TSUKUBA J. MATH.
Vol. 17 No. 2 (1993), 487–490

CHARACTERIZATIONS OF SIMPLY CONNECTED COMPLETE UNITARY-SYMMETRIC KAEHLER MANIFOLDS

Dedicated to Professor Hisao Nakagawa on his sixtieth birthday

By

Hiroshi MORI and Yoshiyuki WATANABE

Abstract. We accomplish the Kaehler version of Choi's characterizations of rotationally symmetric manifolds.

0. Introduction.

A Kaehler manifold M of complex dimension n is said to be unitary-symmetric at a point m of M if the linear isotropy group of automorphisms (that is, holomorphic isometries) of M is the unitary group $U(n)$.

A unitary-symmetric Kaehler manifold is a Kaehler version of a rotationally symmetric manifold (cf. Choi [1], Greene-Wu [2]). The second author [8] has given a characterization of such a Kaehler manifold. Using the result, the present authors have constructed a one parameter family of complete Kaehler metrics on CP^n, the complex projective n-space, which are compatible with the canonical complex structure on it, and have studied the geometry of unitary-symmetric Kaehler manifolds (cf. Watanabe [8], Mori-Watanabe [4], [5], [6]).

Let us fix some notations. Let M be a Kaehler manifold with Kaehler structure (ds^2, J). We denote by ∇ the Levi-Civita connection. The curvature tensor R is defined to be

$$R(X, Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z$$

for any vector fields X, Y, Z on M and the Ricci tensor is denoted by Ric. Further, we denote by Ω the fundamental 2-form, that is, $\Omega(X, Y)=ds^2(JX, Y)$. Let $m\in M$. We define δ to be the distance from the origin O of the tangent space $T_m(M)$ at m to the first conjugate locus $\tilde{Q}_m$ in $T_m(M)$. Define $\tilde{B}_\delta=\{X\in T_m(M) \mid |X|<\delta\}$. Then it is clear that $\tilde{B}_\delta$ becomes a Riemannian manifold equipped with the metric $exp_m^* ds^2$, since $exp_m: \tilde{B}_\delta\to M$ is non-singular. Now we consider the following four conditions.

Received June 1, 1992, Revised January 5, 1993.

(I) (M, ds^2, J) is unitary-symmetric at m.

(II) The metric $exp_m^* ds^2$ and the fundamental 2-form $exp_m^* \Omega$, pulled back under the exponential mapping exp_m, are given by

$$exp_m^* ds^2 = dr^2 + f(r)^2 d\Theta^2 + f(r)^2(f'(r)^2 - 1)\eta \otimes \eta ,$$

$$exp_m^* \Omega = 2f(r)f'(r)\eta \wedge dr + f(r)^2 \Psi$$

on the punctured ball $\tilde{B}_\delta - \{O\}$ of radius δ in $T_m(M)$, where f is a C^∞ odd function on $(-\delta, \delta)$ such that $f'(0)=1$ and $f'(r)>0$. Here we assume that δ is infinite when M is non-compact, and we denote by (r, Θ) the usual polar coordinate system of $C^n \equiv T_m(M)$, by $(d\Theta^2, \phi, \xi, \eta)$ the standard Sasakian structure on the unit sphere S^{2n-1} in $T_m(M)$, and set $\Psi(X, Y) = d\Theta^2(\phi X, Y)$.

(III) The Riemannian curvature tensor R satisfies

$$R(J\gamma', \gamma')\gamma' = h(r)J\gamma' , \qquad R(E(r), \gamma')\gamma' = k(r)E(r) ,$$

where γ' is the tangent vector field of a radial geodesic γ starting from m, $h(r)$, $k(r)$ are functions depending only on the *geodesic distance* r from the origin O, and $E(r)$ is a parallel vector field along γ which is perpendicular to both γ' and $J\gamma'$.

(IV) The exponential image of any complex linear subspace (resp. real subspace spanned by u, w) of $T_m(M)$ is a closed, totally geodesic, complex (resp. real) submanifold of M, where u, Ju and w are orthonormal.

Then our assertion is as follows.

THEOREM. *Let (M, ds^2, J) be a complete, connected, simply-connected, Kaehler manifold of complex dimension $n \geq 2$ and m be a point of M. Then the above conditions I, II, III and IV are equivalent.*

1. Proof of Theorem.

We have already known that (I) is equivalent to (II) (see Watanabe [8]).

We shall show that (III) implies (II). Let γ be a geodesic issuing from m and $E=E(r)$ a parallel vector field along γ such that $E(0)$ is perpendicular to both $\gamma'(0)$ and $J\gamma'(0)$. Then we have the following two kinds of Jacobi fields V and Ξ along γ,

$$V(r) = f(r)E(r) , \qquad \Xi(r) = g(r)J\gamma'$$

for some functions f, g, which satisfy the differential equations

$$f''(r) + k(r)f(r) = 0 , \qquad g''(r) + h(r)g(r) = 0$$

with initial conditions

$$f(0)=0, \qquad f'(0)=1, \qquad g(0)=0, \qquad g'(0)=1,$$

respectively. By applying the Jacobi field argument, a long calculation shows that the Riemannian metric $exp_m^* ds^2$ is given by the form

$$exp_m^* ds^2 = dr^2 + f^2(d\Theta^2 - \eta\otimes\eta) + g^2\eta\otimes\eta$$

(cf. Nakashima-Watanabe [7]). Since (ds^2, J) is a Kaehler structure, we can see that $g=ff'$. Further, since $f>0$ and $f'>0$ on $(0, \delta)$, it follows from the assumption on δ that if $\delta<\infty$, then $f'(\delta)=0$. Thus we have the condition (II).

We shall show that (I) and (II) imply (IV). Let W be a complex linear subspace of $T_m(M)$. Then there exists a unitary matrix φ which leaves W pointwise fixed but $\varphi(X)\neq X$ for every $X\in T_m(M)-W$. From the assumption that (M, ds^2, J) is unitary-symmetric at m, it follows that there exists an automorphims Φ of (M, ds^2, J) such that $(\Phi_*)_m=\varphi$. From this the image of W under the exponential map exp_m is the fixed point set of the isometry Φ of (M, ds^2), which implies that the image set $exp_m(W)$ is a totally geodesic submanifold of (M, ds^2). By restricting the structures (ds^2, J) to the vectors tangent to $exp_m(W)$ we see that $exp_m(W)$ is an almost Hermitian submanifold of (M, ds^2, J). Thus the first assertion is true (see Kobayashi-Nomizu [3], p. 171).

From the first assertion, it suffices to prove the second assertion in the case $n=2$. We adopt a polar coordinate system $\psi(t, \theta_1, \theta_2, \theta_3)=(t\cos\theta_1\cos\theta_2\cos\theta_3,$ $t\cos\theta_1\cos\theta_2\sin\theta_3,\ t\cos\theta_1\sin\theta_2,\ t\sin\theta_1)$, $0<t<\infty$, $-\pi/2<\theta_1,\ \theta_2<\pi/2$, $-\pi<\theta_3<\pi$ for $T_m(M)\equiv C^2$. Then we shall show that the submanifold $exp_m\{\psi(t, 0, \theta_2, 0)\,|\,t\in R,\ -\pi/2\leqq\theta_2\leqq\pi/2\}$ is a closed, totally geoderic submanifold. We find that with respect to local coordinates $w_1=t$, $w_{i+1}=\theta_i$, $i=1, 2, 3$, the components g_{ij} of the Riemannian metric g are given by $g_{1j}=\delta_{1j}$, $j=1, 2, 3, 4$, $g_{i+1i+1}=f(t)^2(\lambda_i+(f'(t)^2-1)\eta_i^2)$, $i=1, 2, 3$, $g_{i+1j+1}=f(t)^2(f'(t)^2-1)\eta_i\eta_j$, $i\neq j$, where $\lambda_1=1$, $\lambda_2=\cos^2 w_2$, $\lambda_3=\eta_3=\cos^2 w_2\cos^2 w_3$, $\eta_1=\sin w_3$, $\eta_2=-\sin w_2\cos w_2\cos w_3$. From this observation it follows that the Christoffel's symbols satisfy $\Gamma^i_{jk}=0$ for $i=2, 4$ and $j, k=1, 3$, when $0<|w_1|<\infty$, $-\pi/2<w_3<\pi/2$ and $w_2=w_4=0$. Thus, the second assertion is true.

Finally, we shall show that (IV) implies (III). Let u, Ju and w be orthonormal vectors in $T_m(M)$ and consider the geodesic $\gamma(r)=exp_m ru$, $r\in R$. Set $P=exp_m\, span\{u, Ju\}$, $Q=exp_m\, span\{u, w\}$. Denote by $E(r)$ a unit vector field (in Q) along $\gamma(r)$ which is perpendicular to $\gamma'(r)$ and satisfies $E(0)=w$. Since P and Q are 2-dimensional totally geodesic submanifolds of M, we find that $E(r)$ is a (uniquely determined) parallel field along γ which is perpendicular to both γ' and $J\gamma'$ and that

$$R(J\gamma'(r), \gamma'(r))\gamma'(r)=h(r)J\gamma'(r), \qquad R(E(r), \gamma'(r))\gamma'(r)=k(r)E(r),$$

for some functions $h(r)$ and $k(r)$. Thus we have the condition (III).

Acknowledgement. The authors would like to thank the referee for some comments.

References

[1] H. I. Choi, Characterizations of simply connected rotationally symmetric manifolds, Trans. Amer. Math. Soc. 275 (1983), 723-727.

[2] R. E. Greene and H. Wu, Function theory on manifolds which possess a pole, Lecture Notes in Math. Springer-Verlag, Berlin and New York, 1979.

[3] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry II, Interscience Tract, 1969.

[4] H. Mori and Y. Watanabe, A Kähler deformation of CP^n, Proc. Amer. Math. Soc., 103 (1988), 910-912.

[5] H. Mori and Y. Watanabe, Geometric properties of unitary-symmetric Kaehler manifolds, Comptes Rendus Math. Acad. Rep. Sci. Canada, 11 (1989), 41-45.

[6] H. Mori and Y. Watanabe, Notes on unitary-symmetaic Kaehler manifolds, Math. Rep. Toyama Univ., 12 (1989), 167-179.

[7] Y. Nakashima and Y. Watanabe, Some constructions of almost Hermitian and quaternion metric structure, Math. J. Toyama Univ., 13 (1990), 119-138.

[8] Y. Watanabe, Unitary-symmetric Kählerian manifolds and pointed Blaschke manifolds, Tsukuba J. Math., 12 (1988), 129-148.

Hiroshi Mori
Department of Mathematics
Joetsu University of
Education Joetsu
Niigata Pref. 943
Japan

Yoshiyuki Watanabe
Department of Mathematics
Faculty of Science
Toyama University
Toyama 930
Japan