REFINABLE MAPS ONTO LOCALLY $n$-CONNECTED COMPACTA

By

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In [4], J. Ford and J. W. Rogers introduced the notion of refinable maps and they proved that each refinable map from a continuum to a locally connected continuum is monotone [4, Corollary 1.2]. In [5, Theorem 2.2], we proved that each refinable map from a compactum to an $FANR$ induces a shape equivalence. In this paper we shall prove that if a map $r: X \to Y$ between compacta is refinable and $Y \in LC^n (n \equiv 0)$, then $r^{-1}(y) \in AC^n$ for each $y \in Y$. Moreover if $Y$ is an $ANR$, then $r$ is a $CE$-map.

It is assumed that all spaces are metrizable and maps are continuous. A connected compactum is a continuum. A map $f: X \to Y$ between compacta is an $\varepsilon$-mapping, $\varepsilon > 0$, if $f$ is surjective and $\text{diam} f^{-1}(y) < \varepsilon$ for each $y \in Y$. If $x$ and $y$ are points of a metric space, $d(x, y)$ denotes the distance from $x$ to $y$. A map $r: X \to Y$ such that $d(r, f) = \sup \{d(r(x), f(x)) | x \in X\} < \varepsilon$. Such a map $f$ is called an $\varepsilon$-refinement of $r$. Note that every refinable map is surjective, every near homeomorphism is refinable and if there is a refinable map from a compactum $X$ to a compactum $Y$, then $X$ is $Y$-like. But simple examples show that any converse assertions of them are not true. A space $X$ is locally $n$-connected $(X \in LC^n)$ if for each $x \in X$ and an open neighborhood $U$ of $x$ in $X$, there is an open set $V$ with $x \in V \subset U$ such that each map $h: S^k \to V$ is null-homotopic in $U$ for $0 \leq k \leq n$, where $S^k$ denotes the $k$-sphere. A compactum $X$ in the Hilbert cube $Q$ is approximately $n$-connected $(X \in AC^n)$ if for each open neighborhood $U$ of $X$ in $Q$ there is an open neighborhood $V \subset U$ of $X$ in $Q$ such that each map $h: S^k \to V$ is null-homotopic in $U$ for $0 \leq k \leq n$ (see [2]). A map $f: X \to Y$ between compacta is a $CE$-map if $f$ is surjective and $f^{-1}(y)$ is an $FAR$ (see [2]) for each $y \in Y$.

The following lemma is well-known.

Lemma 1 ([7, Lemma 1]). Let $f$ be a map from a compactum $X$ to an $ANR$ $Y$ and $\varepsilon > 0$. Then there is a positive number $\delta > 0$ such that if $g$, is any $\delta$-map-

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ping from $X$ to any compactum $Z$, then there is a map $g_\delta : Z \rightarrow Y$ such that $d(f, g_\delta g_\eta) < \varepsilon$.

**Lemma 2.** Let $X$ and $Y$ be closed subsets of $AR$-spaces $M$ and $N$ respectively, and let $f : M \rightarrow N$ be an extension of a map $g : X \rightarrow Y$. If $X$ and $Y$ are locally $n$-connected and $f : (X, x) \rightarrow (Y, f(x))$ induces a zero-homomorphism $\pi_k(f) : \pi_k(X, x) \rightarrow \pi_k(Y, f(x))$ for $0 \leq k \leq n$, then for each open neighborhood $V$ of $Y$ in $N$ there is an open neighborhood $U$ of $X$ in $M$ such that $\pi_k(f|U) : \pi_k(U, x) \rightarrow \pi_k(V, f(x))$ is a zero-homomorphism.

**Proof.** By [3, Theorem 8.7] the natural morphisms $i_k : \pi_k(X, x) \rightarrow \text{pro-}_n \pi_k(X, x)$ and $j_k : \pi_k(Y, y) \rightarrow \text{pro-}_n \pi_k(Y, y)$ are isomorphisms for $0 \leq k \leq n$. Since $j_k \pi_k(f) = \text{pro-}_k \pi_k(f i_k)$, $\text{pro-}_k \pi_k(f) : \pi_k(X, x) \rightarrow \pi_k(Y, y)$ is a zero-homomorphism, which implies the existence of $U$ in the statement of Lemma.

**Theorem.** Let $X$ and $Y$ be compacta and $r : X \rightarrow Y$ be a refinable map. If $Y \in LC^n$ ($n \geq 0$), then $r^{-1}(y) \in AC^n$ for each $y \in Y$. Moreover if $Y$ is an ANR, then $r$ is a CE-map.

**Proof.** Since $X$ is a compactum, $X$ can be embedded into the Hilbert cube $Q$. Let $y \in Y$ and let $G$ be any open neighborhood of $r^{-1}(y)$ in $Q$. Choose a compact ANR $U$ such that $r^{-1}(y) \subseteq \text{Int}_Q U \subseteq U \subseteq G$. Since $U$ is a compact ANR, there is a positive number $\varepsilon_1 > 0$ such that any $\varepsilon_1$-near maps to $U$ are homotopic. Let $\varepsilon_2 = d(r^{-1}(y), Q - U) = \inf \{d(x_1, x_2) \mid x_1 \in r^{-1}(y), x_2 \in Q - U\} > 0$. Since $Y \in LC^n$, there is a sequence $V_1, V_2, V_3, \ldots$ of open sets in $Y$ such that

1. $V_1 \supseteq V_2 \supseteq V_3 \supseteq \ldots$,
2. $\bigcap_{i=1}^\infty V_i = \{y\}$,
3. each map $h : S^k \rightarrow V_{i+1}$ ($0 \leq k \leq n$) is null-homotopic in $V_i$.

Since $r$ is refinable, there are maps $r_i : X \rightarrow Y$ such that each $r_i$ is an $(1/i)$-refinement of $r$ and

4. $r_i(r^{-1}(y)) \subseteq V_{i+1}$ for each $i$.

Then we shall show that $\lim [r_i^{-1}(V_i) \cup r^{-1}(y)] = r^{-1}(y)$. In fact, suppose, on the contrary, that there is a sequence $x_{n_i} \in r_i^{-1}(V_{n_i})$ such that $\lim x_{n_i} = x_0$ and $r(x_0) \neq y$. Choose an open neighborhood $W$ of $x_0$ in $X$ such that $r(W) \subseteq S_\delta y$, where $\delta = (1/4)d(r(x_0), y) > 0$ and for a set $A S_\delta(A)$ denotes the $\delta$-neighborhood of $A$. By (2), choose a sufficiently large integer $n_i$ such that $x_{n_i} \in W$, $d(r, r_{n_i}) < \delta$ and $V_{n_i} \subseteq S_\delta y$. 
Then \( r(x_n) \in S_\delta(r(x_0)) \) and \( r_n(x_n) \in \bar{V}_{n_1} \subset S_\delta(y) \), hence
\[
d(r(x_n), y) \leq d(r(x_0), r(x_n)) + d(r(x_n), r_n(x_n)) + d(r_n(x_n), y)
\]
\[
< \delta + \delta + \delta = 3\delta, \quad \text{which implies the contradiction.}
\]
Let \( 0 < \varepsilon \leq \min \{ \varepsilon_1, \varepsilon_2 \} \). Since \( \lim [r_t(\bar{V}_t)] = r^{-1}(y) \), there is a natural number \( i_0 \) such that
\[
(5) \quad r^{-1}_t(\bar{V}_t) \subset S_{\varepsilon/3}(r^{-1}_t(y)) \quad \text{for each } i \geq i_0.
\]
By Lemma 1, there is a natural number \( m \geq i_0 \) such that there is a map \( g_m : Y \to Q \) such that
\[
(6) \quad d(i_x, g_m r_m) < \varepsilon/3, \quad \text{where } i_x : X \to Q \text{ is the inclusion.}
\]
Then we shall show
\[
(7) \quad g_m(V_m) \subset g_m(\bar{V}_m) \subset U.
\]
In fact, for each \( x \in r_m^{-1}(\bar{V}_m) \), by (5) and (6) we have
\[
d(g_m r_m(x), r^{-1}_t(y)) \leq d(g_m r_m(x), x) + d(x, r^{-1}_t(y)) < \varepsilon/3 + \varepsilon/3 < \varepsilon,
\]
hence \( g_m r_m(x) \in S_{\varepsilon/3}(r^{-1}_t(y)) \subset U \).

Now, take two \( AR \)-spaces \( M \) and \( N \) containing \( V_{m+1} \) and \( V_m \) respectively as closed subsets, and let \( i : M \to N \) be an extension of the inclusion \( i : V_{m+1} \to V_m \). Since \( U \) is an \( ANR \), by (7) there is an open neighborhood \( V'_m \) of \( V_m \) in \( N \) and an extension \( g_m : V'_m \to U \) of \( g_m | V_m : V_m \to U \). Since \( V_{m+1} \), \( V_m \in LC^n \), by Lemma 2 and (3) there is an open neighborhood \( V'_{m+1} \) of \( V_{m+1} \) in \( M \) such that
\[
(8) \quad \pi_k(i'_t | V'_{m+1}) : \pi_k(V'_{m+1}) \longrightarrow \pi_k(V'_{m}) \text{ is a zero-homomorphism}
\]
for each \( 0 \leq k \leq n \).

Let \( U' \) be an open neighborhood of \( r_m^{-1}(\bar{V}_{m+2}) \) in \( Q \) such that \( U' \subset U \) and there is an extension \( i'_m : U' \to V'_{m+1} \) of \( r_m^{-1}(\bar{V}_{m+2}) : r_m^{-1}(\bar{V}_{m+2}) \to V_{m+1} \). Since \( g_m i'r_m | r_m^{-1}(\bar{V}_{m+2}) \) \( = g_m i'_m r_m | r_m^{-1}(\bar{V}_{m+2}) \), by (6) there is an open neighborhood \( U'' \subset U' \) of \( r_m^{-1}(\bar{V}_{m+2}) \) in \( Q \) such that
\[
(9) \quad d(g_m i'_m U'', i''_U) < \varepsilon, \quad \text{where } i''_U : U'' \to U \text{ is the inclusion.}
\]
By (9), we have
\[
(10) \quad g_m i'u_m | U'' \subset i''_U \text{ in } U.
\]
By (8) and (10), \( \pi_k(i'u''_U) : \pi_k(U'') \to \pi_k(U) \) is a zero-homomorphism. Note that
Hisao Kato

Hence $r^{-1}(y) \in AC^n$.

If $Y$ is a compact ANR, the proof is similar. This completes the proof.

**Remark 1.** Note that if $n=0$, Theorem implies the result of J. Ford and J.W. Rogers.

By Theorem and [3, Theorem 8.5], we have the following.

**Corollary 1.** If a map $r: X \rightarrow Y$ between compacta is refinable and $Y \in LC^n$ $(n \geq 1)$, for any compactum $B \subset Y$ and $x \in r^{-1}(B)$, $\rho \cdot \pi_k(r| r^{-1}(B)): \rho \cdot \pi_k(r^{-1}(B), x) \rightarrow \rho \cdot \pi_k(B, r(x))$ is an isomorphism of pro-groups for $1 \leq k \leq n$ and an epimorphism of pro-groups for $k = n+1$.

**Corollary 2.** Let $X$ and $Y$ be compacta and $r: X \rightarrow Y$ be a refinable map. If $Y \in LC^n$ and $Fd(Y) \leq n$ (see [2]), then $r$ induces a shape equivalence.

**Proof.** By [5, Theorem 1.8], $Fd(X) = Fd(Y) \leq n$. By Theorem and the result of [3, Theorem 8.14], [6] or [8], $r$ induces a shape equivalence.

**Corollary 3.** If a map $r: X \rightarrow Y$ between compacta is refinable and $Y$ is a finite-dimensional ANR, then $r$ induces a hereditary shape equivalence, i.e., for any compactum $B$, $r| r^{-1}(B): r^{-1}(B) \rightarrow B$ induces a shape equivalence.

**Corollary 4.** Let $r$ be a map from a $(S_1 \vee S_2 \vee \ldots \vee S_n)$-like continuum onto $S_1 \vee S_2 \vee \ldots \vee S_n$, where $S_1 \vee S_2 \vee \ldots \vee S_n$ denotes the one point union of $n$ circles. Then the followings are equivalent.

1. $r$ is refinable.
2. $r$ is a CE-map.
3. $r$ is monotone.

**Proof.** By [5, Theorem 3.2], (1) and (3) are equivalent. By Theorem, (1) implies (2). Obviously (2) implies (3).

**Remark 2.** In the statement of Theorem, we cannot replace $AC^n$ by $C^n$ ($n$-connected).

**Remark 3.** By [4, p. 264], there is a refinable map $r: X \rightarrow Y$ such that $X$, $Y$ are 1-dimensional continua and $r^{-1}(y) \in AC^n$ for some $y \in Y$ (cf. [5, Example 2.7]). In [5, Example 2.6], for each $n=1, 2, 3, \ldots$, we constructed a refinable map $r: X \rightarrow Y$ such that $X$ and $Y$ are $n$-dimensional continua, $Y \in LC^{n-1}$ and $Sh(X) \neq Sh(Y)$. In fact, for some $y \in Y$, $r^{-1}(y) \in AC^n$. Thus those show that
in the statement of Theorem we cannot replace $LC^n$ by $LC^{n-1}$. Moreover, in [5, Example 2.8], we constructed a near homeomorphism $h: X \rightarrow X$ such that $X$ is a $n$-dimensional continuum, $X \in LC^{n-1}$ and $r$ does not induce a shape equivalence. In fact, for some $y_0 \in X$, $r^{-1}(y_0) \in AC^n$.

References


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