A CHARACTERIZATION OF PARACOMPACTNESS OF LOCALLY LINCOLDOF SPACES

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Abstract. A space $X$ is said to have property $\mathcal{B}$ if every infinite open cover $\mathcal{U}$ of $X$ has an open refinement $\mathcal{V}$ such that every point $x \in X$ has a neighborhood $W$ with $| \{ V \in \mathcal{V} : W \cap V \neq \emptyset \} | < | \mathcal{U} |$. It is proved that a locally Lindelöf space is paracompact if and only if it has property $\mathcal{B}$.

All spaces are assumed to be regular $T_1$.

A well-known problem posed by Arhangel’skii and Tall is: Is every locally compact normal metacompact space paracompact? The problem is affirmative if we assume $V=L$ [10] or if the space is perfectly normal [1] or boundedly metacompact [5] or locally connected [6].

In connection with this problem, in this paper we give a characterization of paracompactness for locally Lindelöf spaces by using property $\mathcal{B}$, and provide another partial answer to the problem.

Property $\mathcal{B}$ was introduced originally by Zenor [12] as a generalization of paracompactness: a space $X$ is said to have property $\mathcal{B}$, if for every monotone increasing open cover $\mathcal{U} = \{ U_\alpha : \alpha \in \kappa \}$ (that is, $U_\alpha \subset U_\beta$ if $\alpha < \beta$) of $X$, there exists a monotone increasing open cover $\mathcal{V} = \{ V_\alpha : \alpha \in \kappa \}$ which is a shrinking of $\mathcal{U}$, i.e., $V_\alpha \subset U_\alpha$ for $\alpha \in \kappa$.

It is proved in [11] that a space $X$ has property $\mathcal{B}$ if and only if every open cover of $X$ of infinite cardinality $\kappa$ has an open refinement $\mathcal{V}$ such that every point $x \in X$ has a neighborhood $W$ with $| \{ V \in \mathcal{V} : V \cap W \neq \emptyset \} | < \kappa$; we say such a refinement $\mathcal{V}$ is locally $\kappa$. It is known from Rudin [9] that normal spaces with property $\mathcal{B}$ are not necessarily paracompact. However, Balogh and Rudin [3] recently proved that a monotonically normal space is paracompact if it has property $\mathcal{B}$. Using the idea in Balogh [2] we now prove the following theorem.

**Theorem 1.** A locally Lindelöf space is paracompact if and only if it has property $\mathcal{B}$.

**Proof.** Let $X$ be a locally Lindelöf space with property $\mathcal{B}$. Suppose $X$ is not paracompact. Then there exists a minimal cardinal $\kappa$ such that we have...
some open cover $\mathcal{U}$ of $X$ of cardinality $\kappa$ which has no locally finite open refinement. We will show $\mathcal{U}$ has, however, a locally finite open refinement. Let $\mathcal{U}=\{U_\alpha : \alpha \in \kappa\}$. Since $X$ is countably paracompact and locally Lindelöf we can assume that $\kappa > \omega$ and each $U_\alpha$ is Lindelöf. There are two cases to consider.

Case 1. $\kappa$ is singular. Then $\text{cf}(\kappa) = \tau < \kappa$. Let $\{\kappa_\mu : \mu \in \tau\}$ be an increasing cofinal subset of $\kappa$ so that $\{\cup U_{\kappa_\mu} : \mu \in \tau\}$ is a monotone increasing open cover of $X$, where $U_\alpha = \{U_\beta : \beta \in \alpha\}$ for every $\alpha \in \kappa$. Since $X$ has property $\mathcal{B}$, there is a monotone increasing open cover $\{V_\mu : \mu \in \tau\}$ of $X$ such that $V_\mu \subset \cup U_{\kappa_\mu}$ for every $\mu \in \tau$. By the definition of $\kappa$, there exists a locally finite open collection $\mathcal{U}$ such that $\mathcal{U}$ refines $\cup U_{\kappa_\mu}$ and $\mathcal{V} \subset \cup \mathcal{U}$. Let us consider the open cover $\mathcal{U} = \cup \{\mathcal{U}_\mu : \mu \in \tau\}$ of $X$. Note that each member of $\mathcal{U}$ has Lindelöf closure, it is easy to check that each member of $\mathcal{U}$ meets at most $\tau$ many other members of $\mathcal{U}$. Using usual chaining argument, we may find some partition $\{\lambda_\alpha : \alpha \in A\}$ of $\mathcal{U}$ such that $(\cup \lambda_a) \cap (\cup \lambda_{a'}) = \emptyset$ if $a, a' \in A$ with $a \neq a'$, and $|\lambda_\alpha| \leq \tau$ for every $\alpha \in A$. By the definition of $\kappa$, $\lambda_a$ has, since $\cup \lambda_a$ is clopen, a locally finite open refinement $\mathcal{K}_a$, so that $\cup \{\mathcal{K}_a : a \in A\}$ is the desired refinement of $\mathcal{U}$.

Case 2. $\kappa$ is regular. Using property $\mathcal{B}$ find an open refinement $\mathcal{U}$ of $\mathcal{U}$ such that every point in $X$ has a neighborhood $V$ with

$$|\{G : G \in \mathcal{U}, G \cap V = \emptyset\}| < \kappa.$$ 

Clearly we may assume $\mathcal{U} = \{G_\alpha : \alpha \in \kappa\}$ with $G_\alpha \subset U_\alpha$ for every $\alpha \in \kappa$. Let us first show that

$$S = \{\alpha \in \kappa : \overline{G_\alpha} \neq \emptyset\}$$ 

is a non-stationary subset in $\kappa$, where $G_\alpha^* = \cup \{G_\beta : \beta \in \alpha\}$ for $\alpha \in \kappa$.

Suppose the contrary that $S$ is stationary. Then for every $\alpha \in S$, pick a point $x_\alpha \in \overline{G_\alpha} \setminus G_\alpha^*$ and let $s(\alpha) = \sup\{\mu \in \kappa : x_\alpha \in G_\mu\}$ which belongs to $\kappa$, since $\kappa$ is regular. Define a subset $C$ of $\kappa$ by

$$C = \{\alpha \in \kappa : \beta \in S \cap \alpha \text{ implies } s(\beta) < \alpha\}.$$ 

Let us check that $C$ is a c.u.b. set in $\kappa$. Indeed, if $\alpha \in C$, then there is a $\beta \in S \cap \alpha$ with $s(\beta) > \alpha$, so that $[\beta, \alpha]$ is a neighborhood of $\alpha$ which misses $C$. To see $C$ is unbounded, let $\alpha \in \kappa$ be given, since $S$ is stationary, we may find an $\alpha_1 \in S$ such that $\alpha < \alpha_1$. Proceeding by induction, find an $\alpha_{n+1} \in S$ so that

$$\alpha_{n+1} > \sup\{s(\mu) : \mu \in S, \mu \leq \alpha_n\}.$$ 

Then we obtain an increasing sequence $\{\alpha_n : n \in \mathbb{N}\}$ such that $\alpha < \sup\{\alpha_n : n \in \mathbb{N}\} \in C$. This concludes that $C$ is a c.u.b. set in $\kappa$. Let $S_1 = S \cap C$ and for every $\alpha \in S_1$ define $m(\alpha) = \min\{\mu \in \kappa : x_\alpha \in G_\mu\}$ so that $\alpha \leq m(\alpha) \leq s(\alpha)$. It follows that
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\[ x_\alpha \notin G_{m(\beta)} \text{ and } x_\beta \notin G_{m(\alpha)} \] whenever \( \alpha, \beta \in S \) with \( \alpha \neq \beta \). This implies that the set \( P = \{ x_\alpha : \alpha \in S \} \) consists of distinct points of \( X \), and \( \{ G_{m(\alpha)} : \alpha \in S \} \) is an open expansion of \( P \), i.e., \( G_{m(\alpha)} \cap P = \{ x_\alpha \} \) for every \( \alpha \in S \). Now for every \( \alpha \in S \), since \( x_\alpha \in \bigcup \{ G_\beta : \beta \in \alpha \} \), there is a \( \beta(\alpha) \in \alpha \) such that \( G_{\beta(\alpha)} \cap G_{m(\alpha)} \neq \emptyset \). By Pressing Down Lemma, there are \( \beta \in \kappa \) and a stationary set \( S_\beta \subset S \) such that \( \beta(\alpha) = \beta \) for all \( \alpha \in S_\beta \), consequently \( G_{\beta} \cap G_{m(\alpha)} \neq \emptyset \) for all \( \alpha \in S_\beta \). This contradicts our assumption that \( G_{\beta} \) is Lindelöf.

Now take a c.u.b. set \( C_1 \) in \( \kappa \) such that \( C_1 \cap S = \emptyset \) and thus \( G^*_{\kappa} \) is clopen for every \( \alpha \in C_1 \). Define \( H_\alpha \) for \( \alpha \in C_1 \) by

\[ H_\alpha = G^*_{\kappa} \setminus \{ \mu \in C_1 \cap C_\alpha \} \]

so that \( X = \bigcup \{ H_\alpha : \alpha \in C_1 \} \). Furthermore for every \( \alpha \in C_1 \), we have

(1) either \( H_\alpha = \emptyset \) or \( H_\alpha = G^*_{\kappa} \setminus G^*_{\mu(\alpha)} \) for some \( \mu(\alpha) \in C_1 \cap C_\alpha \). In fact, if \( H_\alpha = \emptyset \) then there is an \( x \in H_\alpha \), and thus there is \( \gamma \in \alpha \) such that \( x \in G_\gamma \) and \( x \notin G^*_{\mu} \) for any \( \mu \in C_1 \cap C_\gamma \). This shows \( (\gamma, \alpha) \cap C_1 = \emptyset \), because if there is some \( \mu \in (\gamma, \alpha) \cap C_1 \), then \( x \in G_\gamma \subseteq G^*_{\mu} \) which is impossible. Define \( \mu(\alpha) = \sup \{ \mu \leq \gamma : \mu \in C_1 \} \) which belongs to \( C_1 \). Then for every \( \mu \in C_1 \cap C_\alpha \), since \( (\gamma, \alpha) \cap C_1 = \emptyset \), we must have \( \mu \leq \gamma \). This implies \( \mu \leq \mu(\alpha) \) from which it follows that \( H_\alpha = G^*_{\kappa} \setminus G^*_{\mu(\alpha)} \), i.e., (1) holds. By the definition of \( \kappa \), we can find, for every \( \alpha \in C_1 \), a locally finite open cover of \( \mathcal{K}_\alpha \) of \( H_\alpha \) such that every member of \( \mathcal{K}_\alpha \) is contained in some member of \( \mathcal{U} \), so that \( \bigcup \{ \mathcal{K}_\alpha : \alpha \in C_1 \} \) is, since \( X \) is now the union of the disjoint clopen collection \( \{ H_\alpha : \alpha \in C_1 \} \), a locally finite open refinement of \( \mathcal{U} \). Thus the proof is complete.

In [9], by proving that the Navy’s space has property \( \mathcal{B} \), Rudin shows that normality plus property \( \mathcal{B} \) does not imply paracompactness. But the Navy’s space is metacompact [7], in connection with Arhangel’lskii and Tall’s problem, it is natural to ask if the Navy’s space is locally compact. But our Theorem 1 even shows that

**COROLLARY 1.** The Navy’s space is not locally Lindelöf.

Also from Theorem 1 the problem of Arhangel’lskii and Tall can be stated as follows:

**PROBLEM 1.** Does every locally compact normal metacompact space have property \( \mathcal{B} \)?

However note that normal metacompact spaces do not necessarily have property \( \mathcal{B} \), see Example 4.9 (ii) in [4] or [8] for such a counterexample.
With a modification of proof of Theorem 1 we can prove Arhangel'skii's result mentioned above, even we have

**THEOREM 2.** Locally Lindelöf perfectly normal metacompact spaces are paracompact.

**PROOF.** Since normal metacompact spaces are shrinking (thus countably paracompact), $\kappa$ and a point-finite open cover $\mathcal{G} = \{G_\alpha : \alpha \in \kappa\}$ can be defined in the same way as Theorem 1. Clearly we need only consider the case of $\kappa$ being regular, and it suffices to prove that

$$S = \{\alpha \in \kappa : \bigcup_{\beta < \alpha} G_\beta \setminus \bigcup_{\beta < \alpha} G_\beta \neq \emptyset\}$$

is non-stationary.

Suppose indirectly that $S$ is stationary. As in the proof of Theorem 1, define $m(\alpha) \in \kappa$ for every $\alpha \in S$. Without loss of generality, we may assume that there is a $\beta \in \kappa$ such that

$$G_{m(\alpha)} \cap \bigcap_{\beta \in \alpha} G_\beta \neq \emptyset$$

for all $\alpha \in S$.

For every $n \in \omega$ let

$$X_n = \{x \in X : \text{ord}(x, \mathcal{G}) \leq n\}.$$  

Then $X_n$ is closed in $X$. Let

$$S_n = \{\alpha \in S : G_{m(\alpha)} \cap \bigcap_{\beta \in \alpha} X_n \neq \emptyset\}$$

so that $S = \bigcup_{n \in \omega} S_n$ and thus there is a minimal $n \in \omega$ with $|S_n| = \kappa$.

Since

$$G_\beta \cap X_n = G_\beta \cap X_n \cap (X \setminus (G_\beta \cap X_{n-1})) \cup (G_\beta \cap X_{n-1}),$$

we can assume that

$$G_{m(\alpha)} \cap G_\beta \cap X_n \cap (X \setminus (G_\beta \cap X_{n-1})) \neq \emptyset$$

for all $\alpha \in S_n$.

Now every point in $G_\beta \cap X_n \cap (X \setminus (G_\beta \cap X_{n-1}))$ has a neighborhood which meets $G_{m(\alpha)} \cap G_\beta \cap X_n$ for at most finitely may $\alpha \in S_n$. Since $X$ is perfect, the set $G_\beta \cap X_n \cap (X \setminus (G_\beta \cap X_{n-1}))$ is Lindelöf, and hence

$$G_{m(\alpha)} \cap G_\beta \cap X_n \cap (X \setminus (G_\beta \cap X_{n-1})) \neq \emptyset$$

for at most countably many $\alpha \in S_n$, a contradiction proving $S$ is non-stationary. Thus the proof is complete.
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Note that normal submetacompact spaces are shrinking [11], but we do not know whether in Theorem 2 metacompactness can be replaced by submetacompactness, that is

Problem 2. Are locally Lindelöf perfectly normal and submetacompact spaces paracompact?

References


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