AUTOMORPHISMS OF ORDER 4 OF THE SIMPLY CONNECTED COMPACT LIE GROUP $E_6$

By

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Using the theory of Kac-Moody Lie algebras, for compact simple Lie algebra $g$, automorphisms $\rho$ of finite order of $g$ can be classified and the type of Lie subalgebras $g^\rho$ of fixed points are determined [1]. Now for the simply connected compact Lie group $E_6$, we realize automorphisms $\rho$ of order 4 and determine the subgroups $(E_6)^\rho$ of fixed points. Among compact exceptional Lie groups, only $E_6$ has outer automorphisms, so we consider the case of $E_6$. As results, the group $E_6$ has eight inner automorphisms named as $\gamma_1, \gamma_2, \ldots, \gamma_4, \sigma_1, \sigma_2, \sigma_3$, and three outer automorphisms named as $\tau\gamma_1', \tau\gamma_2, \tau\sigma_8$, and the subgroups $(E_6)^\rho$ of fixed points are given as follows.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$(e_6)^\rho$</th>
<th>$(E_6)^\rho$</th>
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</thead>
<tbody>
<tr>
<td>$\gamma_1$</td>
<td>$T^1 \oplus A_1 \oplus A_4$</td>
<td>$(Sp(1) \times S(U(1) \times U(5))) / Z_4$</td>
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<tr>
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<td>$T^1 \oplus A_1 \oplus A_1 \oplus A_4$</td>
<td>$(Sp(1) \times S(U(2) \times U(4))) / Z_4$</td>
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<tr>
<td>$\gamma_2$</td>
<td>$T^1 \oplus A_1 \oplus A_3 \oplus A_3$</td>
<td>$(Sp(1) \times S(U(3) \times U(3))) / Z_5$</td>
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<tr>
<td>$\gamma_3$</td>
<td>$T^1 \oplus A_3$</td>
<td>$(U(1) \times SU(6)) / Z_2$</td>
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</tr>
<tr>
<td>$\gamma_5$</td>
<td>$T^2 \oplus A_1 \oplus A_3^\prime$</td>
<td>$(U(1) \times S(U(2) \times U(4))) / Z_2$</td>
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<tr>
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<td>$T^1 \oplus D_8$</td>
<td>$(U(1) \times Spin(10)) / Z_4$</td>
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<td>$(U(1) \times (Spin(2) \times Spin(8))) / (Z_2 \times Z_4)$</td>
</tr>
<tr>
<td>$\tau\gamma_1'$</td>
<td>$A_1 \oplus D_8$</td>
<td>$(Sp(1) \times SO(6)) / Z_2$</td>
</tr>
<tr>
<td>$\tau\gamma_2$</td>
<td>$T^1 \oplus C_3$</td>
<td>$(U(1) \times Sp(3)) / Z_2$</td>
</tr>
<tr>
<td>$\tau\sigma_8$</td>
<td>$A_1 \oplus B_2$</td>
<td>$(SU(2) \times Spin(7)) / Z_2$</td>
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1. Preliminaries

Let $\mathbb{C} = H \oplus He$ ($H$ is the field of quaternions with the basis $\{1, i, j, k\}$) be the Cayley algebra with the multiplication $(m + ae)(n + be) = (mn - \bar{b}a) + (a\bar{n} + bm)e$, the conjugation $\bar{m + ae} = \bar{m} - ae$, the inner product $(x, y) = (xy + yx)/2$ and the length $|x| = \sqrt{(x, x)}$, and $\mathbb{C}^e$ be its complexification. Let $\mathfrak{Z} = \{ X \in M(3, \mathbb{C}) | X^* = X \}$
be the exceptional Jordan algebra with the multiplication $X \cdot Y = (XY + YX)/2$ and $\mathfrak{J}$ be its complexification. $\mathfrak{J}$ and $\mathfrak{J}^c$ have the inner product $(X, Y) = \text{tr}(XY)\text{tr}(YX) - (X, Y)E)/2$ and the determinant $\det X = (X \times X, X)/3$. $(\mathfrak{J}(3, \mathbf{H}) = \{M \in M(3, \mathbf{H}) | M^* = M\}$ and $\mathfrak{J}(3, \mathbf{H})^c$ are also defined). The complex conjugations of $\mathfrak{J}^c$, $\mathfrak{J}^c$ are denoted by $\tau$. In $\mathfrak{J}^c$, the positive definite inner product $\langle X, Y \rangle$ is defined by $\langle \tau X, Y \rangle$. Now

$$E_+ = \{\alpha \in \text{Isoc}(\mathfrak{J}^c) | \det \alpha X = \det X, \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\}$$

is the simply connected compact Lie group of type $E_8$ [2]. Throughout this paper, we use such notations and theorems in [4] as $E, E_i, F_i(x), i = 1, 2, 3$ of $\mathfrak{J}$, $\mathfrak{J}^c$ and Lie subgroups $F_i = \{\alpha \in E_8 | \alpha = \alpha \tau\} = \{\alpha \in E_8 | aE = E\}$, $\text{Spin}(9) = \{\alpha \in F_i | aE_1 = E_1\}$, $\text{Spin}(10) = \{\alpha \in E_8 | aE_1 = E_1\}$ of $E_8$ etc..

2. Inner automorphisms $\gamma_1, \gamma_2, \cdots, \gamma_s$ of order 4 of $E_8$

The field $\mathbf{H}$ is embedded in $M(2, \mathbf{C})$ by $k : \mathbf{H} = \mathbb{C} \oplus \mathbb{C}j \to M(2, \mathbf{C})$ (where $\mathbb{C} = \{x + yi | x, y \in \mathbb{R}\}$) by $k(a + bj) = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, a, b \in \mathbb{C}$. This $k$ is naturally extended to $\mathbb{R}$-linear mappings $k : M(3, \mathbf{H}) \to M(6, \mathbf{C})$, $k : \mathbf{H}^c \to M(2, 6, \mathbf{C})$. Moreover these $k$ are extended to $\mathbb{C}$-$\mathbb{C}$-linear isomorphisms $k : M(3, \mathbf{H})^c \to M(6, \mathbf{C})$, $k : (\mathbf{H}^c)^c \to M(2, 6, \mathbf{C})$,

$$k(M_1 + iM_2) = k(M_1) + ik(M_2), \quad M_i \in M(3, \mathbf{H}),$$

$$k(a_1 + ia_2) = k(a_1) + ik(a_2), \quad a_i \in \mathbf{H}^c.$$ Finally we define the $\mathbb{C}$-vector space $\mathfrak{J}(6, \mathbf{C})$ by $\{S \in M(6, \mathbf{C}) | S = -S\}$ and the $\mathbb{C}$-$\mathbb{C}$-linear isomorphism $k_J : \mathfrak{J}(3, \mathbf{H})^c \to \mathfrak{J}(6, \mathbf{C})$ by

$$k_J(M_1 + iM_2) = k(M_1)J + ik(M_2)J, \quad M_i \in \mathfrak{J}(3, \mathbf{H})$$

where $J = \text{diag}(J, J, J)$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

In $\mathfrak{J}(3, \mathbf{H})^c \oplus (\mathbf{H}^c)^c$, we define the Freudenthal multiplication [2] as

$$(M + a) \times (N + b) = (M \times N - \frac{1}{2} (a^*b + b^*a)) - \frac{1}{2} (aN + bM).$$

Then $\mathfrak{J}^c$ is isomorphic to $\mathfrak{J}(3, \mathbf{H})^c \oplus (\mathbf{H}^c)^c$ by the correspondence
Automorphisms of order 4 of the Lie group $E_6$

\[
\begin{pmatrix}
\xi_1 & x_1 & \xi_2
\\
x_2 & \xi_1 & \xi_2
\\
\xi_1 & x_2 & \xi_1
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
\xi_1 & m_1 & \overline{m}_2
\\
\overline{m}_1 & \xi_2 & m_3
\\
m_2 & \overline{m}_1 & \xi_3
\end{pmatrix} + (a_1, a_2, a_3)
\]

(where $x_1 = m_1 + a_1 e$, $m_1, a_1 \in H^c$) as Freudenthal algebra [4]. Hereafter we identify $3^c$ and $3(3, H)^c \oplus (H^3)^c$. We define an involutive $C$-linear mapping $\gamma : 3^c \rightarrow 3^c$ by

$$\gamma(M + a) = M - a, \quad M + a \in 3(3, H)^c \oplus (H^3)^c = 3^c.$$ 

Then $\gamma \in E_6$ and $\gamma^2 = 1$.

**Proposition 2.1.** $(E_6)^{\gamma} \cong (Sp(1) \times SU(6)) / Z_2$, $Z_2 = \{(1, E), (-1, -E)\}$.

**Proof.** Let $Sp(1) = \{p \in H | \bar{p}p = 1\}$ and $SU(6) = \{A \in M(6, C) | A^*A = E, \det A = 1\}$. Now the mapping $\phi : Sp(1) \times SU(6) \rightarrow (E_6)^\gamma$,

$$\phi(p, A)(M + a) = k^{-1}(Ak(M)A^*A + \bar{k}^{-1}(k(A)A^*), \quad M + a \in 3^c$$

induces the required isomorphism. The details of proof are in [2] or [4].

**Remark.** $\phi : Sp(1) \times SU(6) \rightarrow (E_6)^\gamma$ satisfies $\gamma = \phi(-1, E)$ and $\tau \phi(p, A) = \phi(p, -J\bar{A})$.

Using $\phi : Sp(1) \times SU(6) \rightarrow (E_6)^\gamma$ of Proposition 2.1, we define

$$\gamma_1 = \phi(1, i1), \quad I_1 = \text{diag} (-1, 1, 1, 1, 1, 1),$$
$$\gamma_2 = \phi(1, i1), \quad I_2 = \text{diag} (-1, -1, 1, 1, 1, 1),$$
$$\gamma_3 = \phi(1, 1),$$
$$\gamma_4 = \phi(e, eI_1), \quad e = (1 + i) / \sqrt{2}, \quad I_1 = \text{diag} (i, 1, 1, 1, 1, 1),$$
$$\gamma_5 = \phi(1, I_2), \quad I_2 = \text{diag} (-1, -1, 1, 1, 1, 1).$$

Then $\gamma_i \in E_6$ and the order of $\gamma_i$ is 4, for $i = 1, 2, \ldots, 5$.

**Theorem 2.2.** (1) $(E_6)^{\gamma_1} \cong (Sp(1) \times SU(U(1) \times U(5))) / Z_2$,
(2) $(E_6)^{\gamma_2} \cong (Sp(1) \times SU(U(3) \times U(3))) / Z_3$,
(3) $(E_6)^{\gamma_3} \cong (U(1) \times SU(6)) / Z_2$,
(4) $(E_6)^{\gamma_5} \cong (U(1) \times SU(U(2) \times U(4))) / Z_2$

where $Z_2 = \{(1, E), (-1, -E)\}$ in any case.

**Proof.** (1) Since $\gamma_1^2 = \gamma$, we have $(E_6)^{\gamma_1} \subset (E_6)^\gamma$. Hence, for $\alpha \in (E_6)^{\gamma_1}$ there exist $p \in Sp(1), A \in SU(6)$ such that $\alpha = \phi(p, A)$ (Proposition 2.1). From the
condition $\gamma_1 a = a \gamma_1$, we have $\phi(p, iA) = \phi(p, iA_1)$, that is, $I_1 A = A_1 I$, therefore $A \in S(U(1) \times U(5))$. Thus we have the required isomorphism.

(2), (3), (4) are proved to be similar to (1).

**Theorem 2.3.** $(E_6)^4 \cong (U(1) \times S(U(1) \times U(5)))/Z_{\mathbb{Z}}$, $Z_{\mathbb{Z}} = \{(1, E), (-1, -E)\}$.

**Proof.** Since the operation of $\gamma_4$ on $\mathfrak{g}^c = \mathfrak{g}(3, H)^c \oplus (H)^c$ is given by

$$
\gamma_4 \left( \begin{array}{c} \xi_1 \\ m_3 \\ \bar{m}_3 \\ \xi_2 \\ m_1 \\ \bar{m}_1 \\ \xi_3 \\ m_2 \\ \bar{m}_2 \\ \end{array} \right) = \left( \begin{array}{c} -\xi_1 \\ i\epsilon m_3 \\ i\epsilon \bar{m}_3 \\ i\epsilon \xi_2 \\ im_1 \\ i\epsilon \bar{m}_1 \\ i\epsilon \xi_3 \\ im_2 \\ i\epsilon \bar{m}_2 \\ \end{array} \right) + \langle \epsilon a_i \xi, -\epsilon a_i \xi, -\epsilon a_i \xi \rangle
$$

(where $\epsilon = (1+i)/\sqrt{2}$), the eigen $C$-vector spaces $(\mathfrak{g}^c)_\nu, \nu = 1, -1, i, -i$ with respect to $\gamma_4$ are

$$(\mathfrak{g}^c)_1 = \{M + a \in \mathfrak{g}(3, H)^c \oplus (H)^c | \gamma_4(M + a) = M + a\}$$

$$= \{ (a_1 (i-i), (i+i) a_2, (i+i) a_3) | a_1 \in C_j, a_2, a_3 \in H \},$$

$$(\mathfrak{g}^c)_{-1} = \{M + a \in \mathfrak{g}(3, H)^c \oplus (H)^c | \gamma_4(M + a) = -M - a\}$$

$$= \{ (i, i)a_1 (i-i), (i+i) a_2, (i+i) a_3) | a_1 \in C_j, a_2, a_3 \in H \},$$

$$(\mathfrak{g}^c)_i = \{M + a \in \mathfrak{g}(3, H)^c \oplus (H)^c | \gamma_4(M + a) = i(M + a)\}$$

$$= \{ (0, (i-i)a_3, (i+i) a_3) | a_1 \in C, a_2, a_3 \in H \},$$

$$(\mathfrak{g}^c)_{-i} = \{M + a \in \mathfrak{g}(3, H)^c \oplus (H)^c | \gamma_4(M + a) = -i(M + a)\}$$

$$= \{ (a_1, (i-i)a_3, (i+i) a_3) | a_1 \in C, a_2, a_3 \in H \},$$

where $C_j = \{ sj + tk | s, t \in R \}$. These spaces are invariant under the group $(E_6)^4$.

We shall show that $(H)^c$ is invariant under $(E_6)^4$. From the forms of $(\mathfrak{g}^c)_n$, it is sufficient to show that $aa(\mathfrak{g}(H)^c$ for $a \in (E_6)^4$ and $a = (a(i+i), 0, 0) = F_i((a(i+i)) e) (a \in C_j)$. Now, in fact,
Automorphisms of order 4 of the Lie group $E_6$

$$aF_i((a+i)\alpha)=4\alpha((F_3((i+i)\alpha)\times F_1(1))\times F_3(\alpha))$$

$$=4(aF_i((i\alpha)\times aF_1(1))\times \tau\alpha aF_3(\alpha))$$

$$\in 4(3(3, H)^c \times 3(3, H)^c)\times (H^c)^c \subset 3(3, H)^c \times (H^c)^c \subset (H^c)^c.$$ 

Thus we see that $(H^c)^c$ is invariant under $(E_6)^4$, hence $3(3, H)^c = (H^c)^c$ for all $Y \in (H^c)^c$. Consequently, $\alpha \in (E_6)^4$ commutes with $\gamma$, that is, $(E_6)^c \subset (E_6)^c$. Hence, for $\alpha \in (E_6)^4$, there exist $p \in Sp(1), A \in SU(6)$ such that $\alpha = \phi(p, A)$ (Proposition 2.1). From the condition $\gamma \alpha = \alpha \gamma$, we have $\phi(\mathbf{ep}, e\Gamma, A) = \phi(p, A\epsilon\Gamma)$, that is, $\mathbf{ep} = le, \Gamma, A = A\Gamma$ (or $\mathbf{ep} = -pe, \Gamma, A = -A\Gamma$ (which is impossible)), therefore $p \in U(1), A \in S(U(1) \times U(5))$. Thus we have the required isomorphism.

3. Inner automorphisms $\sigma, \sigma_2, \sigma_3$ of order 4 of $E_6$

Let $U(1) = \{\theta \in C | (\tau \theta) \theta = 1\}$ (where $C = R^c$) and we define an embedding $\phi: U(1) \rightarrow E_6$ by

$$\phi(\theta) = \left(\begin{array}{ccc}
\xi_1 & x_3 & \bar{x}_2 \\
\bar{x}_3 & \xi_2 & x_1 \\
x_2 & \bar{x}_1 & \xi_3
\end{array}\right) = \left(\begin{array}{ccc}
\theta^4 \xi_1 & \theta x_3 & \theta \bar{x}_2 \\
\theta \bar{x}_3 & \theta^2 \xi_3 & \theta^2 x_1 \\
\theta x_2 & \theta^{-2} \bar{x}_1 & \theta^{-2} \xi_3
\end{array}\right)$$

and put $\sigma = \phi(-1) \in E_6$.

The group $Spin(10)$ is defined by $(E_6)^e = \{a \in E_6 | aE_1 = E_1\}$ which is the covering group of $SO(10) = SO(V^{10})$ where $V^{10} = \{X \in \mathbb{R}^c | 2E_1 \times X = -\tau x\} = \{x E_2 - \tau E_3 + F_i(x) | \xi \in C, x \in \mathbb{C}\}$. Note that $Spin(10)$ leaves invariant $\{X \in \mathbb{R}^c | E_1 \times X = 0\} = \{F_i(x) + F_j(y) | x, y \in \mathbb{C}\}$.

**Proposition 3.1.** $(E_6)^e \equiv (U(1) \times Spin(10))/\mathbb{Z}_4, \mathbb{Z}_4 = \{1, 1, (-1, \sigma), (i, \phi(-i)), (-i, \phi(i))\}$.

**Proof.** The mapping $\phi: U(1) \times Spin(10) \rightarrow (E_6)^e$,

$$\phi: (\theta, \beta) = (\theta, \beta)$$

induces the required isomorphism. The details of proof are in [2] or [4].

Using $\phi: U(1) \rightarrow E_6$ or $\phi: Sp(1) \times SU(6) \rightarrow E_6$ of Proposition 2.1, we define

$$\sigma_1 = \phi(i) = \phi(-1, \Gamma), \quad \Gamma = \text{diag}(1, 1, i, i, i, i),$$

$$\sigma_2 = \gamma \sigma_1 = \phi(1, \Gamma).$$

Then $\sigma_1 \in E_6$ and $\sigma_1^4 = \sigma, \sigma_1^4 = 1$ for $i = 1, 2$.

**Theorem 3.2.** $(E_6)^e \equiv (U(1) \times Spin(10))/\mathbb{Z}_4, \mathbb{Z}_4 = \langle i, \phi(-i) \rangle$. 
Proof is clear from Proposition 3.1, because $\sigma_1=\phi(i)$ commutes with any elements of $U(1)$ and $Spin(10)$.

**Theorem 3.3.** $(E_8)^{\ast\ast} \cong (Sp(1) \times SU(2) \times U(4))/Z_3$, $Z_3 = \{(1, E), (-1, -E)\}$.

**Proof.** Since $\sigma_1=\sigma$, we have $(E_8)^{\ast\ast} \subset (E_8)^{\ast\ast}$. Hence, for $\alpha \in (E_8)^{\ast\ast}$, there exist $\theta \in U(1)$, $\beta \in Spin(10)$ such that $\alpha = \phi(\theta)\beta$ (Proposition 3.1). In particular, $\alpha$ commutes with $\sigma_1=\phi(i)$. Therefore, from the condition $\sigma_2\alpha = a\sigma_2$, that is, $\gamma_1\alpha = a\gamma_1$, we have $\gamma_2 = a\gamma_2$, namely $\alpha \in (E_8)^{\ast\ast}$. Hence there exist $p \in Sp(1)$, $A \in SU(6)$ such that $\alpha = \phi(p, A)$ (Proposition 2.1). Moreover from the condition $\alpha_1=\alpha\sigma_1$, we have $\phi(p, \Gamma_4A) = \phi(p, A\Gamma_4)$, that is, $\Gamma_4A = A\Gamma_4$, therefore $A \in S(U(2) \times U(4))$. Thus we have the required isomorphism.

**Remark.** The group $(E_8)^{\ast\ast}$ also has the following expression

$$(E_8)^{\ast\ast} \cong (U(1) \times Sp(1) \times (SU(2) \times SU(4)))/(Z_3 \times Z_4)$$

where $Z_3 = \langle (1, -1, -I_4) \rangle$, $Z_4 = \langle (-i, 1, -i) \rangle$. In fact, for $\alpha = \phi(p, A) \in (E_8)^{\ast\ast}$, $p \in Sp(1)$, $A = (P, Q) = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \in S(U(2) \times U(4))$, the condition that $\alpha$ belongs to the group $(E_8)^{\ast\ast}$, that is, $\phi(p, (P, Q)) E_1 = E_1$, is $p \in Sp(1)$, $P \in SU(2)$, $Q \in SU(4)$. From this we have easily the required isomorphism.

The field $\mathbb{C}$ of complex numbers is embedded in $\mathbb{C}$ as $\mathbb{C} = \{x + ye \mid x, y \in \mathbb{R}\}$ and put $C^4 = \{t \in \mathbb{C} \mid (t, C) = 0\}$. Let $Spin(2) = \{a \in \mathbb{C} \mid \bar{a}a = 1\} (\cong U(1))$ and we define an embedding $D : Spin(2) \rightarrow E_8$ by

$$D_a \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_2 & \xi_3 & x_1 \\ \bar{x}_1 & \xi_2 & x_2 \end{pmatrix} \begin{pmatrix} x_3 & \bar{x}_2 & \xi_3 \\ \bar{x}_3 & x_2 & \xi_2 \\ -x_1 & \bar{x}_1 & \xi_1 \end{pmatrix} = \begin{pmatrix} a \bar{x}_2 & \bar{x}_3 & ax_1a \\ \bar{x}_3 & x_2 & \bar{x}_1a \\ -x_1 & \bar{x}_1 & \xi_1 \end{pmatrix}.$$ 

Put $\sigma_8 = D_{-e}$. Then $\sigma_8 \in E_8$ and $\sigma_8^2 = \sigma$, $\sigma_8^3 = 1$.

The group $Spin(8)$ is defined by

$$(E_8)_{E_8} = \{a \in E_8 \mid aE_1 = E_1, \alpha F_i(s) = F_i(s) \text{ for all } s \in \mathbb{C}^+\}$$

$$= \{a \in Spin(10) \mid aF_i(1) = F_i(1), \alpha F_i(\phi) = F_i(\phi)\}$$

which is the covering group of $SO(8) = SO(V^8)$ where $V^8 = (V^8)_{E_8} = \{x_3 - \tau x_3 + F_i(t) | \xi \in C, t \in C^+\}$.

**Lemma 3.3.** $D_a (a \in Spin(2))$ and $\beta \in Spin(8)$ commute with each other.
Automorphisms of order 4 of the Lie group $E_6$  

Proof.  

\[ \beta D_\alpha F_i(z) = \beta F_i(a z a) = \beta F_i(a^2 s + t) \quad (z = s + t \in C^C \otimes (C^+)^C = C^C) \]

\[ = F_i(a^2 s) + \beta F_i(t) = F_i(a^2 s + (\xi_1 E_8 + \xi_2 E_7 + F_i(t'))) \quad (\xi_i \in C, t' \in (C^+)^C) \]

\[ = D_\alpha (F_i(s) + \xi_1 E_8 + \xi_2 E_7 + F_i(t')) = D_\alpha (F_i(s) + \beta F_i(t)) \]

\[ = D_\alpha \beta F_i(s + t) = D_\alpha \beta F_i(z). \]

\[ \beta D_\alpha F_i(z) = \beta F_i(\alpha z) = 4 \beta (\beta F_i(1) \times F_i(z)) \times \beta F_i(a) = 4 (\beta F_i(1) \times \beta F_i(z)) \times \tau \beta \tau F_i(a) \]

\[ = 4 (F_i(1) \times (F_i(x) + F_i(y)) \times F_i(a) \quad (for \ some \ x, y \in C^C) \]

\[ = F_i(\alpha x) + F_i(y \alpha) = D_\alpha (F_i(x) + F_i(y)) = D_\alpha \beta F_i(z). \]

Similarly $\beta D_\alpha F_i(z) = D_\alpha \beta F_i(z)$. Clearly $D_\alpha \beta = \beta D_\alpha$ on $E_6$.

\[ D_\alpha \beta E_8 = D_\alpha (\xi_1 E_8 + \xi_2 E_7 + F_i(t)) = \xi_1 E_8 + \xi_2 E_7 + F_i(t) = \beta E_8 = \beta D_\alpha E_8 \]

(for some $\xi_i \in C$, $t' \in (C^+)^C$). Similarly $D_\alpha \beta E_8 = \beta D_\alpha E_8$. Thus we have $D_\alpha \beta = \beta D_\alpha$.

Lemma 3.4. Let $\beta \in \text{Spin}(10)^*$. Then we can put $\beta F_i(1) = F_i(s)$, $\beta F_i(\epsilon) = F_i(\epsilon s)$, $s \in C$, $|s| = 1$.

Proof. Since the group $(\text{Spin}(10)^*)^2$ acts on $\{F_i(s)| s \in C\} = \{X \in \mathfrak{z}^C| \sigma_3 X = -X, 2E_1 \times X = -\tau X\}$, we can put

\[ \beta F_i(1) = F_i(s), \quad \beta F_i(\epsilon) = F_i(s'), \quad s, s' \in C, |s| = |s'| = 1. \]

Operate $\tau \beta \tau$ on the relation $F_i(1) \times F_i(\epsilon) = -(1, \epsilon)E_1 = 0$, then $0 = \tau \beta \tau (F_i(1) \times F_i(\epsilon)) = \beta F_i(1) \times \beta F_i(\epsilon) = F_i(s) \times F_i(s') = -(s, s')E_1$, hence $(s, s') = 0$. Together with $|s| = |s'| = 1$, we have $s' = \epsilon s$ or $s' = -\epsilon s$. The latter case is impossible. In fact, choose $s \in C$ such that $a^2 = s$ and put $\delta = D_\alpha \beta$, then $\delta F_i(1) = F_i(1), \delta F_i(\epsilon) = -F_i(\epsilon)$.

Then

\[ \delta F_i(\epsilon) = \delta F_i(1) = \sigma_3 \delta F_i(1) = \sigma_3 F_i(F_i(x) + F_i(y)) \quad (for \ some \ x, y \in C^C) \]

\[ = F_i(\epsilon x) + F_i(\epsilon y), \]

\[ \delta F_i(1) = 2 \delta (F_i(1) \times F_i(1)) = 2 \tau \delta \tau F_i(1) \times \tau \delta \tau F_i(1) \]

\[ = 2 F_i(1) \times (F_i(\tau x) + F_i(\tau y)) = F_i(\tau x) + F_i(\tau y). \]

Therefore we have

\[ F_i(\epsilon) = \delta F_i(-\epsilon) = 2 \delta (F_i(\epsilon) \times F_i(1)) = 2 \tau \delta \tau F_i(\epsilon) \times \tau \delta \tau F_i(1) \]

\[ = 2 (F_i(\epsilon x) + F_i(\epsilon y)) \times (F_i(\tau x) + F_i(\tau y)) \]

\[ = F_i(- x((\tau x)x) - (\epsilon(\tau y)y) + \epsilon E_8 + s E_8). \]
Compare the coefficients of $e$, then we have $1 = -|x_1|^2 - |x_2|^2 - |y_1|^2 - |y_2|^2$ (where $x = x_1 + iy_2$, $y = y_1 + iy_2$, $x_1, y_1 \in \mathbb{C}$), which is a contradiction. Thus Lemma 3.4 is proved.

**Theorem 3.5.** $(E_6)^{\sigma} \cong (U(1) \times Spin(2) \times Spin(8))/\langle Z_2 \times Z_4 \rangle$,

$Z_2 \times Z_4 = \langle (1, -1, 1), (1, -1, 1), (1, -1, 1), (1, -1, 1), (1, -1, 1) \rangle.$

**Proof.** We define a mapping $\phi: U(1) \times Spin(2) \times Spin(8) \to (E_6)^{\sigma}$ by

$\phi(\theta, a, \delta) = \phi(\theta)D_a \delta.

Obviously $\phi$ is well-defined. Since $\phi(\theta)(\theta \in U(1))$, $D_a (a \in Spin(2))$ and $\delta \in Spin(8)$ commute with one another (Lemma 3.3), $\phi$ is a homomorphism. We shall show that $\phi$ is onto. (Although it suffices to show dim $((E_6)^{\sigma}) = 30$, we will give a direct proof). Since $\sigma^2 = \sigma$, we have $(E_6)^{\sigma} \subset (E_6)^{\sigma}$. Hence, for $\alpha \in (E_6)^{\sigma}$, there exist $\theta \in U(1), \beta \in Spin(10)$ such that $\alpha = \phi(\theta)\beta$ (Proposition 3.1). From $\sigma \alpha = \alpha \sigma$, we have $\beta \in (Spin(10))^\sigma$. Hence we can put $\beta F(1) = F(s), \beta F(e) = F(es), s \in C, |s| = 1$ (Lemma 3.4). Choose $a \in C$ such that $a^2 = s$ and put $\delta = D_a^{-1} \beta$, then $\delta F(1) = F(1), \delta F(e) = F(e)$, that is, $\delta \in Spin(8)$. Hence we have a presentation such that

$\alpha = \phi(\theta)D_a \delta, \quad \theta \in U(1), a \in Spin(2), \delta \in Spin(8).

Then $\phi$ is onto. Ker $\phi = Z_2 \times Z_4$ is easily obtained. Thus we have the required isomorphism.

**4. Outer automorphisms $\tau_4'$, $\tau_3'$ of order 4 of $E_6$**

Using $\phi: Spin(1) \times SU(6) \to (E_6)^\sigma$ of Proposition 2.1, we define $\gamma_s = \phi(1, J)$ and consider an automorphism $\tau_4'$ of $E_4: E_4 \to \tau_4' \alpha \tau_4'^{-1} \tau_4 \in E_6$. Then $(\tau_4')^4 = 1$.

**Theorem 4.1.** $(E_6)^{\tau_4'} \cong (Spin(1) \times SO(6))/Z_2$, $Z_2 = \{(1, E), (-1, -E)\}$.

**Proof.** Since $(\tau_4')^4 = 1$, we have $(E_6)^{\tau_4'} \subset (E_6)^\sigma$. Hence, for $\alpha \in (E_6)^{\tau_4'}$, there exist $\rho \in Spin(1), A \in SU(6)$ such that $\alpha = \phi(\rho, A)$ (Proposition 2.1). Since $\tau_4' \alpha \tau_4'^{-1} \tau_4 = \alpha$, we have $\tau_4 \phi(\rho, -JA) \tau_4 = \phi(\rho, A)$, that is, $\phi(\rho, A) = \phi(\rho, A)$ (Remark of Proposition 2.1) then $A = \tau_4$, hence $A \in SO(6)$. Thus we have the required isomorphism.
We use $\gamma_3=\phi(i, E)$ of Section 2 and consider an automorphism $\tau_\gamma$ of $E_\theta: E_\theta \\equiv\alpha\rightarrow\tau_\gamma a\tau_\gamma^{-1}a\in E_\theta$. Then $(\tau_\gamma a)^9=\tau$, $(\tau_\gamma a)^3=1$.

**Theorem 4.2.** $(E_\theta)^{\gamma_3}\equiv(U(1)\times Sp(3))/Z_2 = \{(1, E), (-1, -E)\}$.

**Proof.** Since $(\tau_\gamma a)^9=\tau$, we have $(E_\theta)^{\gamma_3}\subset (E_\theta)^{\gamma}$. Hence, for $\alpha\in (E_\theta)^{\gamma_3}$, there exist $p\in Sp(1), A\in SU(6)$ such that $\alpha=\phi(p, A)$ (Proposition 2.1). Since $\tau_\gamma a\tau_\gamma^{-1}a=a$, we have $\phi(-ip, A)\alpha'=\phi(p, A)$ (Remark of Proposition 2.1), then

$p=-ip, A=-J\bar{A}J$ or $p=ip, A=J\bar{A}J$.

In the first case $p\in U(1) = \{p\in C | \bar{p}p=1\}$, $A\in Sp(3) = \{A\in M(6, \mathbb{C}) | JA=\bar{A}J, \quad A^*A=E, \quad (\det A=1)\}$ (where $C\subset H$). The latter case is impossible. In fact, consider $Al$ where $l=\text{diag}(1, -1, 1, -1, -1, -1)$, then $Al\in Sp(3)$, hence $\det(Al)=1$. On the other hand, $\det(Al)=\det(A)\det(I)=1(-1)^{-1}=-1$, which contradicts $\det(Al)=1$. Thus we have the required isomorphism.

5. **Outer automorphism $\tau_\sigma$ of order 4 of $E_\theta$**

As in Section 3, we embed the field $C$ of complex numbers in $\mathbb{C}$ as $C = \{x+ye | x, y\in \mathbb{R}\}$. Let $SU(2) = \{A\in M(2, C) | A^*A=E, \det A=1\}$ and we define a mapping $\phi: SU(2) \rightarrow E_\theta$ by

\[\phi(A)X = (\rho_s(A))X(\rho_s(A))^*, \quad X\in \mathfrak{g}^c\]

where $\rho_s(A) = \Gamma^{-1}A^{\prime}\Gamma$, $\Gamma = \text{diag}(i, i, i)$, $A^{\prime} = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$. Explicitly, for $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

\[\phi(A)E_1 = E_1, \quad \phi(A)(E_x+E_z) = (|a|^2-|b|^2)(E_x+E_z)-2iF_1(ab), \quad \phi(A)(E_x-E_z) = E_x-E_z, \quad \phi(A)F_2(z) = -iF_2(bz)+F_2(\bar{z}a).\]

Note that $D_\alpha = \phi \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}$ for $a\in C$, $|a|=1$, in particular, $\phi \begin{pmatrix} -e & 0 \\ 0 & e \end{pmatrix} = D_\alpha = \sigma_3$.

**Lemma 5.1.** $\phi$ is well-defined, that is, for $A\in SU(2)$ we have $\phi(A)\in E_\theta$.

**Proof.** Define $\rho \in E_\theta$ by $\rho X = \Gamma X \Gamma$, $X \in \mathfrak{g}^c$. We consider an embedding
SU(3)⊂F₄⊂E₆, in which SU(3)⊂F₄ is given by \( h : SU(3) \to F₄ \),

\[
h(A)(X+M) = AXA^* + MA^*, \quad X+M \in \mathfrak{f}(3, \mathbb{C}) \oplus M(3, \mathbb{C}) = \mathfrak{f}^c\]

(as for notations see [5]). The \( \phi(A) \) is nothing but \( \phi(A) = \rho^{-1}h(A')\rho \in E₆ \).

We use \( \sigma \) of Section 3 and consider an automorphism \( \tau \sigma \) of \( E₆ : E₆ = \alpha \to \tau \sigma \alpha \tau^{-1} \tau \in E₆ \). Then \( (\tau \sigma)^2 = \sigma \), \( (\tau \sigma)^4 = 1 \).

**Lemma 5.2.** For \( \alpha \in (E₆)^{\tau \sigma} \), we have \( \alpha E₁ = E₁ \). In particular, \( (E₆)^{\tau \sigma} \subset (E₆)_{E₁} = Spin(10) \).

**Proof.** Since \( (\tau \sigma)^2 = \sigma \), we have \( (E₆)^{\tau \sigma} \subset (E₆)^{\sigma} \). Hence, for \( \alpha \in (E₆)^{\tau \sigma} \), there exists \( \xi \in C \), \( (\tau \xi)² = 1 \) such that \( \alpha E₁ = \xi E₁ \) (Proposition 3.1). From \( \xi E₁ = \alpha E₁ = \alpha \tau \sigma E₁ = \tau \sigma \alpha E₁ = \tau \sigma (\xi E₁) = \tau \xi E₁ \), we have \( \tau \xi = \xi \), that is, \( \xi \in \mathbb{R} \), hence \( \xi = ±1 \). The case of \( \xi = -1 \) is impossible. (Although it follows from the connectedness of \( (E₆)^{\tau \sigma} \), we will give an elementary proof). Suppose \( \xi = -1 \). Let \( \alpha = \phi(\theta)\beta \), \( \theta \in U(1) \), \( \beta \in Spin(10) \). Then \( -E₁ = \alpha E₁ = \phi(\theta)\beta E₁ = \theta^4 E₁ \), hence \( \theta^4 = -1 \). Now, for \( t \in C^+ \), we can put \( \beta F₁(t) = \eta E₂ - \tau \eta E₃ + F₁(x) \), \( \eta \in C \), \( x \in \mathbb{R} \). Then \( \phi(\phi)\beta F₁(t) = \pm i \eta E₂ \mp i \tau \eta E₃ \pm i F₁(x) \). Since \( \tau \sigma \alpha = \alpha \tau \sigma \), we have \( \eta \in i \mathbb{R} \), \( x \in C \). This shows that for \( V = \{ F₁(t) | t \in C^+ \} \), \( \dim V = 6 \), \( \dim (\alpha V) \leq 3 \), which contradicts the regularity of \( \alpha \). Thus Lemma 5.2 is proved.

The group \( Spin(7) \) is defined by

\[
(E₆)_{E₁, E₂, F₁(\mathbb{C})} = \{ \alpha \in E₆ | \alpha E₁ = E₁, \quad \alpha E = E, \quad \alpha F₁(s) = F₁(s) \quad \text{for all} \quad s \in C \}
\]

\[
= \{ \alpha \in F₁ | \alpha E₁ = E₁, \quad \alpha F₁(s) = F₁(s) \quad \text{for all} \quad s \in C \}
\]

\[
= \{ \alpha \in Spin(9) | \alpha F₁(1) = F₁(1), \quad \alpha F₁(0) = F₁(0) \}
\]

which is the covering group of \( SO(7) = SO(V^7) \) where \( V^7 = \{ \xi(E₂ - E₃) + F₁(t) | \xi \in \mathbb{R}, \ t \in C^+ \} \).

**Lemma 5.3.** \( \phi(A) \in \phi(SU(2)) \) and \( \beta \in Spin(7) \) commute with each other.

**Proof.** For \( A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in SU(2), \ \beta \in Spin(7) \), we shall show

\[
\beta \phi(A) X = \phi(A) \beta X, \quad X \in \mathfrak{g}.
\]

\[
\beta \phi(A)(E₂ - E₃) = \beta(E₂ - E₃) = \xi(E₂ - E₃) + F₁(t) \quad \text{(for some} \ \xi \in C, t \in (C^+)³ \}
\]

\[
= \phi(A)(\xi(E₂ - E₃) + F₁(t)) = \phi(A)\beta(E₂ - E₃),
\]
Automorphisms of order 4 of the Lie group $E_6$

$$\beta \phi(A)(E_x+E_y) = \beta( |a|^2 - |b|^2 (E_x+E_y) - 2i F_1(\overline{a}b)) = \phi(A)(E_x+E_y)$$

$$\phi(A)\beta(E_x+E_y).$$

Thus (i) is true for $X = E_x, E_y$. For $X = E_1$, (i) is trivial.

$$\beta \phi(A)F_1(z) = \beta \phi(A)F_1(s+t) = \beta(-2i(s, \overline{b}a)(E_x+E_y) + F_1(a^2s - b\overline{a}x + F_1(t))$$

$$= -2i(s, \overline{b}a)(E_x+E_y) + F_1(a^2s - b\overline{a}x + (\xi(E_x-E_y) + F_1(t')$$

(for some $\xi \in C, t' \in (C^+)^C$)

$$= \phi(A)\beta F_1(z) = \phi(A)\beta F_1(z).$$

$$\beta \phi(A)F_1(z) = \beta(\overline{F_1}(\overline{a}z) - iF_1(\overline{b}b)) = 4\beta(\overline{F_1}(1)\times F_1(z))\times F_1(a) - 4i\beta(\overline{F_1}(1)\times F_1(\overline{a}))\times F_1(\overline{b})$$

$$= 4(\beta F_1(1) \times \beta F_1(z)) - 4i(\beta F_1(1) \times \beta F_1(\overline{a})) \times \beta F_1(\overline{b})$$

(put $\beta F_1(z) = F_1(x) + F_1(y), x, y \in C^C, then \beta F_1(z) = \beta(2F_1(1) \times F_1(z)) = 2F_1(1) \times \beta F_1(z)$)

$$= 2F_1(1) \times (F_1(x) + F_1(y)) = F_1(z) + F_1(\overline{z})$$

$$= 4(\overline{F_1}(1) \times (F_1(x) + F_1(y))) \times F_1(a) - 4i(\overline{F_1}(1) \times (F_1(x) + F_1(\overline{z}))) \times F_1(\overline{b})$$

$$= F_1(\overline{a}x) + F_1(\overline{y}a) - iF_1(\overline{a}b) + F_1(b\overline{y})$$

$$= F_1(\overline{a}x) - iF_1(\overline{a}b) - iF_1(b\overline{y}) + F_1(y\overline{a})$$

$$= \phi(A)\beta F_1(z) + \phi(A)F_1(y) = \phi(A)(F_1(x) + F_1(y)) = \phi(A)\beta F_1(z).$$

Similarly, $\beta \phi(A)F_1(z) = \phi(A)\beta F_1(z).$ Thus (i) is proved.

**Lemma 5.4.** $\phi(SU(2))$ and $Spin(7)$ are contained in $(E_6)^{e_4}.$

**Proof.** $\phi(SU(2)) \subset (E_6)^{e_4}$ is clear, noting that

$$= \left( \begin{array}{cc} a & b \\ \overline{b} & \overline{a} \end{array} \right), \tau = -i \text{ and } b, i \text{ appear simultaneously in } \phi(A)X. \text{ Next, } \beta \in Spin(7) \subset F_4 \text{ implies } \tau \beta = \beta \tau \text{ and } \sigma_3 \beta = \beta \sigma_3 \text{ (Lemma 5.3). Hence } Spin(7) \subset (E_6)^{e_4}.$$

**Theorem 5.5.** $(E_6)^{e_4} \cong (SU(2) \times Spin(7))/\mathbf{Z}_2, \mathbf{Z}_2 = \{(E, 1), (-E, 1)\}$.

**Proof.** We define a mapping $\phi: SU(2) \times Spin(7) \rightarrow (E_6)^{e_4}$ by

$$\phi(A, \beta) = \phi(A)\beta.$$  

Then $\phi$ is well-defined (Lemma 5.4) and is a homomorphism (Lemma 5.3). We
shall show that $\phi$ is onto. (Although it suffices to show $\dim((e_4)^{\tau_3}) = 24$, we will give a direct proof). Let $\alpha \in (E_8)^{\tau_3}$. Then we can put

$$\alpha(i(E_2+E_3)) = i\tau(E_2+E_3) + F_i(s), \quad \tau^2 + |s|^2 = 1, \quad \tau \in R, \quad s \in C.$$ 

In fact, since $\alpha \in (E_8)^{\tau_3} \subset (E_8)_{E_7} = \text{Spin}(10)$ (Lemma 5.2), we can put $\alpha(i(E_2+E_3)) = \xi E_2 - \tau \xi E_3 + F_i(s+t), \quad \xi \in C, \quad s+t \in C \oplus C^\perp = \mathbb{C}$. From $\tau \sigma_a = \alpha \tau \sigma_3$, we have $\xi \in iR$, $t = 0$. And $2(\tau^2 + |s|^2) = \langle \alpha(i(E_2+E_3)), \alpha(i(E_2+E_3)) \rangle = \langle i(E_2+E_3), i(E_2+E_3) \rangle = 2$. Now put

$$P = \frac{1}{\sqrt{2(1-\tau)}} \begin{pmatrix} \xi & 1-\tau \\ 1+\tau & s \end{pmatrix} \quad (\text{if } \tau = 1, \text{ put } P = E).$$

Then $P \in SU(2)$ and $\phi(P)i\tau(E_2+E_3) = i(E_2+E_3)$. Hence $\phi(P)\alpha \in (E_8)_{E_7} = \text{Spin}(9)$, moreover $(\text{Spin}(9))^{\tau_3} = (\text{Spin}(9))^{\tau_3}$ $\subset (\text{Spin}(10))^{\tau_3}$. Hence we can put

$$\delta F_i(1) = F_i(s_0), \quad \delta F_i(e) = F_i(es_0), \quad s_0 \in C, \quad |s_0| = 1$$

(Lemma 3.4). Choose $a \in C$ such that $a^2 = \xi$ and put $\beta = D_ao \beta$, then $\beta(i(E_2+E_3)) = i(E_2+E_3)$, $\beta F_i(1) = F_i(1)$, $\beta F_i(e) = F_i(e)$, that is, $\beta = D_ao \phi(P) \alpha \in \text{Spin}(7)$. Therefore we have a presentation such that

$$\alpha = \phi(P^{-1}A) \beta, \quad P^{-1}A \in SU(2) \quad (\text{where } A = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}, \quad \beta \in \text{Spin}(7)).$$

Thus $\phi$ is onto. $\ker \phi = Z_3 = \{(E, 1), (-E, \sigma)\}$. In fact, let $\phi(A)\beta = 1$, $A = \begin{pmatrix} a & -\bar{b} \\ b & a \end{pmatrix} \in SU(2), \quad \beta \in \text{Spin}(7)$. Then $E = \phi(A)\beta E = \phi(A)E = E_1 + (|a|^2 - |b|^2)(E_2 + E_3)$. Hence $|a|^2 - |b|^2 = 1$. Together with $|a|^2 + |b|^2 = 1$, we have $b = 0$, so $A = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}$. From $F_i(1) = \phi(A)\beta F_i(1) = \phi(A)F_i(1) = F_i(a^2)$, we have $a^2 = 1$, hence $a = \pm 1$, so $A = \pm E$. Then $\beta = \phi(E) = 1$ or $\beta = \phi(-E) = \sigma$. Thus we have the required isomorphism.

References

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Automorphisms of order $4$ of the Lie group $E_6$


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