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TAME TRIANGULAR MATRIX ALGEBRAS OVER SELF-INJECTIVE ALGEBRAS

By

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Dedicated to Professor Hisao Tominaga on his 60th birthday

Throughout this note, we will work over a fixed algebraically closed field k. The notations and the terminologies will be the same as in [6], [9] and [10]. Let Λ be a finite dimensional self-injective algebra and assume that Λ is basic, connected and non-simple. For an integer $p\geqq 2$, denote by $T_p(\Lambda)$ the algebra of the $p\times p$ upper triangular matrices over Λ. We ask when $T_p(\Lambda)$ is tame. So we assume further that Λ is representation-finite. Otherwise, $T_p(\Lambda)$ has to be wild [13]. Then, as well known, the universal cover of the stable Auslander-Reiten quiver of Λ is isomorphic to a Dynkin-translation-quiver $\mathbf{Z}\Delta$ [7], where $\Delta=A_q$ $(q\geqq 1)$, $\mathbf{D}_q(q\geqq 4)$ or $\mathbf{E}_q(6\leqq q\leqq 8)$, and Δ is called the Dynkin class of Λ. Our aim is to prove the following

THEOREM. *Let Λ be as above. Then, $T_2(\Lambda)$ is tame if and only if Λ has Dynkin class $\mathbf{A}_3$.*

REMARK 1. *The case $p>2$ is rather easy. Denote by $J_p(\Lambda)$ the ideal of $T_p(\Lambda)$ consisting of the strictly upper triangular matrices. Suppose $p>2$ and $p\geqq r\geqq 2$. Then, $T_p(\Lambda)/J_p(\Lambda)^r$ is tame if and only if Λ is a Nakayama algebra of Dynkin class $\mathbf{A}_q$ and $(p, q, r)=(3, 2, 2)$, $(4, 1, 3)$ or $(4, 1, 4)$ (cf. [11]).*

REMARK 2. *For a Dyinkin-translation-quiver $\mathbf{Z}\Delta$, the mesh category $k(\mathbf{Z}\Delta)$ is known to be locally bounded [2], and it is not difficult to check the following: i) $k(\mathbf{Z}\Delta)$ is locally representation-finite if $\Delta=\mathbf{A}_q(q\leqq 4)$; ii) $k(\mathbf{Z}\Delta)$ is locally support-finite and tame if $\Delta=\mathbf{A}_5$ or $\mathbf{D}_4$; iii) $k(\mathbf{Z}\Delta)$ has a finite quotient which is wild if Δ is otherwise.*

Proof of Theorem

Let us consider first the case where Λ is a Nakayama algebra. Suppose that Λ is a Nakayama algebra of Dynkin class $\mathbf{A}_q$. Then, $T_2(\Lambda)$ has the following universal Galois covering U:

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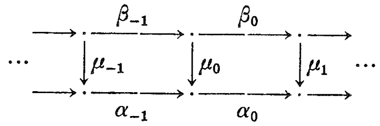
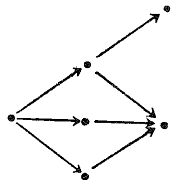


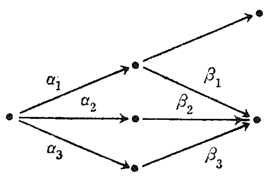
with $\alpha_i\mu_i-\mu_{i+1}\beta_i=\alpha_{i+q}\cdots\alpha_{i+1}\alpha_i=\beta_{i+q}\cdots\beta_{i+1}\beta_i=0$ for all $i\in\mathbf{Z}$. If $q\leqq 2$ then U is locally representation-finite [4], if $q=3$ then U is locally support-finite and tame [12], and if $q\geqq 4$ then U has a finite quotient which is wild [12]. Thus, in this case, $T_2(\Lambda)$ is tame if and only if $q=3$.

In what follows, we assume that Λ is not a Nakayama algebra. Then, there is no DTr-invariant module [5]. Notice also that Λ is a Nakayama algebra if Λ has Dynkin class $\mathbf{A}_q(q\leqq 2)$.

Consider next the case where Λ has Dynkin class $\mathbf{A}_q(q\geqq 4)$, $\mathbf{D}_q(q\geqq 4)$ or $\mathbf{E}_q$ $(6\leqq q\leqq 8)$. Then, as easily seen, the Auslander-Reiten quiver of Λ has the following full subquiver:



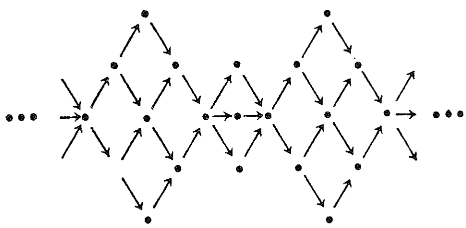
and, as a quotient, the Auslander algebra over Λ has the following algebra:

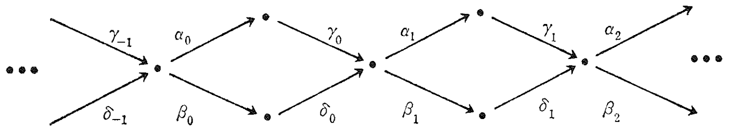


with $\beta_1\alpha_1+\beta_2\alpha_2+\beta_3\alpha_3=0$.

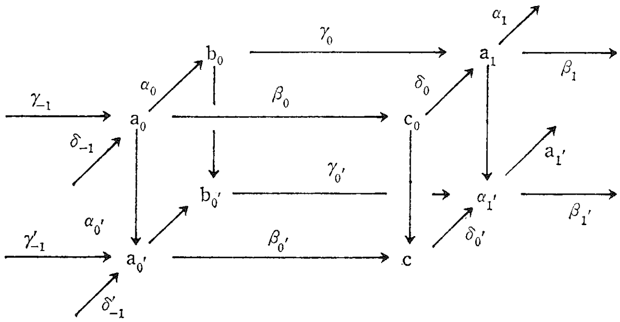
This is a concealed hereditary algebra of type $\tilde{\tilde{\mathbf{D}}}_4$. Notice that $T_2(\Lambda)$ is representation equivalent to the Auslander algebra over Λ [1], because Λ is assumed to be representation-finite. Thus $T_2(\Lambda)$ is wild.

It only remains the case of Λ having Dynkin class $\mathbf{A}_3$. Then, the universal cover of the Auslander-Reiten quiver of Λ is the following:



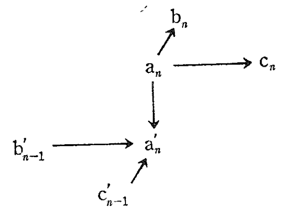


Thus, since Λ is standard [8], Λ has the following universal Galois covering: with $\gamma_i\alpha_i-\delta_i\beta_i=\alpha_{i+1}\delta_i=\beta_{i+1}\gamma_i=0$ for all $i\in\mathbf{Z}$ [4]. Hence, by [3], it suffices to prove that the following locally bounded category U is locally support-finite and tame:



with $\alpha_{i+1}\delta_i=\beta_{i+1}\gamma_i=\alpha'_{i+1}\delta'_i=\beta'_{i+1}\gamma'_i=0$ for all $i\in\mathbf{Z}$ and with all the squares commutative.

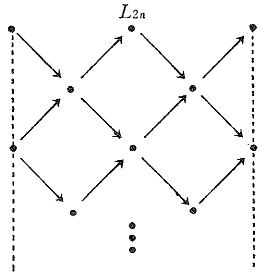
For each $n\in\mathbf{Z}$, let A_{2n} be the following full subcategory



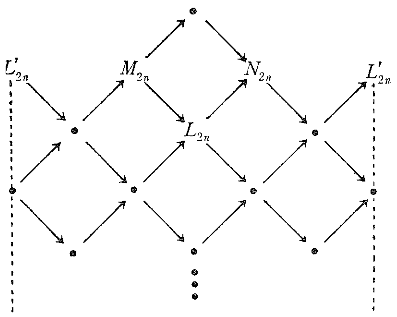
this is a hereditary algebra of type $\tilde{\mathbf{D}}_5$, and let B_{2n} and B^*_{2n} be the full subcategories obtained from A_{2n} by adding b_{n-1} and b'_n respectively, these are tilted algebras of type $\tilde{\mathbf{E}}_6$. Then, the full subcategory $B_{2n}\cup B^*_{2n}$ consisting of the objects of B_{2n} and B^*_{2n} is, as an algebra, isomorphic to

$$\begin{bmatrix} k & DL_{2n} & k \\ 0 & A_{2n} & L_{2n} \\ 0 & 0 & k \end{bmatrix}$$

where $L_{2n}={}_{1}{}_{0}^{1}{}^{1}{}^{0}$ is a regular module:



The vector space categories Hom(L_{2n}, mod A_{2n}) and Hom(mod A_{2n}, L_{2n}) belong to the pattern ($\tilde{\mathbf{D}}_5$, 2), and ind $(B_{2n}\cup B^*_{2n})=P_{2n}\uplus R_{2n}\uplus Q_{2n}$, where P_{2n} consists of the objects of ind B^*_{2n} with restriction to A_{2n} being preprojective, Q_{2n} consists of the objects of ind B_{2n} with restriction to A_{2n} being preinjective and R_{2n} consists of the regular objects of ind A_{2n} except that the above tube changes to the following:

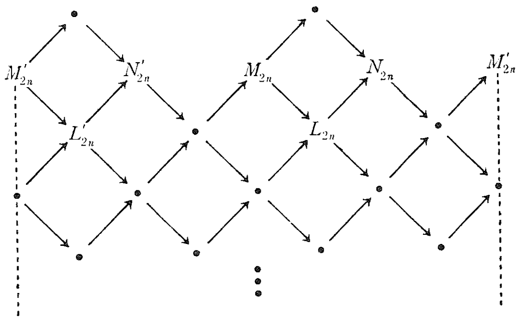


Thus, $B_{2n}\cup B^*_{2n}$ is tame.

Further, let C_{2n} and C^*_{2n} be the full subcategories obtained from B_{2n} and B^*_{2n} by adding c_{n-1} and c'_n respectively, these are tilted algebras of type $\tilde{\mathbf{E}}_7$. Then, $C_{2n}\cup C^*_{2n}$ is isomorphic to

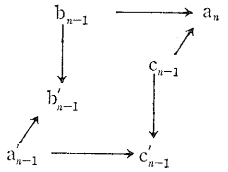
$$\begin{bmatrix} k & DL'_{2n} & k \\ 0 & B_{2n}\cup B^*_{2n} & L'_{2n} \\ 0 & 0 & k \end{bmatrix}$$

and the vector space categories Hom(L'_{2n}, mod$(B_{2n}\cup B^*_{2n})$) and Hom (mod$(B_{2n}\cup B^*_{2n})$, L'_{2n}) are isomorphic to Hom(L'_{2n}, modB_{2n}) and Hom(mod B^*_{2n}, L'_{2n}) respectively. Hence, both of them belong to the pattern ($\tilde{\mathbf{E}}_6$, 3). We have ind $(C_{2n}\cup C^*_{2n})=P'_{2n}\uplus R'_{2n}\uplus Q'_{2n}$, where P'_{2n} consists of the objects of ind C^*_{2n} with restriction to $B_{2n}\cup B^*_{2n}$ lying in P_{2n}, Q'_{2n} consists of the objects of ind C_{2n} with restriction to $B_{2n}\cup B^*_{2n}$ lying in Q_{2n} and R'_{2n} coincides with R_{2n} except that the above tube changes to the following:



Thus, $C_{2n} \cup C^*_{2n}$ is tame.

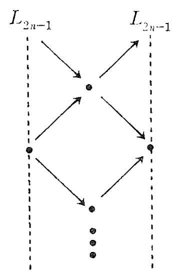
Similarly, for each $n \in \mathbf{Z}$, let A_{2n-1} be the following full subcategory



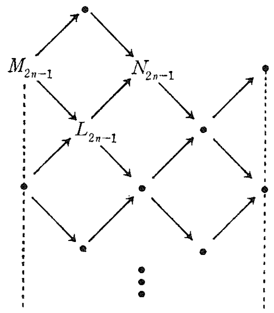
this is a hereditary algebra af type $\tilde{\mathbf{A}}_{33}$, and let B_{2n-1} and B^*_{2n-1} be the full subcategories obtained from A_{2n-1} by adding a_{n-1} and a'_n respectively, these are tilted algebras of type $\tilde{\mathbf{E}}_6$. Then, $B_{2n} \cup B^*_{2n}$ is isomorphic to

$$\begin{bmatrix} k & DL_{2n-1} & k \\ 0 & A_{2n-1} & L_{2n-1} \\ 0 & 0 & k \end{bmatrix}$$

where $L_{2n-1} = {}_1{}^{1}_{1}{}^{1}_{1}{}^{1}$ is a regular module:



The vector space categories $\mathrm{Hom}(L_{2n-1}, \mathrm{mod}\, A_{2n-1})$ and $\mathrm{Hom}(\mathrm{mod}\, A_{2n-1}, L_{2n-1})$ belong to the pattern $(\tilde{\mathbf{A}}_{33}, 1)$, and $\mathrm{ind}(B_{2n-1} \cup B^*_{2n-1}) = P_{2n-1} \uplus R_{2n-1} \uplus Q_{2n-1}$, where P_{2n-1} consists of the objects of ind B^*_{2n-1} with restriction to A_{2n-1} being preprojective, Q_{2n-1} consists of the objects of ind B_{2n-1} with restriction to A_{2n-1} being preinjective and R_{2n-1} consists of the regular objects of ind A_{2n-1} except that the above tube changes to the following:

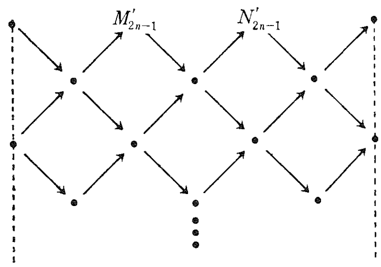


Thus, $B_{2n-1}\cup B^*_{2n-1}$ is tame.

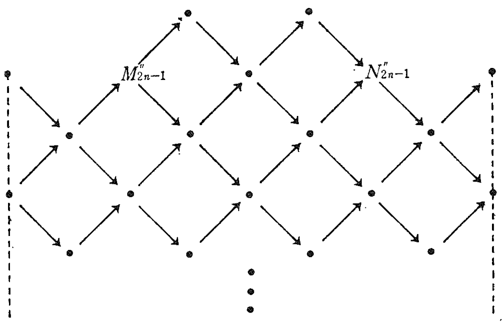
Further, let C_{2n-1} and C^*_{2n-1} be the full subcategories obtained from B_{2n-1} and B^*_{2n-1} by adding b'_{n-2} and b_n respectively, these are tilted algebras of type $\tilde{\mathbf{E}}_7$. Then, $C_{2n-1}\cup C^*_{2n-1}$ is isomorphic to

$$\begin{bmatrix} k & DN'_{2n-1} & 0 \\ 0 & B_{2n-1}\cup B^*_{2n-1} & M'_{2n-1} \\ 0 & 0 & 0 \end{bmatrix}$$

where $M'_{2n-1}=\begin{smallmatrix} & 0 & & 0 \\ 0 & 1 & 0 & 0 \\ 1 & & 0 & \end{smallmatrix}$ and $N'_{2n-1}=\begin{smallmatrix} & 1 & & 1 \\ 0 & 0 & 0 & 0 \\ 0 & & 0 & \end{smallmatrix}$ are regular modules:



The vector space categories $\mathrm{Hom}(M'_{2-n1}, \mathrm{mod}(B_{2n-1}\cup B^*_{2n-1}))$ and $\mathrm{Hom}(\mathrm{mod}(B_{2n-1}\cup B^*_{2n-1}), N'_{2n-1})$ are isomorphic to $\mathrm{Hom}(M'_{2n-1}, \mathrm{mod}\, B_{2n-1})$ and $\mathrm{Hom}(\mathrm{mod}\, B^*_{2n-1}, N'_{2n-1})$ respectively, thus belong to the pattern $(\tilde{\mathbf{E}}_6, 3)$. We have $\mathrm{ind}(C_{2n-1}\cup C^*_{2n-1})=P'_{2n-1}\uplus R'_{2n-1}\uplus Q'_{2n-1}$, where P'_{2n-1} consists of the objects of ind C^*_{2n-1} with restriction to $B_{2n-1}\cup B^*_{2n-1}$ lying in P_{2n-1}, Q'_{2n-1} consists of the objectes of ind C_{2n-1} with restriction to $B_{2n-1}\cup B^*_{2n-1}$ lying in Q_{2n-1} and R'_{2n-1} coincides with R_{2n-1} except that the above tube changes to the following:

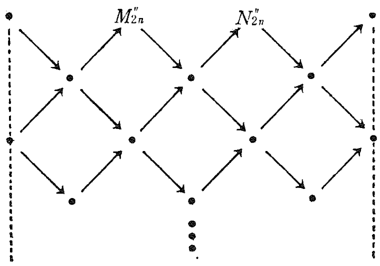


Thus, $C_{2n-1} \cup C^*_{2n-1}$ is tame.

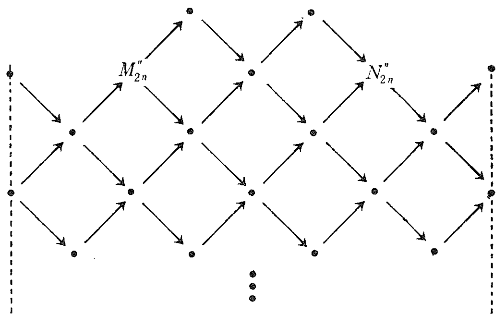
Now, for any l, $m \in \mathbf{Z}$ with $l \leqq m$, let $A_{l,m}$ be the full subcategory consisting of the objects of the A_n, $l \leqq n \leqq m$. Then, for each $n \in \mathbf{Z}$, $A_{2n-1,\ 2n+1}$ is isomorphic to

$$\begin{bmatrix} k & DN''_{2n} & 0 \\ 0 & C_{2n} \cup C^*_{2n} & M''_{2n} \\ 0 & 0 & k \end{bmatrix}$$

where $M''_{2n} = \begin{smallmatrix} & & 0 & & \\ 0 & 0 & & 0 & 0 \\ 1 & 1 & & 0 & \\ & 1 & & & \end{smallmatrix}$ and $N''_{2n} = \begin{smallmatrix} & & 1 & & \\ 0 & 0 & & 0 & 1 \\ 0 & 0 & & 0 & \\ & 0 & & & \end{smallmatrix}$ are regular modules:



The vector space categories $\mathrm{Hom}(M''_{2n}, \mathrm{mod}(C_{2n} \cup C^*_{2n}))$ and $\mathrm{Hom}(\mathrm{mod}(C_{2n} \cup C^*_{2n}), N''_{2n})$ are ismorphic to $\mathrm{Hom}(M''_{2n}, \mathrm{mod}\ C_{2n})$ and $\mathrm{Hom}(\mathrm{mod}\ C^*_{2n}, N''_{2n})$ respectively, thus belong to the pattern $(\tilde{\mathbf{E}}_7, 3)$. We have ind $A_{2n-1,\ 2n+1} = P''_{2n} \uplus R''_{2n} \uplus Q''_{2n}$, where P''_{2n} consists of the objects of $\mathrm{ind}(C^*_{2n} \cup A_{2n+1})$ with restriction to $C_{2n} \cup C^*_{2n}$ lying in P'_{2n}, Q''_{2n} consists of the objects of $\mathrm{ind}(A_{2n-1} \cup C_{2n})$ with restriction to $C_{2n} \cup C^*_{2n}$ lying in Q'_{2n} and R''_{2n} coincides with R'_{2n} except that the above tube changes to the following:

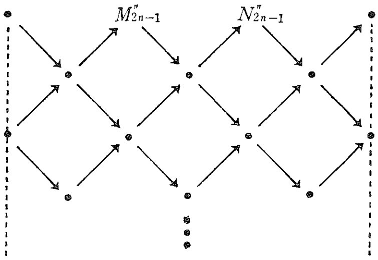


Thus, $A_{2n-1,\ 2n+1}$ is tame.

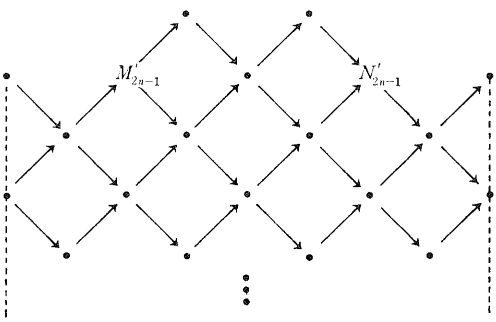
Similarly, for each $n\in\mathbf{Z}$, $A_{2n-2,\ 2n}$ is isomorphic to

$$\begin{bmatrix} k & DN''_{2n-1} & 0 \\ 0 & C_{2n-1}\cup C^{*}_{2n-1} & M''_{2n-1} \\ 0 & 0 & k \end{bmatrix}$$

where $M''_{2n-1}={}_0\,{}^{0\,0\,0\,0}_{1\,0\,1\,0}$ and $N''_{2n-1}={}_0\,{}^{0\,0\,1\,0}_{0\,0\,1\,0}$ are regular modules:



The vector space categories $\mathrm{Hom}(M''_{2n-1}, \mathrm{mod}(C_{2n-1}\cup C^{*}_{2n-1}))$ and $\mathrm{Hom}(\mathrm{mod}(C_{2n-1}\cup C^{*}_{2n-1}), N''_{2n-1})$ are isormorphic to $\mathrm{Hom}(M''_{2n-1}, \mathrm{mod}\ C_{2n-1})$ and $\mathrm{Hom}(\mathrm{mod}\ C^{*}_{2n-1}, N''_{2n-1})$ respectively, thus belong to the pattern $(\tilde{\mathbf{E}}_7, 3)$. We have $\mathrm{ind}\ A_{2n-2,\ 2n}=P''_{2n-1}\uplus R''_{2n-1}\uplus Q''_{2n-1}$, where P''_{2n-1} consists of the objects of $\mathrm{ind}(C^{*}_{2n-1}\cup A_{2n})$ with restriction to $C_{2n-1}\cup C^{*}_{2n-1}$ lying in P'_{2n-1}, Q''_{2n-1} consists of the objects of $\mathrm{ind}(A_{2n-2}\cup C_{2n-1})$ with restriction to $C_{2n-1}\cup C^{*}_{2n-1}$ lying in Q'_{2n-1} and R''_{2n-1} coincides with R'_{2n-1} except that the above tube changes to the following:



Thus, $A_{2n-2,\,2n}$ is tame.

Finally, for any l, $m\in\mathbf{Z}$ with $l\leqq m$, $B_l\cup A_{l,m+1}$ is the one-point extension of $A_{l,m+1}$ by M_l and the vector space category Hom(M_l, mod $A_{l,m+1}$) is isomorphic to Hom(L_l, mod A_l). We have ind$(B_l\cup A_{l,m+1})=$ind$(B_l\cup B_l^*)\cup$ind $A_{l,m+1}$. Next, $C_l\cup A_{l,m+1}$ is the one-point extension of $B_l\cup A_{l,m+1}$ by M_l' and the vector space category Hom(M_l', mod($B_l\cup A_{l,m+1}$)) is isomorphic to Hom(M_l', mod($B_l\cup C_l^*$)) and belongs to the pattern ($\tilde{\mathbf{E}}_6$, 3). We have ind$(C_l\cup A_{l,m+1})=$ind$(C_l\cup C_l^*)\cup$ind$A_{l,m+1}$. Finally, $A_{l-1,m+1}$ is the one-point extension of $C_l\cup A_{l,m+1}$ by M_l'' and the vector space category Hom(M_l'', mod($C_l\cup A_{l,m+1}$)) is isomorphic to Hom (M_l'', mod($C_l\cup C_l^*$)). We have ind $A_{l-1,m+1}=$ind $A_{l-1,l+1}\cup$ind $A_{l,m+1}$. Therefore, ind $U=\cup_{n\in\mathbf{Z}}$ ind $A_{n-1,n+1}$. We are done.

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