ON THE INTERVALS BETWEEN CONSECUTIVE NUMBERS THAT ARE SUMS OF TWO PRIMES

By

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1. Introduction.

It is the well known conjecture of H. Cramér that

\[ p_{n+1} - p_n \ll (\log p_n)^2 \]

where \( p_n \) is the \( n \)-th prime. In 1940 P. Erdös proposed the problem to estimate the sum

\[ \sum_{p_n \leq x} (p_{n+1} - p_n)^2, \]

and A. Selberg showed that it is

\[ \ll x(\log x)^3 \]

under the Riemann hypothesis. This problem has been stimulating the several authors, vide [2, 3, 10, 11, 13].

Let \((g_n)\) denote in ascending order even integers that are representable as the sum of two primes. The Goldbach conjecture is then interpreted as that

\[ g_{n+1} - g_n = 2 \]

for all \( n \). In 1952 Ju. V. Linnik [7] proved, on assuming the Riemann hypothesis, that

\[ g_{n+1} - g_n \ll (\log g_n)^{5+\varepsilon} \]

for any \( \varepsilon > 0 \) and all \( n \). Also see [1]. In this paper we shall estimate the third moment of it.

**Theorem.**

\[ \sum_{g_n \leq x} (g_{n+1} - g_n)^3 \ll x(\log x)^{3+\varepsilon}. \]

**Corollary.** For \( 0 \leq \gamma < 3 \), we have

\[ \sum_{g_n \leq x} (g_{n+1} - g_n)^{\gamma} = (2^{\gamma-1} + o(1))x. \]

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Our assertion should be compared with the known results in Goldbach's problem. Let \( E(x) \) be the number of even integers not exceeding \( x \) that may not be expressed as a sum of two primes, and \( D(x) \) be the maximum of \( (g_{n+1} - g_n) \) for \( g_n \leq x \). It was proved by H. L. Montgomery and R. C. Vaughan [9] that

\[
E(x) \ll x^{1-\delta}
\]

with some \( \delta > 0 \). As for \( D(x) \), the argument in [9] runs as follows. Suppose that one knows the equi-distribution of primes in intervals \([x, x+x^\theta]\) for almost all \( x \), and in \([x, x+x^\theta]\) for all \( x \). Then,

\[
(1.1) \quad D(x) \ll x^{\theta/2}.
\]

By an elementary consideration, see section 3, we find

\[
\sum_{g_n \leq x} (g_{n+1} - g_n)^3 = 2x + O(D(x)E(x)).
\]

It seems that no unconditional result leads \( \theta \theta \leq d \).

Our argument is based upon Linnik's method [6, 7] and D. Wolke's trick [13]. The limitation of our estimate comes from A. E. Ingham's bound [4] for zeros of the Riemann zeta-function.

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2. Notation and Lemmas.

We use the standard notation in number theory. \( \rho \) stands for the non-trivial zeros of the Riemann zeta-function. For \( 1/2 \leq \sigma \leq 1 \) and \( T > 0 \), \( N(\sigma, T) \) denotes the number of \( \rho \) such that \( \sigma \leq \Re(\rho) \) and \( |\Im(\rho)| \leq T \).

**Lemma 1.** Uniformly for \( x, T \geq 3 \), we have

\[
\sum_{\pm \Im(\rho) \leq T} x^\rho = -\frac{T}{\pi} \Lambda'(x) + O(x(\log x T)^2)
\]

where \( \Lambda'(x) \) is equal to the von Mangoldt function if \( x \) is an integer, and \( \Lambda'(x) \) \( = 0 \) otherwise.

This is a formula of E. Landau [5]. Though his estimate is not uniform for \( x \), it is easy to alter the proof of [5] to be suitable for our aim. The following Lemma 2 is due to Ingham [4] and Montgomery [8, Theorem 1]. Lemma 3 follows from [8, Theorem 2].
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Lemma 2. For \( T > 2 \), we have

\[
N(\sigma, T) \ll T^{(4(e^{1-\sigma}))(\log T)^{13}}
\]

where

\[
\lambda(\sigma) = \begin{cases} 
\frac{3}{2-\sigma} & \text{if } 1/2 \leq \sigma \leq 4/5 \\
\frac{2}{\sigma} & \text{if } 4/5 \leq \sigma \leq 1.
\end{cases}
\]

Lemma 3. If \( 9/10 \leq \sigma \leq 1 \), then

\[
N(\sigma, T) \ll T^{(12-\sigma)(\log T)^{13}}
\]

where \( c \) is a positive absolute constant.

In sections 3 and 4 we use the convention \( L = \log X \). For a real \( x \), write \( e(x) = e^{2\pi ix} \) and \( * \) mean that \( f \ast g(x) = \int_{-\infty}^{\infty} f(x-y) g(y) dy \) and \( f(x) = \int_{-\infty}^{\infty} f(y) \cdot e(-xy) dy \), respectively. The implied constants in \( O \) and \( \ll \) are absolute, except for the proof of Corollary.

3. Reduction of the problem

In this section we first reduce the proof of Theorem to that of Lemma 4 below. Lemma 4 will be verified in section 5. Next we derive Corollary from Theorem. Put \( d_n = g_{n+1} - g_n \), for simplicity.

Proof of Theorem. It is sufficient to prove

\[
F(x) = \sum_{x' < \xi < x} d_n^3 \ll (\log x)^{300}
\]

for all large \( x \) and \( x' = (5/7)x \). We have

\[
F(x) \ll \sum_{x' < \xi < x} d_n^3 + (\log x) \sup_{d_n \in (\log x)^{1/6}} \sum_{x' < \xi < x} d_n^3.
\]

Because of (1.1), \( \delta D(x) \ll x^{1/6} \). Put

\[
\Gamma(x, \delta) = \{ g_n : x' < g_n \leq x, \delta < d_n \leq 2\delta, g_{n+1} \leq x \}.
\]

Then,

\[
(3.1) \quad F(x) \ll (\log x)^{300} \sum_{\xi_n \leq x} d_n + (\log x) \sup_{(\log x)^{1/6} \leq \xi \leq x^{1/6}} \left( \sum_{\xi_n \in \Gamma'(x, \delta)} d_n^3 + \delta^2 D(x) \right) \ll x(\log x)^{300} + (\log x) \sup_{(\log x)^{1/6} \leq \xi \leq x^{1/6}} \delta^2 \sum_{\xi_n \in \Gamma'(x, \delta)} d_n.
\]

Here we state our main lemma.
**Lemma 4.** Let $X$ be a large parameter,
\[
(5/2)X \leq x \leq (7/2)X \quad \text{and} \quad (1/2)(\log X)^{140} < \Delta < X^{1/6}.
\]
There exists a function $R(x, \Delta)$ such that
\[
(3.2) \quad \int_{(5/2)X}^{(7/2)X} |R(x, \Delta)|^2 \, dx \ll X^9 (\log X)^{290}
\]
and
\[
(3.3) \quad \sum_{X < n < X + x} A(m)A(n) = \Delta(X - |x - 3X|) + O(\Delta X(\log X)^{-1}) + R(x, \Delta),
\]
uniformly for $X$, $x$ and $\Delta$.

Now, if $t \in [(g_n + g_{n+1})/2, g_{n+1})$ for $g_n \in \Gamma(x, \delta)$ then
\[
t - \frac{\delta}{2} > \frac{g_n + g_{n+1}}{2} - \frac{d_n}{2} = \tilde{g}_n.
\]
Namely the interval $(t - \Delta/2, t]$ contains no sum of two primes. By (3.3) in Lemma 4 with $(7/2)X = x$ and $2\Delta = \delta$, we therefore have
\[
R(t, \delta/2) = -\frac{\delta}{2} \left( \frac{2}{7} x - \left| t - \frac{6}{7} x \right| \right) + O(\delta x(\log x)^{-1})
\]
for all $t \in [(g_n + g_{n+1})/2, g_{n+1})$ with $g_n \in \Gamma(x, \delta)$. Since these intervals are mutually disjoint, we have
\[
\sum_{g_n \in \Gamma(x, \delta)} \left( g_n + g_{n+1} \right) (\delta x)^2 \ll \sum_{g_n \in \Gamma(x, \delta)} \int_{\tilde{g}_n + g_{n+1}/2}^{\tilde{g}_n + g_{n+1}} |R(t, \delta/2)|^2 \, dt
\]
\[
\leq \int_{x}^{x+\delta} |R(t, \delta/2)|^2 \, dt.
\]
Hence (3.2) in Lemma 4 yields that
\[
\delta^2 \sum_{g_n \in \Gamma(x, \delta)} d_n \ll (\log x)^{290}
\]
uniformly for $\delta, (\log x)^{140} < \delta < x^{1/6}$. Combining this with (3.1) we obtain
\[
F(x) \ll (\log x)^{200},
\]
as required.

**Proof of Corollary.** With the notation in section 1, we easily see that
\[
\sum_{g_n \in \Gamma} d_n = x + O(D(x)),
\]
and
\[
\sum_{g_n \in \Gamma} 1 = \frac{1}{2} x + O(1) - E(x).
\]
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By subtraction, we have

\[ (3.4) \quad \sum_{\sigma(n) \leq x} d_n \ll D(x) + E(x) \ll E(x), \]

or

\[ (3.5) \quad \sum_{\sigma(n) \leq x} 1 = \frac{1}{2} x + O(E(x)). \]

Now, if \( 0 \leq \gamma \leq 1 \) then

\[ \sum_{\sigma(n) \leq x} d_n^\gamma = 2^\gamma \sum_{\sigma(n) \leq x} 1 + O\left( \sum_{\sigma(n) \leq x} d_n \right) = 2^{\gamma-1} x + O(E(x)) \]

by (3.4) and (3.5). It is known [12; Kap. VI. Satz 7.1] that

\[ (3.6) \quad E(x) \ll x(\log x)^{-A} \]

for any \( A > 0 \). Hence we get Corollary in case \( 0 \leq \gamma \leq 1 \).

Suppose \( 1 < \gamma < 3 \). Let \( D \) be a positive constant, which will be specified later. Then,

\[
\sum_{\sigma(n) \leq x} d_n^\gamma = \sum_{d_n \leq x} + \sum_{2 \leq d_n < (\log x) D} + \sum_{d_n > (\log x) D} \]

\[= 2^{\gamma-1} x + O(E(x)) + O\left((\log x)^{(\gamma-1)D} \sum_{\sigma(n) \leq x} d_n \right) + O\left((\log x)^{-1-(3-\gamma)D} \sum_{\sigma(n) \leq x} d_n^3 \right) \]

\[= 2^{\gamma-1} x + O(E(x)(\log x)^{(\gamma-1)D} + x(\log x)^{3-3-(3-\gamma)D}) \]

because of (3.4), (3.5) and Theorem. On taking \( D = \frac{301}{3-\gamma} \) we get, by (3.6), that

\[ \sum_{\sigma(n) \leq x} d_n^\gamma = 2^{\gamma-1} x + O(x(\log x)^{-1}), \]

as required.

4. Proof of Lemma 4, preliminaries.

We begin with modifying the explicit formula:

\[ (4.1) \quad \sum_{n \leq x} A(n) = x - \sum_{\Im(\rho) \leq T} \frac{x^\rho}{\rho} + O\left((1 + \frac{x}{T})(\log x)^s\right) \]

uniformly for \( x, T \geq 3 \). For \( T \geq 3 \), define

\[ q_n = q_n(T) = \int_{n-1/2}^{n+1/2} \sum_{\Im(\rho) \leq T} \frac{y^{\rho-1} dy}{\rho} \]

if \( n \leq 5 \), and \( q_n = 0 \) otherwise. Moreover we determine \( r_n = r_n(T) \) by the relation

\[ (4.3) \quad A(n) = 1 - q_n - r_n. \]
Lemma 1 then gives

\[ q_n, r_n \ll (\log nT)^2. \]

For large \( x \), it follows from the prime number theorem, (4.1) and (4.2) that

\[ \sum_{n \leq x} q_n = \sum_{\operatorname{Im}(\rho) \leq T} \left( \left\lceil \frac{x}{2} \right\rceil + \frac{1}{2} \rho \right) + O\left( \frac{1}{\rho} \right), \]

\[ \ll x \exp\left( -\frac{\log x}{\sqrt{\log T}} \right) + \left( 1 + \frac{x}{T} \right) (\log xT)^2. \]

Similarly,

\[ \sum_{n \leq x} r_n \ll \left( 1 + \frac{x}{T} \right)(\log xT)^2 \]

by (4.1), (4.2) and (4.3).

Now, on choosing

\[ T = \frac{X}{\Delta} L^*, \]

we consider the sum in question:

\[ G = \sum_{x-\Delta < m + n \leq x} A(m)A(n). \]

By (4.3),

\[ A(m)A(n) = 1 + q_m q_n - (q_m + q_n) r_m A(n) - A(m)r_n - r_m r_n. \]

Accordingly,

\[ G = G_1 + G_2 - 2G_3 - 2G_4 - G_5, \]

say.

\[ G_1 = \sum_{x-\Delta < m + n \leq x} 1 \]

\[ = \sum_{x-\Delta < m + n \leq x} \# \{ n : x - m - \Delta < n \leq x - m \} + O(\Delta^2) \]

\[ = \Delta \sum_{x-\Delta < m + n \leq x} 1 + O(X) \]

\[ = \Delta (X - |x - 3X|) + O(X). \]

On writing

\[ Z(y) = Z(y, T) = \sum_{\operatorname{Im}(\rho) \leq T} y^{\rho-1}, \]

we have
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\[ G_3 = \sum_{\substack{d \leq m, \ n \leq 2X \atop x - \Delta < m + n \leq x}} \int_{x - \Delta < m + n \leq x} Z(u)Z(v)du dv \]

\[ = \int_D Z(u)Z(v)du dv, \quad \text{say.} \]

We replace the domain \( D \) by

\[ D = D(X, \ x, \ \Delta) := \{ (u, v) \in [X, 2X]^2 : x - \Delta \leq u + v \leq x \}. \]  

The resulting error is

\[ \ll \int_{(D - D) \setminus (D - D)} |Z(u)Z(v)| dudv \ll XL^\delta, \]

because of Lemma 1.

\[ G_3 = \sum_{\substack{d \leq m, \ n \leq 2X \atop x - \Delta < m + n \leq x}} q_m \]

\[ = \sum_{\substack{d \leq m, \ n \leq 2X \atop x - \Delta - \Delta < m + n < x - \Delta - \Delta}} q_m (\Delta + O(1)) + O(\Delta^2 L^\delta) \]

\[ \ll \Delta \sup_{m \leq X} \sum_{n \leq X} q_m |m + X|^\delta \]

\[ \ll \Delta XL^{-4}, \]

by (4.4) and (4.5). Also, (4.4) and (4.6) give that

\[ G_4 = \sum_{\substack{d \leq m, \ n \leq 2X \atop x - \Delta < m + n \leq x}} A(m)r_n \]

\[ \ll \sum_{m \leq X} A(m) \sup_{N \leq X} \sum_{n \leq N} r_n \]

\[ \ll XL^\delta \left(1 + \frac{X}{T}\right)L^\delta \]

\[ \ll \Delta XL^{-4}. \]

Similarly,

\[ G_5 \ll \Delta XL^{-4}. \]

On summing up the above estimates (4.7)-(4.14) we obtain

\[ G = \Delta(X - |x - 3X|) + O(\Delta XL^{-4}) + \int_D Z(u)Z(v)du dv \]

where \( Z \) and \( D \) are defined by (4.9) and (4.10), respectively.

Put

\[ N^*(\sigma, T) = T^{\frac{\lambda}{2}}(1-\sigma) L^{13} \]

where \( \lambda \) is defined in Lemma 2. Since

\[ T^{\frac{\lambda}{2}(1/2)} = T^{\frac{\lambda}{2}(1)} = \left(\frac{X}{\Delta} L^s\right)^2 < X^4 L^{-280} < \left(\frac{X}{\Delta} L^s\right)^{12/5} = T^{\lambda(3/4)} < T^{\lambda(4/5)}, \]

there exist \( r \) and \( t \) such that \( 1/2 < r < 3/4 \), \( 4/5 < t < 1 \) and

\[ T^{\lambda(r)} = T^{\lambda(t)} = X^2 L^{-280}. \]

Define \( s = \min(t, 9/10) \), and \( I = [r, s) \). We then see

\[ T^{\lambda(\sigma)} \leq X^2 L^{-280} \quad \text{for all } \sigma \in [1/2, 9/10) \setminus I, \]

and

\[ T^{\lambda(\sigma)} \geq X^2 L^{-280} \quad \text{for all } \sigma \in I. \]

Now, we divide the sum \( Z(y) \), which is defined by (4.9).

\[ Z(y) = \sum_{\Re(\rho) \in I} + \sum_{\Re(\rho) \notin I} = z_1(y) + z(y), \quad \text{say}. \]

We first consider \( z_1 \). By a familiar way,

\[ J = \int_{X}^{2X} |z_1(y)|^2 dy \ll L^5 \sum_{\Re(\rho) \subset I} X^{\sigma \Re(\rho) - 1} \ll L^5 \sum_{\Re(\rho) \subset I} 1 + L^5 \sup_{1/2 < \sigma < 1} \sup_{\sigma / 2 < \rho} X^{\sigma - 1} N(\sigma, T). \]

Here, because of the zero-free region \([12; \text{Kap. VIII. Satz } 6.2]\), the above supremum may be taken over \( \sigma \leq 1 - \eta(T) \) only, where \( \eta(T) = (\log T)^{-\epsilon/6} \). Lemmas 2 and 3 yield that

\[ J \ll L^3 T + L^{14} X \left\{ \sup_{\sigma / 2 < \rho} T^{\lambda(\sigma)} \right\}^{1-\sigma} + L^{14} X \left\{ \sup_{\sigma / 10 < \rho} T^{\lambda(\sigma)} \right\}^{T^{\lambda(1-\sigma)}} \]

\[ \ll L^3 T + X^{\lambda - 1} + L^{14} X \left\{ (X^{-280})^{1/10} + T^{-\epsilon \eta(T)} \right\} \]

\[ \ll X L^{-12}, \]

by (5.1).

We turn to the double integral in (4.15). Since

\[ Z(u)Z(v) = z(u)z(v) + z_1(u)Z(v) + Z(u)z_1(v) - z_1(u)z_1(v), \]
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\[ \int_{B} \left( |Z(u)Z(v)| - |z(u)z(v)| \right) du \, dv \]
\[ \ll \int_{x < \Delta u < x + \Delta} |z(u)| |Z(v)| + |z(v)| du \, dv \]
\[ \ll L^2 \Delta \int \left| z(y) \right| \, dy + \Delta \int \left| z(y) \right| ^2 \, dy \]
\[ \ll L^2 \Delta (X^2 L^{-1}) 1/2 + \Delta XL^{-2} \]
\[ \ll \Delta XL^{-4} , \]
by Lemma 1 and (5.4). Combining this with (4.15) we reach (3.3);

\[ G = \Delta (X - |x - 3X|) + O(\Delta XL^{-4}) + R(x, \Delta) \]

where

\[ (5.5) \]
\[ R(x, \Delta) = \int_{B} z(u)z(v) du \, dv . \]

It remains to prove (3.2). First we define \( z(y) = 0 \) if \( y \notin [X, 2X] \). Next we split up \( z(y) \). Let \( z_0(y) \) be the partial sum of \( z(y) \) restricted by \( \sigma \leq \text{Re} (\rho) < \sigma (1+1/L) \). Then,

\[ z(y) = \sum_{n \geq 0} z_0(y) . \]

Furthermore let \( \chi(x) \) denote the characteristic function of \([0, \Delta]\). Thus we may rewrite (5.5) as

\[ R(x, \Delta) = \int_{-X}^{+X} \int_{-X}^{+X} \chi(x-u-v)z(u)z(v) du \, dv \]
\[ = \chi \ast z \ast z(x) . \]

Now, by Plancherel's relation, we have

\[ (5.6) \]
\[ I = \int_{(\ell/2)X}^{(\ell/2)X} \left| R(x, \Delta) \right| ^2 \, dx \leq \int_{-X}^{+X} \left| \chi \ast z \ast z(x) \right| ^2 \, dx \]
\[ = \int_{-X}^{+X} \left| \chi \ast z \ast z(x) \right| ^2 \, dx \]
\[ = \int_{-X}^{+X} \left| \tilde{\chi}(x) \right| \left| \tilde{z}(x) \right| ^4 \, dx . \]

Here we see

\[ \left| \tilde{\chi}(x) \right| = \left( \frac{\sin \pi \Delta x}{\pi x} \right) ^2 , \]
and, on using Hölder's inequality,
Therefore (5.6) becomes

\[ I \ll L^4 \Delta^2 \sup_{\sigma \in \mathbb{I}} \left( \sup_x \left| \dot{z}_\sigma(x) \right|^2 \right) \int_{-\infty}^{\infty} \left| z_\sigma(x) \right|^2 dx, \]

by Plancherel's relation again.

We proceed to estimate the square integral of \( z_\sigma \).

\[ \int_{-\infty}^{\infty} \left| z_\sigma(y) \right|^2 dy = \int_{x}^{2x} \left| \sum_{\sigma \leq \text{Re} \rho < \sigma} \sum_{\text{Re} \rho > \sigma} \text{Re} \left( \frac{1}{y+\text{Re} \left( \frac{1}{\text{Im} \rho \log y - 2\pi xy} \right)} \right) \right|^2 \]

\[ \ll L^2 X^{2\sigma - 1} N(\sigma, T). \]

We turn to \( \dot{z}_\sigma \). The simplest saddle point method [12; Kap IX, Lemma 4.2] leads that

\[ \dot{z}_\sigma(x) = \sum_{\text{Re} \rho \leq \sigma} \int_{\rho \in \mathbb{R}} y^{\text{Re} \left( \frac{1}{\text{Im} \rho \log y - 2\pi xy} \right)} \exp \left( \text{Re} \left( x \text{Im} \rho \log y - 2\pi xy \right) \right) \]
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This completes our proof.

References


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