A NOTE ON THE HEREDITARY PROPERTIES
IN THE PRODUCT SPACE

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In this note we shall investigate some hereditary properties of a subspace of a product space.

Let $X_\alpha$ be a topological space for each $\alpha \in I$ and $A$ be a subset of $I$. $p_A$ is the projection: $\prod_{\alpha \in I} X_\alpha \to \prod_{\alpha \in A} X_\alpha$, i.e. $p_A(x)$ is the restricted function of $x$ whose domain is $A$. $A$ is co-countable if $I-A$ is countable.

The family of sets is linked if each pair of its members has non-empty intersection. The space has $(K)$-property (precaliber $\aleph_0$) (caliber $\aleph_0$), if any uncountable family of non-empty open subsets of $X$ includes an uncountable subfamily which is linked (has the finite intersection property) (has non-empty intersection).[2]

**THEOREM.** Let $X_\alpha$ be second-countable for $\alpha \in I$ and $X$ be a subspace of $\prod_{\alpha \in I} X_\alpha$ and $\phi$ be one of the properties:

1) the countable chain condition, 2) $(K)$-property, 3) precaliber $\aleph_0$, 4) caliber $\aleph_0$, 5) the separability, 6) the Lindelöf property.

Then, $X$ satisfies the hereditarily $\phi$ if and only if for any subspace $Y$ of $X$, there exists co-countable subset $A$ of $I$ such that $p_A^*Y$ satisfies $\phi$.

**LEMMA 1.** (N. A. Sanin) [1] Let $\Gamma$ be an uncountable set of finite sets, then $\Gamma$ includes an uncountable subfamily $\Delta$ which is quasi-disjoint i.e. $x \cap y \subseteq \bigcap \Delta$ for each different $x$ and $y$ of $\Delta$.

See [1] for the proof.

**LEMMA 2.** Let $f$ be a continuous function whose domain is $X$ and $X$ satisfies $\phi$ in the theorem. Then, the range of $f$ also satisfies $\phi$.

**Proof.** Easy to check.

Let $\{V^*_n; n<\omega\}$ be a base of $X_\alpha$. Then, $\{p_A^*V^*_n \times \cdots \times V^*_m; A = \{\alpha_1, \ldots, \alpha_m\}, m<\omega\}$ is a base of $\prod_{\alpha \in I} X_\alpha$. The domain of the basic open set $V=(p_A^*V^*_n \times \cdots \times V^*_m)$ is $A$.

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...×V_{\omega}\) is A, which is finite, and is denoted by dom V.

**Lemma 3.** Let \(\theta\) be an uncountable subfamily of \(\mathcal{V}\). Then, \(\theta\) includes an uncountable subfamily \(\Phi\) which has the following properties:

a) \(\{\text{dom } V; V \in \Phi\}\) is quasi-disjoint,

b) \(p_\alpha^n V = p_\alpha^n W\) for each \(V, W \in \Phi\), where \(A = \cap \{\text{dom } V; V \in \Phi\}\).

**Proof.** By Lemma 1, \(\theta\) includes an uncountable subfamily \(\theta'\) such that \(\{\text{dom } V; V \in \theta'\}\) is quasi-disjoint.

Let \(A = \cap \{\text{dom } V; V \in \theta'\}\). Then, \(\{p_\alpha^n V; V \in \theta'\}\) is countable. Hence, some uncountable subfamily \(\Phi\) of \(\theta'\) has the properties in the lemma.

**Proof of Theorem.** The necessity is clear and so we shall prove the sufficiency. Suppose that \(X\) does not satisfy the hereditarily \(\psi\). Then, there exists a subset \(\{x_\alpha; \alpha < \omega_1\}\) and a family \(\{O_\alpha; \alpha < \omega_1\}\) of open subsets of \(X\) such that \(x_\alpha \subseteq O_\alpha\) for each \(\alpha < \omega_1\), and

i) \(x_\alpha \subseteq \bigcap_{\beta \neq \alpha} O_\beta\) for any \(\beta \neq \alpha\),

ii) for any uncountable subset \(S\) of \(\omega_1\), there exists a pair \(\alpha, \beta\) of \(S\);

\[O_\alpha \cap O_\beta \cap \{x_\alpha; \alpha < \omega_1\} = \emptyset,\]

iii) for any uncountable subset \(S\) of \(\omega_1\), there exists a finite subset \(F\) of \(S\);

\[\bigcap_{\alpha \in F} O_\alpha \cap \{x_\alpha; \alpha < \omega_1\} = \emptyset,\]

iv) for any uncountable subset \(S\) of \(\omega_1\), \(\bigcap_{\alpha \in S} O_\alpha \cap \{x_\alpha; \alpha < \omega_1\} = \emptyset,\)

v) \(x_\alpha \subseteq \bigcap_{\beta > \alpha} O_\beta\) for any \(\beta > \alpha\), or

vi) \(x_\alpha \subseteq \bigcap_{\beta < \alpha} O_\beta\) for any \(\beta < \alpha\), according to \(\psi\) is 1), 2), 3), 4), 5) or 6), respectively.

We may take the above \(O_\alpha(\alpha < \omega_1)\) from \(\mathcal{V}\). By Lemma 3, without a loss of generality we can assume that \(\{O_\alpha; \alpha < \omega_1\}\) satisfies the conditions a) and b) of Lemma 3.

Now, we apply the assumption and Lemma 2 to \(\{x_\alpha; \alpha < \omega_1\}\). Then, there exists a co-countable subset \(A\) of \(I\) such that \(p_\alpha^n \{x_\alpha; \alpha < \omega_1\}\) satisfies \(\psi\) and \(\bigcap \{\text{dom } O_\alpha; \alpha < \omega_1\} \cap A\) is empty. Since \(\{\text{dom } O_\alpha; \alpha < \omega_1\}\) is quasi-disjoint, we may assume \(\text{dom } O_\alpha \cap \{\text{dom } O_\alpha; \alpha < \omega_1\} \subseteq A\) for \(\alpha < \omega_1\). There exists(s)

i) \(\alpha\) such that \(p_\alpha^n x_\alpha \in p_\beta^n \bigcap \bigcap \{\text{dom } O_\beta; \alpha < \omega_1\} = \emptyset\) for some \(\beta \notin \gamma\),

ii) an uncountable subset \(S\) of \(\omega_1\) such that

\[p_\alpha^n O_\alpha \cap \bigcap_{\beta \in S} \{x_\alpha; \alpha < \omega_1\} = \emptyset\] for each distinct \(\alpha, \beta \in S\),

iii) an uncountable subset \(S\) of \(\omega_1\) such that \(\bigcap_{a \in F} \{x_\alpha; \alpha < \omega_1\} = \emptyset\)

for any finite \(F \subseteq S\),
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iv) $\alpha$ and an uncountable subset $S$ of $\omega_1$ such that $p_A(x_\alpha) \subseteq \bigcap_{\beta \in S} p'_A O_{\alpha \beta}$,

v) $\alpha$ such that $p_A(x_\alpha) \subseteq p'_2 O_\beta$ for some $\beta > \alpha$, or

vi) $\alpha$ such that $p_A(x_\alpha) \subseteq p'_2 O_\beta$ for some $\beta < \alpha$,

according that $\phi$ is 1), 2), 3), 4), 5) or 6), respectively.

By the assumption of $A$ and the fact $x_\alpha \in O_\alpha$, $p_A(x_\alpha) \subseteq p'_2 O_\beta$ holds if and only if $x_\alpha \in O_\beta$ holds, for each $\alpha, \beta$. So, i)′, …, or vi)′ contradicts to i), …, or vi) respectively.

Now, the proof is complete.

Since the hereditary separability is equivalent to the hereditary caliber-$\aleph_1$-property, it is a little interesting to compare the two cases 4) and 5) in the theorem.

References


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