delta t \lambda = \tau^\alpha \delta\) (Proposition 1.2.3). Note that \(\phi\) defined in (1) satisfies \(\gamma_c \phi(B) \gamma_c = \phi(JB)\), \(B \in SU^*(8, C^0)\). In fact, since \(h \bar{B} = -J(hB)J\) and \(\bar{B} = -JB, \tau_c \phi(B)\gamma_c = \tau_c \phi(hB)\gamma_c = \phi(J(hB)) = \phi(h\bar{B}) = \phi(\bar{B}) = \phi(JB)\). Define \(\varphi : SL(8, C) \to (E_7^\circ)^{27}\) by \(\varphi(A) = \phi(JA)\) where \(f : SL(8, C) \to SU^*(8, C^0), f(A) = \varepsilon A - \varepsilon JAJ\) where \(\varepsilon = \frac{1}{2}(1 + i)\) (Lemma 0.3).

Now let \(\alpha \in (\tau^\alpha \gamma_c)^{27}, \alpha = \varphi(A), A \in SL(8, C)\). From \(\tau^\alpha \gamma_c \alpha = \tau^\alpha \gamma_c \alpha\), we have \(\varphi(\tau A) = \varphi(A)\). Thus \((E_{1\ell})^{27} = (\tau^\alpha \gamma_c)^{27} \cong (SL(8, R) \cup (i I)SL(8, R))/Z_2 = SL(8, R)/Z_2 \times 2\). \((\varphi(JA) = \gamma_c\).

### 4.6. Subgroups of type \(A_1 \oplus D_6\) of Lie groups of type \(E_7\)

We define \(C\)-linear transformations \(\kappa, \mu\) of \(\Psi^0\) by

\[
\begin{pmatrix}
X \\
Y \\
\xi \\
\eta
\end{pmatrix} =
\begin{pmatrix}
-\kappa_1 X \\
\kappa_1 Y \\
-\xi \\
\eta
\end{pmatrix}, \quad \kappa_1 X = (E_1, X) E_1 - 4E_1 \times (E_1 \times X),
\]
\[ \mu \begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix} = \Phi(0, E_1, E_1, 0) \begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 2E_1 \times Y + \eta E_1 \\ 2E_1 \times Y + \xi E_1 \\ (E_1, Y) \\ (E_1, X) \end{pmatrix}, \]

respectively. Their explicit forms are

\[ \kappa(X, Y, \xi, \eta) = \kappa \left( \begin{array}{cccc} \xi_1 & x_2 & \bar{x}_2 & \eta_1 \\ \eta_2 & x_1 & \bar{x}_1 & \xi_2 \end{array} \right) \]

\[ \mu(X, Y, \xi, \eta) = \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ -\xi_1 & 0 & 0 & 0 \\ 0 & \eta_3 & -\eta_1 & 0 \\ 0 & -\bar{x}_1 & \eta_1 & \xi_3 \end{array} \right) \]

**Lemma 4.6.1.** \( \kappa \mu = -\mu \kappa, \)

\[ \begin{cases} \kappa \sigma = \sigma \kappa \\ \mu \sigma = \sigma \mu \\ \kappa \lambda = -\lambda \kappa \\ \mu \lambda = -\lambda \mu \\ \kappa \iota = \iota \kappa \\ \mu \iota = -\iota \mu \end{cases} \]

We define subgroups \((E_7, c)^{\sigma, \kappa, \mu}, ((E_7, c)^{\sigma, \kappa, \mu})_B\) of \((E_7, c)^{\sigma}\) by

\[ (E_7, c)^{\sigma, \kappa, \mu} = \{ \alpha \in (E_7, c)^{\sigma} | \kappa \alpha = \alpha \kappa, \mu \alpha = \alpha \mu \} , \]

\[ ((E_7, c)^{\sigma, \kappa, \mu})_B = (\sigma, \kappa, \mu)_B \]

\[ = \{ \alpha \in (E_7, c)^{\sigma, \kappa, \mu} | \alpha(0, E_1, 0, 1) = (0, E_1, 0, 1) \} . \]

Their Lie algebras are given as follows.

**Proposition 4.6.2.** (1) \((e_7)^{\sigma} = \{ \Phi \in e_7 | \sigma \Phi = \phi \sigma \} \)

\[ = \{ \Phi(\phi, A, B, \nu) \in e_7 | \phi \in (e_7)^{\sigma}, A, B \in (3^c)_s, \nu \in C \} . \]

(2) \((e_7)^{\sigma, \kappa, \mu} = \{ \Phi \in (e_7)^{\sigma} | \kappa \Phi = \Phi \kappa, \mu \Phi = \Phi \mu \} \)

\[ = \left\{ \Phi(\phi, A, B, \nu) \in e_7 | \phi \in (e_7)^{\sigma}, A, B \in (3^c)_s, (E_1, A) = (E_1, B) = 0, \nu = -\frac{3}{2}(\phi E_1, E_1) \right\} , \]

(3) \(((e_7)^{\sigma, \kappa, \mu})_B = \{ \Phi \in ((e_7)^{\sigma, \kappa, \mu})_B | \Phi(0, E_1, 0, 1) = 0 \} \)

\[ = \{ \Phi(\phi, A, -2E_1 \times A, 0) \in e_7 | \phi \in e_7, \phi E_1 = 0, A \in (3^c)_s, (E_1, A) = 0 \} . \]
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Proof. (1) is easy and (3) is also easy under (2).

(2) Let \( \Phi = \Phi(\phi, A, B, \nu) \in \mathfrak{e}^C \) satisfy \( \kappa \Phi = \Phi \kappa, \mu \Phi = \Phi \mu \). Compare the \( \eta \)-term of \( \kappa \Phi P = \Phi \kappa P, \ P = (X, Y, \xi, \eta) \in \mathfrak{p}^C \), then

\[-(A, Y) = (A, \kappa Y), \ (B, X) = -(B, \kappa X), \ X, Y \in \mathfrak{z}^C.\]

In particular, \( (A, E_1) = (B, E_1) = 0 \). Next compare the \( \eta \)-term of \( \mu \Phi P = \Phi \mu P \), then

\[ (E_1, \phi X) = -\frac{2}{3} \nu(E_1, X), \quad X \in \mathfrak{z}^C. \]

Since \( \phi \in (e^C)^e \), we can put \( \phi E_1 = b E_1, \ k \in C \) (Lemma 3.6.1). Put \( X = E_1 \) in the above, then we have \( k = -\frac{2}{3} \nu \). The converse follows from

Lemma 4.6.3. (1) If \( A \in (\mathfrak{z}^c)_a \), then \( \kappa_1(A \times X) = \kappa_1 A \times \kappa_1 X, \ X \in \mathfrak{z}^c. \)

(2) If \( A \in (\mathfrak{z}^c), \ (E_1, A) = 0 \), then \( \kappa_1 A = -A \).

(3) If \( \phi \in (e^C)^e \), then \( \kappa_1 \phi = \phi \kappa_1 \).

(4) If \( A, B \in (\mathfrak{z}^c)_a, \ (E_1, A) = (E_1, B) = 0 \), then

\[ 4B \times (E_1 \times X) + (E_1, X) A = 4E_1 \times (A \times X) + (B, X) E_1, \quad X \in \mathfrak{z}^c. \]

For \( \nu \in C \), we define a \( C \)-linear transformation \( \phi(\nu) \) of \( \mathfrak{z}^c \) by \( \phi(\nu) = 2\nu E_1 \nu E_1 \), that is,

\[ \phi(\nu) X = \frac{\nu}{3} \begin{pmatrix}
4\xi_1 & x_3 & \xi_2 \\
x_3 & -2\xi_2 & -2x_1 \\
x_3 & -2\xi_1 & -2\xi_2
\end{pmatrix} \]

(c.f. Proposition 3.6.5). Then \( \phi(\nu) \in (e^C)_e \).

Proposition 4.6.4. (1) \( a^C = \{ \Phi(\phi(\nu), a E_1, b E_1, \nu) \in e^C \mid a, b, \nu \in C \} \) is a Lie subalgebra of \( (e^C)^e \) and is isomorphic to the Lie algebra \( \mathfrak{sl}(2, C) = \{ D \in M(2, C) \mid \text{tr}(D) = 0 \} \).

(2) \( (e^C)^e \cong a^C \oplus (e^C)^{e.e, \nu} \) (as Lie algebras).

Proof. (1) The correspondence

\[ \mathfrak{sl}(2, C) \ni \begin{pmatrix} \nu & a \\ b & -\nu \end{pmatrix} \rightarrow \Phi(\phi(\nu), a E_1, b E_1, \nu) \in a^C \]

gives an isomorphism as Lie algebras.

(2) The mapping \( \phi_* : (e^C)_e \rightarrow a^C \oplus (e^C)^{e.e, \nu}, \)

\[ \phi_*(\Phi(\phi, A, B, \nu)) = \Phi(\phi(\nu'), a E_1, b E_1, \nu') + \Phi(\phi - \phi(\nu'), A - a E_1, B - b E_1, \nu - \nu') \]

where \( \nu' = \frac{1}{3} \nu + \frac{1}{2} (E_1, \phi E_1), \ a = (E_1, A), \ b = (E_1, B) \), gives an isomorphism of Lie algebras.
We define a 12-dimensional $\mathbb{C}$-vector space $(V^\mathbb{C})^{12}$ by

$$(V^\mathbb{C})^{12} = \{ P \in \mathbb{C} | \kappa P = P \}$$

$$= \{ (X, \eta, E_1, 0, \eta) \in \mathbb{C} | X \in \mathbb{H}, 4E_1 \times (E_1 \times X) = X, \eta, \eta \in \mathbb{C} \}$$

$$= \left\{ \begin{pmatrix} 0 & 0 & 0 & \eta_1 & 0 & 0 \\ 0 & \xi_2 & x & 0 & 0 & 0 \\ 0 & \bar{x} & \xi_2 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_2, \xi_2, \eta_1, \eta \in \mathbb{C} \end{pmatrix} \right\}$$

with the norm

$$(P, P)_\mu = \frac{1}{2} \{ \mu P, P \} = x \vec{x} - \xi_2 \vec{\xi}_2 + \eta_1 \eta$$

and an 11-dimensional $\mathbb{C}$-vector space $(V^\mathbb{C})^{11}$ by

$$(V^\mathbb{C})^{11} = \{ P \in (V^\mathbb{C})^{12} | P \times (0, E_1, 0, 1) = 0 \}$$

$$= \{ (X, -\eta E_1, 0, \eta) \in \mathbb{C} | X \in \mathbb{H}, 4E_1 \times (E_1 \times X) = X, \eta \in \mathbb{C} \}$$

$$= \left\{ \begin{pmatrix} 0 & 0 & 0 & -\eta & 0 & 0 \\ 0 & \xi_2 & x & 0 & 0 & 0 \\ 0 & \bar{x} & \xi_2 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_2, \xi_2, \eta \in \mathbb{C} \end{pmatrix} \right\}$$

with the norm $$(P, P)_\mu = \frac{1}{2} \{ \mu P, P \} = x \vec{x} - \xi_2 \vec{\xi}_2 - \eta^2.$$
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\((E^+_1)^{g,e}\) is connected.

Proof ([14]). Put \((S^c)^{10} = \{ P \in (V^c)^{11} \mid (P, P)_{\mu} = 1 \}\) (which is a 10-dimensional complex sphere). The group \((\sigma, \kappa, \mu)_{\mathbb{R}}\) acts on \((S^c)^{10}\) (Lemma 4.1.2). We show that this action is transitive. To prove this, it suffices to show that any element \(P \in (S^c)^{10}\) can be transformed to \((0, -E_1, 0, 1) \in (S^c)^{10}\). Now for a given

\[
P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_z & x \\ 0 & \bar{x} & \xi_z \end{pmatrix} \begin{pmatrix} -\eta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \eta) \in (S^c)^{10},
\]

choose \(a \in \mathbb{R}, 0 \leq a \leq \frac{\pi}{2}\) such that \(\tan 2a = \frac{2\text{Re}(\eta)}{\text{Re}(\xi_z + \bar{\xi}_z)}\) (if \(\text{Re}(\xi_z + \bar{\xi}_z) = 0\), then let \(a = \frac{\pi}{4}\)). Operate \(\alpha_{\text{rev}}(a) = (\alpha(a)\alpha^s(A) = \exp(\Phi(0, a(E_z + E_3), -a(E_z + E_3), 0))\) (Lemma 4.3.4) \(\in (\sigma, \kappa, \mu)_{\mathbb{R}}\) (Proposition 4.6.2.(3)) on \(P\), then the real part of \(\eta\)-term of \(\alpha_{\text{rev}}(a)P\) is \(\frac{1}{2}(\xi_z + \bar{\xi}_z)\sin 2a - \eta \cos 2a = 0\). Again choose \(b \in \mathbb{R}, 0 \leq b \leq \frac{\pi}{4}\), such that \(\tan 2b = \frac{2\text{Im}(\eta)}{\text{Im}(\xi_z + \bar{\xi}_z)}\) (if \(\text{Im}(\xi_z + \bar{\xi}_z) = 0\), then \(b = \frac{\pi}{4}\)), then the \(\eta\)-term of \(\alpha_{\text{rev}}(b)\alpha_{\text{rev}}(a)P\) is 0. Hence

\[
P' = \alpha_{\text{rev}}(b)\alpha_{\text{rev}}(a)P \in (S^c)^{10}.
\]

Since \(\text{Spin}(10, C) \subset (\sigma, \kappa, \mu)_{\mathbb{R}}\) acts transitively on \((S^c)^{10}\) (Lemma 3.6.3), there exists \(\beta \in \text{Spin}(10, C)\) such that

\[
\beta P' = (E_z + E_3, 0, 0, 0).
\]

Operate again \(\alpha_{\text{rev}}\left(-\frac{\pi}{4}\right)\) on it, then

\[
\alpha_{\text{rev}}\left(-\frac{\pi}{4}\right)\beta P' = (0, -E_1, 0, 1).
\]

This shows the transitivity of \((\sigma, \kappa, \mu)_{\mathbb{R}}\). The isotropy subgroup of \((\sigma, \kappa, \mu)_{\mathbb{R}}\) at \((0, -E_1, 0, 1)\) is \(\text{Spin}(10, C)\) (Lemma 4.6.5). Thus we have the homomorphism \((\sigma, \kappa, \mu)_{\mathbb{R}}/\text{Spin}(10, C) \cong (S^c)^{10}\).

Proposition 4.6.7. \((E^+_1)^{g,e} \cong \text{Spin}(11, C)\).

Proof. Since the group \((\sigma, \kappa, \mu)_{\mathbb{R}}\) is connected (Lemma 4.6.6), we can define a homomorphism \(\pi : (\sigma, \kappa, \mu)_{\mathbb{R}} \to SO(11, C) = SO((V^c)^{11})\) by \(\pi(\alpha) = \alpha | (V^c)^{11}\). \(\ker \pi = (1, 1) = Z_2\). Hence \(\pi\) induces a monomorphism \(d\pi : ((E^+_1)^{g,e})/Z_2 \to SO(11, C)\). Since \(\dim((E^+_1)^{g,e}) = 45 + 10\) (Lemma 4.6.2.(3)) = 55 = \(\dim SO(11, C)\), \(d\pi\) is onto, hence \(\pi\) is also onto. Thus \((\sigma, \kappa, \mu)_{\mathbb{R}}/Z_2 \cong SO(11, C)\). Therefore
belongs to the group \((E^C)^{a, \kappa, \mu}\).

**Proof.** ([14]). For \(\phi(\nu)=2\nu E_1 \vee E_1 \in (e^\nu)^*\), \(\nu \in C\), \(\Phi(\phi(\nu), 0, 0, -2\nu) \in (e^\nu)^*\) and \(\beta(\nu)=\exp \Phi(\phi(\nu), 0, 0, -2\nu)\). Hence \(\beta(\nu) \in (\sigma, \kappa, \mu)\).

**Lemma 4.6.9.** \((E^C)^{a, \kappa, \mu}/\text{Spin}(11, C) \cong (S^C)^{11}\). In particular, the group \((E^C)^{a, \kappa, \mu}\) is connected.

**Proof ([14]).** Put \((S^C)^{11}=\{P \in (V^C)^{11} \mid (P, P)_n=1\}\) (which is an 11-dimensional complex sphere). The group \((\sigma, \kappa, \mu)\) acts on \((S^C)^{11}\) (Lemma 4.1.2). We show that this action is transitive. To prove this, it suffices to show that any element \(P \in (S^C)^{11}\) can be transformed to \((0, E_1, 0, 1) \in (S^C)^{11}\). For a given

\[
P = \begin{pmatrix}
0 & 0 & 0 \\
0 & \xi_2 & x \\
0 & \bar{x} & \xi_3
\end{pmatrix}, \quad \begin{pmatrix}
\eta_1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad (0, \eta) \in (S^C)^{11},
\]

we shall show that there exists \(\alpha \in (\sigma, \kappa, \mu)\) such that \(\alpha P \in (S^C)^{10}\).

(1) Case \(\eta_1 \neq 0, \eta \neq 0\). Choose \(\nu \in C\) such that \(-e^{2\nu} \eta_1 = e^{2\nu} \eta\). Operate \(\beta(\nu)\) of Lemma 4.6.8 on \(P\), then \(\beta(\nu)P \in (S^C)^{10}\).

(2) Case \(\eta_1 = 0, \eta \neq 0, \xi_3 \neq 0\). Operate \(\alpha = \exp \Phi(0, E_3, 0, 0) \in (\sigma, \kappa, \mu)\) on \(P\), then

\[
\alpha P = (*, \xi_3 E_1, 0, \eta)
\]

which is reduced to the case (1).

(3) Case \(\eta_1 = 0, \eta \neq 0, \xi_3 = 0\) is similar to the case (2).

(4) Case \(\eta_1 = \xi_3 = 0, \eta \neq 0\). Operate \(\alpha = \exp \Phi(0, tF_i(x), 0, 0) \in (\sigma, \kappa, \mu)\) \((t \in R)\) on \(P = (F_i(x), 0, 0, \eta)\), then
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\[ \alpha P = (*, -(2t+r^2)E_1, 0, \eta) \]

which is reduced to the case (1) for some \( t \in \mathbb{R} \).

(5) Case \( \eta_1 \neq 0, \eta = 0, \xi_2 \neq 0 \). Operate \( \alpha = \exp \Phi(0, 0, E_3, 0) \in (\sigma, \kappa, \mu) \) on \( P \), then

\[ \alpha P = (*, \eta_1 E_1, 0, \xi_2) \]

which is reduced to the case (1).

(6) Case \( \eta_1 \neq 0, \eta = 0, \xi_2 = 0 \) is similar to the case (5).

(7) Case \( \eta_1 \neq 0, \eta = 0, \xi_2 = \xi_3 = 0 \). Operate \( \alpha = \exp \Phi(0, 0, tF_1(x), 0) \in (\sigma, \kappa, \mu) \) \((t \in \mathbb{R})\) on \( P = (F_1(x), \eta_1 E_1, 0, 0) \), then

\[ \alpha P = (*, \eta_1 E_1, 0, 2t - \eta_1 t^2) \]

which is reduced to the case (1) for some \( t \in \mathbb{R} \).

(8) Case \( \eta_1 = \eta = 0 \). In this case \( P \in (S^c)^9 \subset (S^c)^{15} \).

Now since the group \( \text{Spin}(10, C)/(\sigma, \kappa, \mu) \) acts transitively on \( (S^c)^{10} \) (Lemma 4.6.6), there exists \( \beta \in \text{Spin}(10, C) \) such that

\[ \beta \alpha P = (0, iE_1, 0, -i) \]

Operate again \( \beta \left( \frac{i\pi}{4} \right) \in (\sigma, \kappa, \mu) \) of Lemma 4.6.8 on it, then

\[ \beta \left( \frac{i\pi}{4} \right) \beta \alpha P = (0, E_1, 0, 1) \]

This shows the transitivity of \((\sigma, \kappa, \mu)\). The isotropy subgroup of \((\sigma, \kappa, \mu)\) at \((0, E_1, 0, 1)\) is \( \text{Spin}(11, C) \) (Proposition 4.6.7). Thus we have the homeomorphism \((\sigma, \kappa, \mu)/Z_2 \approx (S^c)^{11} \).

**PROPOSITION 4.6.10.** \((E_7^c)^{\sigma, \kappa, \mu} \approx \text{Spin}(12, C)\).

**PROOF.** Since the group \((\sigma, \kappa, \mu)\) is connected (Lemma 4.6.9), we can define a homomorphism \( \pi: (\sigma, \kappa, \mu) \rightarrow SO(12, C) = SO((V^c)^{12}) \) by \( \pi(\alpha) = \alpha \mid (V^c)^{12} \). Ker \( \pi = \{1, \sigma\} = Z_2 \). Since \( \epsilon_7 \mid \epsilon_7 \cdot \epsilon_7 = 46 + 10 + 10 \) (Lemma 4.6.2.(2)) = \( 66 = \dim e_8 \sigma(12, C) \), \( \pi \) is onto. Thus \( (\sigma, \kappa, \mu)/Z_2 \approx SO(12, C) \). Therefore \((\sigma, \kappa, \mu)\) is \( \text{Spin}(12, C) \) as the universal covering group of \( SO(12, C) \).

**PROPOSITION 4.6.11.** The group \( E_7^c \) has a subgroup \( \phi(SL(2, C)) \) which is isomorphic to the group \( SL(2, C) \). Where \( \phi(A), A \in SL(2, C) \), is the \( C \)-linear transformation of \( \mathcal{V}^c \) defined by

\[ \phi(A)(X, Y, \xi, \eta) = (X', Y', \xi', \eta'), \]
\[
\begin{pmatrix}
\xi'_1 \\
\eta'_1
\end{pmatrix} = A \begin{pmatrix}
\xi_1 \\
\eta_1
\end{pmatrix} ,
\begin{pmatrix}
\xi'_2 \\
\eta'_2
\end{pmatrix} = A \begin{pmatrix}
\xi_2 \\
\eta_2
\end{pmatrix} ,
\begin{pmatrix}
\xi'_3 \\
\eta'_3
\end{pmatrix} = A \begin{pmatrix}
\xi_3 \\
\eta_3
\end{pmatrix} ,
\begin{pmatrix}
\xi'_4 \\
\eta'_4
\end{pmatrix} = A \begin{pmatrix}
\xi_4 \\
\eta_4
\end{pmatrix} .
\]

\[
\begin{pmatrix}
x'_1 \\
y'_1
\end{pmatrix} = \tau A \begin{pmatrix}
x_1 \\
y_1
\end{pmatrix} ,
\begin{pmatrix}
x'_2 \\
y'_2
\end{pmatrix} = \tau A \begin{pmatrix}
x_2 \\
y_2
\end{pmatrix} ,
\begin{pmatrix}
x'_3 \\
y'_3
\end{pmatrix} = \tau A \begin{pmatrix}
x_3 \\
y_3
\end{pmatrix} ,
\begin{pmatrix}
x'_4 \\
y'_4
\end{pmatrix} = \tau A \begin{pmatrix}
x_4 \\
y_4
\end{pmatrix} .
\]

**Proof ([14]).** The action of \( \Phi(\phi(\nu), aE_1, bE_1, \nu) \in aE_1(a, b, \nu \in C) \) on \( \Psi_c \) is

\[
\Phi(\phi(\nu), aE_1, bE_1, \nu)(X, Y, \xi, \eta) = (X', Y', \xi', \eta')
\]

\[
\begin{pmatrix}
\xi'_1 \\
\eta'_1
\end{pmatrix} = \begin{pmatrix}
\nu & a \\
b & -\nu
\end{pmatrix} \begin{pmatrix}
\xi_1 \\
\eta_1
\end{pmatrix} ,
\begin{pmatrix}
\xi'_2 \\
\eta'_2
\end{pmatrix} = \begin{pmatrix}
\nu & a \\
b & -\nu
\end{pmatrix} \begin{pmatrix}
\xi_2 \\
\eta_2
\end{pmatrix} ,
\begin{pmatrix}
\xi'_3 \\
\eta'_3
\end{pmatrix} = \begin{pmatrix}
\nu & a \\
b & -\nu
\end{pmatrix} \begin{pmatrix}
\xi_3 \\
\eta_3
\end{pmatrix} ,
\begin{pmatrix}
\xi'_4 \\
\eta'_4
\end{pmatrix} = \begin{pmatrix}
\nu & a \\
b & -\nu
\end{pmatrix} \begin{pmatrix}
\xi_4 \\
\eta_4
\end{pmatrix} .
\]

Hence for \( A = \exp \begin{pmatrix}
\nu & a \\
b & -\nu
\end{pmatrix} \in SL(2, C) \) we have \( \phi(A) = \exp \Phi(\phi(\nu), aE_1, bE_1, \nu) \in \phi(SL(2, C)) \in E_c^\nu. \)

**Lemma 4.6.12.** \( \phi : SL(2, C) \rightarrow E_c^\nu \) of Proposition 4.6.11 satisfies

\[
\tau \phi(A) \tau^{-1} = \phi(\tau A), \quad \lambda \phi(A) \lambda^{-1} = \phi(\lambda A), \quad \iota \phi(A) \iota^{-1} = \rho \phi(A) \rho = \phi(IAI),
\]

\[
\gamma \phi(A) \gamma^{-1} = \sigma \phi(A) \sigma = \phi(A).
\]

**Theorem 4.6.13.** \( (E_c)\cong(SL(2, C) \times Spin(12, C))/Z_2, Z_2 = \{(E, 1), (-E, -\sigma)\}. \)

**Proof.** We define \( \phi : SL(2, C) \times Spin(12, C) \rightarrow (E_c)\) by

\[
\phi(A, \beta) = \phi(A) \beta.
\]

Since the algebras \( a_c \) and \( c_c \) are elementwisely commutative (Proposition 4.6.4.(2)), \( A \in SL(2, C) \) and \( \beta \in Spin(12, C) \) are commutative. Hence \( \phi \) is a homomorphism. \( (E_c)\) is connected (Lemma 0.7) and \( \dim(c_c) = 3 + 66 \) (Proposition 4.6.4.(2)) = \( \dim (\mathfrak{sl}(2, C) \oplus \mathfrak{so}(12, C)) \), hence \( \phi \) is onto. \( \ker \phi = \{(E, 1), (-E, -\sigma)\} \) is easily obtained \( (\phi(-E) \equiv \phi(SL(2, C))) \) coincides with \( -\sigma \equiv Spin(12, C) \) (Proposition 4.6.11). Thus we have the required isomorphism.

**Theorem 4.6.14.** (1) \( (E_\tau) \equiv (SU(2) \times Spin(12))/Z_2 \equiv (\tau \lambda \sigma) \equiv (E_{\tau = 0}) \).

(2) \( (E_{\tau = 0}) \equiv (\tau \lambda \sigma) \equiv (SU(2) \times Spin(8, 4))/Z_2. \)

**Proof.** (1) Let \( \alpha \equiv (E_\tau) \equiv ((E_c)\tau^k)\). By Theorem 4.6.13, there exist \( A \in SL(2, C), \beta \in Spin(12, C) \) such that \( \alpha = \phi(A) \beta \). From the condition \( \tau \lambda \alpha = \alpha \tau \lambda \), we have \( \phi(A) \beta = \alpha = \tau \lambda \alpha \tau^{-1} = \tau \lambda \phi(A) \beta \lambda^{-1} \tau = \tau \lambda \phi(A) \lambda^{-1} \tau \lambda \beta \lambda^{-1} \tau = \phi(\tau^k A^{-1}) \tau \lambda \beta \lambda^{-1} \tau. \)
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(Lemma 4.6.12). Hence

\[
\begin{cases}
\tau^t A^{-1} = A \\
\tau \lambda \beta \lambda^{-1} \tau = \beta \\
\tau \lambda \beta \lambda^{-1} \tau = -\sigma \beta
\end{cases}
\]

The latter case is impossible because \((\tau^t A) A = -E\) is false. Therefore \((\tau^t A) A = E\), that is, \(A \in SU(2) = \{ A \in M(2, C) \mid (\tau^t A) A = E, \det A = 1 \}\). To determine the group \(((E_7, C)^{\sigma, \kappa, \mu})^{\tau^t} = (\sigma, \kappa, \mu)^{\tau^t}\), consider an \(R\)-vector space

\[
V^{12} = \langle \mathbb{H}_C, \mu \lambda \rangle = \{ P \in (V^{12})^4 \mid \mu \tau \lambda P = P \}
\]

with the norm \((P, P)_\mu = \frac{1}{2} \{ \mu P, P \} = x \bar{x} + \xi (\tau \xi) + \eta (\tau \eta)\). The group \((\sigma, \kappa, \mu)^{\tau^t}\) acts on \(V^{12}\). Since \((\sigma, \kappa, \mu)^{\tau^t}\) is connected (Lemma 0.7), we can define a homomorphism \(\pi : (\sigma, \kappa, \mu)^{\tau^t} \rightarrow SO(12) = SO(V^{12})\) by \(\pi(\alpha) = \alpha \mid V^{12}\). Ker \(\pi = \{ 1, \sigma \} = Z_2\).

Since \(((E_7, C)^{\sigma, \kappa, \mu})^{\tau^t} = \{ \Phi \in (E_7, C)^{\sigma, \kappa, \mu} \mid \tau \lambda \Phi = \Phi \tau \lambda \} = \{ \Phi(\phi, A, -\tau \lambda, \nu) \in C^2 \mid \phi \in (E_7, C)^{\sigma, \kappa, \mu}, A \in (E_7, C), (E_1, A) = 0, \nu = -\frac{3}{2} (\phi E_1, E_1) \}\) (Propositions 4.3.3.1, 4.6.2.2),

\[
\dim((E_7, C)^{\sigma, \kappa, \mu})^{\tau^t} = 46 + 20 = 66 = \dim \mathfrak{so}(12),\]

hence \(\pi\) is onto. Hence \((\sigma, \kappa, \mu)^{\tau^t} / Z_2 \cong SO(12)\). Therefore \((\sigma, \kappa, \mu)^{\tau^t}\) is \(Spin(12)\) as the universal covering group of \(SO(12)\). Thus \(E(\tau^t, -\lambda) \cong Spin(12)\).

(2) \(E(\tau^t, -\lambda) \cong Spin(12)\) (Theorem 4.4.5.3) \(\cong (E_7, C)^{\tau^t} \sigma\) because \(\sigma \sim \sigma'\) under \(\delta : F_1 \subset E_3 \subset E_7 : \delta \sigma = \sigma' \delta, \delta \tau \lambda = \tau \lambda \delta\) (Proposition 2.2.3). Let \(\alpha \in (\tau \lambda \sigma)\), \(\alpha = \phi(\lambda) \beta, A \in SL(2, C), \beta \in Spin(12, C)\). From \(\tau \lambda \sigma' = \alpha \tau \lambda \sigma'\), we have \(\phi(\tau \lambda A^{-1}) \tau \lambda \sigma' \beta' \sigma' \tau \lambda \tau = \phi(\lambda) \beta\). As similar to (1), \(A \in SU(2)\). To determine the group \(((E_7, C)^{\sigma, \kappa, \mu})^{\tau^t} \sim (\tau \lambda \sigma)^{\tau^t} = (\tau \lambda)^{\tau^t}\), consider an \(R\)-vector space

\[
V^{8, 4} = \langle \mathbb{H}_C, \mu \tau \lambda \sigma \rangle = \{ P \in (V^{12})^4 \mid \mu \tau \lambda \sigma P = P \}
\]

with the norm \((P, P)_\mu = \frac{1}{2} \{ \mu P, P \} = -x \bar{x} + \xi (\tau \xi) + \eta (\tau \eta)\). The group \((\sigma, \kappa, \mu)^{\tau^t} \sigma'\) acts on \(V^{8, 4}\). Since the group \((\sigma, \kappa, \mu)^{\tau^t} \sigma'\) is connected (Lemma 0.7), we can define a homomorphism \(\pi : (\sigma, \kappa, \mu)^{\tau^t} \sigma' \rightarrow O(8, 4) = O(V^{8, 4})\) with Ker \(\pi = \{ 1, \sigma \}\).
\(Z_4\). Since \(\dim((\xi, \eta, \pi)^{2\pi}) = 66 = \dim \mathfrak{o}(8, 4)\), \(\pi\) is onto. Therefore \((\sigma, \kappa, \mu)^{2\pi}\) is denoted by \(\text{spin}(8, 4)\) (not simply connected) as a double covering group of \(O(8, 4)_5\). Thus \((E_{7\times -5})^\sigma \simeq (\tau \lambda \sigma')^\sigma \simeq (SU(2) \times \text{spin}(8, 4))/Z_2\).

**Theorem 4.6.15.** \((E_{7\times -5})^\sigma \simeq (\tau \lambda \rho)^\sigma \simeq (SU(2) \times \text{spin}^*(12))/Z_2 \simeq (\tau \lambda \gamma \rho)^\sigma \simeq (E_{7\times -5})^\sigma\).

**Proof.** \(E_{7\times -5} \cong (E_{7\times -5})^{2\pi}\) (Theorem 4.4.5.2) \(\equiv (E_{7\times -5})^{2\pi}\),

because \(\gamma \sim \rho\) under \(\delta \in E_\sigma \subset E_7: \delta \gamma = \gamma \rho \delta, \delta t = t \rho \delta\) (Proposition 3.2.3). Let \(\alpha \in (E_{7\times -5})^{2\pi}\), \(\alpha = \phi(A)\beta, A \in SL(2, C), \beta \in \text{Spin}(12, C)\). From \(\tau \lambda \rho \alpha \beta = \tau \lambda \rho \phi\), we have \(\phi(\tau^A A^{-1} \tau \lambda \rho \beta \rho t^{-1} \lambda^{-1} \tau = \phi(A)\beta\). Hence

\[
\begin{align*}
\tau^A A^{-1} &= A & \text{or} & \tau^A A^{-1} &= -A \\
\tau \lambda \rho \beta \rho t^{-1} \lambda^{-1} \tau &= \beta & \tau \lambda \rho \beta t^{-1} \lambda^{-1} \tau &= -\beta
\end{align*}
\]

The latter case is impossible (cf. Theorem 4.6.14). Therefore \(A \in SU(2)\). To determine the group \((SU(2) \times \text{spin}^*(12))/Z_2 \simeq O^*(12) = O^*(12)\), consider \(\alpha \in \text{Iso}_C((V^C)^{12})\), \(\alpha = (A)\beta, A \in SL(2, C)\), \(\beta \in \text{Spin}(12, C)\). From \(\alpha A = A\), \(\alpha P\beta = \alpha P \beta\), we have \(\alpha P^\gamma = \alpha P\beta = \alpha P \beta\).

As in Theorem 3.6.10, by the coordinate transformation

\[
\xi = s_1 + is_2, \quad \xi_2 = -s_1 + is_2, \quad \eta = s_3 + is_4, \quad \eta_3 = s_3 - is_4,
\]

we have \((P, P)_a = (s, x) E(s, x)\), \((P, P)_{\lambda \rho} = (s, \lambda \rho) S(s, x)\) where \(s = (s_1, s_2, s_3, s_4)\) and \(S = -2i J \in M(12, C)\). This shows that we have an isomorphism

\[
\{ \alpha \in \text{Iso}_C((V^C)^{12}) | (\alpha P, \alpha P)_a = (P, P)_a, \alpha P \beta = \alpha P \beta \}
\]

\[
\cong \{ A \in M(12, C) | \lambda AA = A, J A = (\tau A) J \} = O^*(12) = O^*(12).
\]

Since the group \((\sigma, \kappa, \mu)^{2\pi}\) is connected, we can define a homomorphism \(\pi: (\sigma, \kappa, \mu)^{2\pi} \to SO^*(12) = O^*(12)_5\) by \(\pi(\alpha) = \alpha | (V^C)^{12}\). Ker \(\pi = \{ 1, \sigma \} = Z_2\). As similar to Theorem 4.6.14, \((\sigma, \kappa, \mu)^{2\pi}\) \((Z_2 \times SO^*(12))\). Therefore \((\sigma, \kappa, \mu)^{2\pi}\) is denoted by \(\text{spin}^*(12)\) (not simply connected) as a double covering group of \(SO^*(12)\). Thus \((\tau \lambda \rho)^\sigma \simeq (SU(2) \times \text{spin}^*(12))/Z_2\).

\(E_{7\times -5} \cong (E_{7\times -5})^{2\pi}\) (Theorem 4.4.5.3) \(\equiv (E_{7\times -5})^{2\pi}\),

because \(\sigma \sim \gamma \rho\) under \(\delta \in E_\sigma \subset E_7: \delta \sigma = \gamma \rho \delta, \delta t = t \rho \delta\) (Proposition 3.2.3). Let \(\alpha \in (E_{7\times -5})^{2\pi}\), \(\alpha = \phi(A)\beta, A \in SL(2, C), \beta \in \text{Spin}(12, C)\). From \(\alpha t \lambda \rho \beta = \tau \lambda \rho \phi\), we have \(\phi(\tau^A A^{-1} \tau \lambda \rho \beta \rho t^{-1} \lambda^{-1} \tau = \phi(A)\beta\). As similar to (1), \(A \in SU(2)\). To determine
the group \( \langle \sigma, \kappa, \mu \rangle^{2} \), consider a \( C \)-vector space \( (V^{c})^{12} = (B^{c})_{c} \) with norms \( (P, P)_{\mu} \) and \( \langle P, P \rangle_{x_{\gamma}^{T} P} \) which is

\[
\mathbb{I} \in \tau \lambda \gamma P, P = (x_{\xi}^{\xi_{2}} (x_{\xi_{2}}^{\xi_{2}} - 2(i \pi x_{x} x, x) - (\tau \eta_{1}) \eta_{1} + (\tau \eta) \eta).
\]

Since \( J \) and \( - J \) are conjugate in \( O(2) \), by a suitable coordinate transformation, \( \langle P, P \rangle_{x_{\gamma}^{T} P} = (x_{s} \tau x')S(x_{\xi}^{\xi_{2}}) \), therefore we have \( (\sigma, \kappa, \mu)^{2} \gamma = \text{spin}^{*}(12) \) (cf. Theorem 3.6.10). Thus \( (\tau \lambda \gamma P)^{\sigma} \equiv (SU(2) \times \text{spin}^{*}(12))/Z_{2} \).

**Theorem 4.6.16.** (1) \( (E_{7(7)})^{\sigma} = (SL(2, R) \times \text{spin}(6, 6))/Z_{2} \times 2 \).

(2) \( (E_{7(7)})^{\gamma} = (SL(2, R) \times \text{spin}(2, 10))/Z_{2} \).

**Proof.** (1) Let \( \alpha, \beta = \phi(A) \beta, A \in SL(2, C) \), \( \beta \equiv \text{Spin}(12, C) \). From \( \tau \gamma \alpha = \alpha \tau \gamma \), we have \( \phi(\tau \gamma) \beta \gamma \tau = \phi(A) \beta \). Hence we have

\[
\begin{aligned}
\tau A = A & \quad \text{or} \quad \tau A = - A \\
\tau \gamma \beta \gamma \tau = \beta & \quad \text{or} \quad \tau \gamma \beta \gamma \tau = - \beta \beta.
\end{aligned}
\]

In the first case, \( A \in SL(2, R) \). To determine the group \( ((E_{7})^{c})^{(\kappa, \mu)}^{\gamma} = (\sigma, \kappa, \mu)^{\gamma} \), consider an \( R \)-vector space

\[
V^{c} = (B^{c})_{c, \gamma_{\gamma}} = \{ P \in (V^{c})^{12} | \tau \gamma P = P \}
\]

with the norm \( (P, P)_{\mu} = \frac{1}{2} \{ \mu P, P \} = x_{x} - x_{x} + 0 + 0 \). As similar to Theorem 4.6.14, the group \( (\sigma, \kappa, \mu)^{\gamma} \) is connected and \( (\sigma, \kappa, \mu)^{\gamma}/Z_{2} \equiv O(6, 6)_{0} = O(V^{c})_{0} \). Therefore \( (\sigma, \kappa, \mu)^{\gamma} \) is denoted by \( \text{spin}(6, 6) \) (not simply connected) as a double covering group of \( O(6, 6)_{0} \). We consider the latter case. \( \rho_{e} \equiv E_{8}^{c} \subset E_{8}^{c} \) of Theorem 3.4.5 (4) satisfies \( \sigma \rho_{e} = \rho \sigma, \kappa \rho_{e} = \rho \kappa, \mu \rho_{e} = - \rho \mu \), hence \( l = \sqrt{\sigma \rho \rho_{e}} \) satisfies \( \sigma l = l \sigma, \kappa l = l \kappa, l \mu = \mu l \) (Lemma 4.6.1), that is, \( l \in (\sigma, \kappa, \mu) = \text{Spin}(12, C) \) and \( l \) satisfies \( \tau \gamma l \tau = - \sigma l \). The explicit form of \( l \) is

\[
l(X, Y, \xi, \eta) = \begin{pmatrix}
i \xi \xi & e x e & -i e \xi \xi \\
e x e & -i \xi \xi & e x_1 \\
i e x e & -i \xi \xi & e x_1
\end{pmatrix}
\]

Hence we can put \( A = (i I) B, B \in SL(2, C), B = l \beta, \beta \in \text{spin}(6, 6) \). Thus \( (E_{7(7)})^{c} \equiv (SL(2, R) \times \text{spin}(6, 6) \cup (i I) SL(2, R) \times \text{spin}(6, 6))/Z_{2} = (SL(2, R) \times \text{spin}(6, 6))/Z_{2} \).
\( (\phi(iI, i) = \rho) \).

(2) Let \( \alpha \in (E_{7(-5)}^\prime)^{\omega} = (\tau)^{\omega} \), \( \alpha = \phi(A) \beta \), \( A \in SL(2, C) \), \( \beta \in \text{Spin}(12, C) \). From \( \tau \alpha = \alpha \tau \), we have \( \phi(\tau A) = \phi(A) \beta \). Hence we have

\[
\begin{cases}
\tau A = A \\
\tau \beta = \beta
\end{cases}
\quad \text{or} \quad
\begin{cases}
\tau A = -A \\
\tau \beta = -\sigma \beta.
\end{cases}
\]

In the first case, \( A \in SL(2, R) \). To determine the group \( (E_{7}^\prime)^{\omega, \tau, \rho} = (\sigma, \kappa, \mu)^{\omega} \), consider an \( R \)-vector space

\[
V^{\tau, -1} = (\mathcal{V})_{\tau} = \{ P \in (V^{\mathcal{V}})_{\tau} | \tau P = P \}
\]

\[
= \left\{ \left( \begin{array}{ccc}
0 & 0 & 0 \\
0 & \xi & x \\
0 & 0 & 0
\end{array} \right), \left( \begin{array}{ccc}
\eta & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array} \right) \right\}_{x, \xi, \eta, \eta \in R}
\]

with the norm \( (P, P)_{\mu} = \frac{1}{2} \{ \mu P, P \} = x \xi - \xi \eta + \eta \cdot \eta \). As similar to (1), the group \( (\sigma, \kappa, \mu)^{\omega} \) is connected and \( (\sigma, \kappa, \mu)^{\omega}/Z_2 \cong O(2, 10)_{\rho} = O(V^{\tau, -1})_{\rho} \). Therefore \( (\sigma, \kappa, \mu)^{\omega} \) is denoted by \( \text{Spin}(2, 10) \) (not simply connected) as a double covering of \( O(2, 10)_{\rho} \). The latter case is impossible. In fact, since \( \beta \) acts on \( V^{\tau, -1} \), \( \beta \) induces a matrix \( B \in M(12, C) \) such that \( \tau B = -B, \) \( \delta B = B = I_8 \). Put \( B = iB', B' \in M(12, R) \), then \( B' I_8 B' = -I_8 \), which is false because the signature of both sides are different. Thus \( (E_{7(-5)}^\prime)^{\omega} = (SL(2, R) \times \text{Spin}(2, 10))/Z_2 \).

We define a subgroup \( SL_1(2, R) \) of \( SL(2, C) \) by \( A \in SL(2, C) | \tau A^{-1} = I A I \).

**Lemma 4.6.17.** \( SL_1(2, R) \cong SL(2, R) \).

**Proof.** The correspondence \( SL_1(2, R) \equiv A \rightarrow \Gamma \alpha \Gamma^{-1} \in SL(2, R) \) where \( \Gamma = \frac{1}{2} \left( \begin{array}{rr}
1 & i \\
i & 1
\end{array} \right) \) gives an isomorphism. (Note \( \Gamma i \Gamma^{-1} = \mathcal{J} \).)

**Theorem 4.6.18.** \( (E_{7(-5)}^\prime)^{\omega, \tau, \rho} \cong (SL(2, R) \times \text{Spin}^{\ast}(12))/Z_2 \times 2 \).

**Proof.** \( E_{7(-5)}^\prime \cong (E_7^\prime)^{\omega, \tau, \rho} \) (Theorem 4.4.5.(3)) \( \equiv (E_7^\prime)^{\omega, \tau, \rho} \) because \( \sigma \sim \gamma \rho \) under \( \delta \in E_7 \subset E_7 : \delta \sigma = \gamma \rho \sigma \), \( \delta \tau \lambda = \tau \delta \lambda \) (Proposition 3.2.3). Let \( \alpha \in (\tau \lambda \rho)^{\omega}, \alpha = \phi(A) \beta \), \( A \in SL(2, C) \), \( \beta \in \text{Spin}(12, C) \). From \( \tau \lambda \rho \alpha = \alpha \tau \lambda \rho \), we have \( \phi(I^A A^{-1}) = \tau \lambda \rho \beta \rho \gamma \lambda^{-1} \tau = \phi(A) \beta \). Hence we have

\[
\begin{cases}
I^A A^{-1} = A \\
\tau \lambda \rho \beta \rho \gamma \lambda^{-1} \tau = \beta
\end{cases}
\quad \text{or} \quad
\begin{cases}
I^A A^{-1} = -A \\
\tau \lambda \rho \beta \rho \gamma \lambda^{-1} \tau = -\sigma \beta.
\end{cases}
\]

In the first case, \( A \in SL_1(2, R) \). To determine the group \( (\sigma, \kappa, \mu)^{\omega, \tau, \rho} \), consider
the $C$-vector space $(V^C)^a=(C^C)_a$ with the norms $(P, P)_\rho$ and
\[
\langle P, P \rangle_{x, y} = - \{ \tau_\gamma \rho P, P \} = \langle (x_\xi)_{\xi_1} - (\xi_\xi)_{\xi_2}, 2(i \tau_\gamma x, x) + (\tau_\gamma)_\eta, (y_\eta)_{\eta_1} - (\tau_\eta)_\eta \rangle
\]
as is seen in Theorem 4.4.15. Hence $(\sigma, \kappa, \mu)^{\tau_\gamma \rho \tau_\gamma \rho} \equiv \text{spin}^*(12)$ (cf. Theorem 3.6.10).
We consider the second case. $a_i = a_i(\frac{\pi}{2}) = \exp \Phi(0, \frac{\pi}{2} E_1, - \frac{\pi}{2} E_1, 0)$ (Lemma 4.3.4) satisfies $\sigma a_i = a_i, \kappa a_i = - \alpha_1, \mu a_i = - \alpha_1 \mu$, hence $l_i = \gamma_c \lambda a_i$ satisfies $\sigma l_i = l_i, \kappa l_i = l_i, \mu l_i = l_i, \mu$ (Lemma 4.6.1), that is, $l_i \in (\sigma, \kappa, \mu) = \text{Spin}(12, C)$ and $l_i$
satisfies $\tau_\gamma \rho l_i, \rho \gamma \lambda \tau = - \sigma l_i$. (The explicit form of $l_i$
is
\[
l_i(X, Y, \xi, \eta) = \begin{pmatrix} -\xi & \gamma_c y_3 & \gamma_c \xi_2 \\ \gamma_c y_3 & \xi_2 & -\gamma_c x_1 \\ \gamma_c x_2 & -\gamma_c \xi_1 & \xi_2 \end{pmatrix},
\]
Hence we can put $A = (iI)B, B \in SL(2, R), B = l_i B', B' \in \text{spin}^*(12)$. Thus $(\tau_\gamma \rho)^a$
$\equiv (SL(2, R) \times \text{spin}^*(12)) \cup (iI) SL(2, R) \times l_i \text{spin}^*(12))/Z_2 \equiv (SL(2, R) \times \text{spin}^*(12))/Z_2 \times 2$ (Lemma 4.6.17). (The explicit form of $\phi(iI, l_i)$
is
\[
\phi(iI, l_i(X, Y, \xi, \eta)) = \begin{pmatrix} -i\xi & \gamma_c y_3 & \gamma_c \xi_2 \\ \gamma_c y_3 & i\xi_2 & i\gamma_c x_1 \\ \gamma_c x_2 & i\gamma_c \xi_1 & i\xi_2 \end{pmatrix},
\]
Appendix

The Cartan decompositions of the exceptional universal linear Lie groups of type $E_7$ are given as follows.

$E_7 :$ simply connected compact Lie group of type $E_7$,

$E_7^{\mathbb{C}} \simeq E_7 \times \mathbb{R}^{133}$,

$E_7(7) = SU(8)/\mathbb{Z}_2 \times \mathbb{R}^{56}$,

$E_7(4) \simeq (SU(2) \times \text{Spin}(12))/\mathbb{Z}_2 \times \mathbb{R}^{34}$,

$E_7(-28) \simeq (U(1) \times E_8)/\mathbb{Z}_2 \times \mathbb{R}^{34}$.

References

[14] Yasukura, O.-Yokota, I., Subgroup $(SU(2) \times Spin(12))/\mathbb{Z}_2$ of compact simple Lie group $E_7$ and non-compact simple Lie group $E_{7,0}$ of type $E_{7(-5)}$, Hiroshima Math., J. 12 (1982), 59-76.

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