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ON SPAN AND INVERSE LIMITS

By

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1. Introduction.

A compact metric space is called a *compactum* and a connected compactum is called a *continuum*. All maps in this paper are continuous. Let $f: X\rightarrow Y$ be a map between continua. Ingram [2] and Lelek [11] defined the *span*, *semispan*, *surjective span*, and *surjective semispan* of f by the following formulas (the map $p_i: X\times X\rightarrow X$ denotes the projection to the i-th factor, $i=1, 2$).

$\tau=\sigma, \sigma_0, \sigma^*, \sigma_0^*$.

$$\tau(f)=\sup\left\{c\geqq 0 \;\middle|\; \begin{array}{l}\text{there exists a continuum } Z\subset X\times X \text{ such} \\ \text{that } Z \text{ satisfies the condition } \tau) \text{ and} \\ d(f(x), f(y))\geqq c \text{ for each } (x, y)\in Z\end{array}\right\},$$

where the condition $\tau)$ is:

$$p_1(Z)=p_2(Z) \text{ if } \tau=\sigma, \quad p_1(Z)\supset p_2(Z) \text{ if } \tau=\sigma_0,$$
$$p_1(Z)=p_2(Z)=X \text{ if } \tau=\sigma^*, \quad p_1(Z)=X \text{ if } \tau=\sigma_0^*.$$

The span of a continuum X is defined by $\sigma(id_X)$. The other cases are similar. In the same way, we can define the *symmetric span* of f by the formula

$$s(f)=\sup\left\{c\geqq 0 \;\middle|\; \begin{array}{l}\text{there exists a continuum } Z\subset X\times X \text{ such that} \\ Z \text{ is symmetric (i.e. } (x, y)\in Z \text{ iff } (y, x)\in Z) \\ \text{and } d(f(x), f(y))\geqq c \text{ for each } (x, y)\in Z\end{array}\right\}.$$

It is a mapping version of symmetric span of a continuum due to J. F. Davis [1].

Let $X=\varprojlim(X_n, p_{n\,n+1})$ be a continuum, where $p_{n\,n+1}: X_{n+1}\rightarrow X_n$. Ingram [2] and [4] showed that $\sigma(X)=0$ if and only if there exists a cofinal subsequence $(n_i)_{i\geqq 1}$ such that $\lim_j \sigma(p_{n_i n_j})=0$ for each $i\geqq 1$. In section 2 of this paper, we will prove a mapping version of this theorem. H. Cook proved essentially that the symmetric span of the dyadic solenoid is zero ([1], p. 134), while its span is positive. The author wishes to thank to the referee for pointing out this fact. In section 3, we generalize this to the poly-adic solenoid. Let f and $g: X\rightarrow Y$ be maps. $d(f, g)$ denotes $\sup\{d(f(x), g(x))\mid x\in X\}$.

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2. Span and a limit of maps.

Let $X=\varprojlim(X_n, p_{n\,n+1})$ and $Y=\varprojlim(Y_n, q_{n\,n+1})$ be compacta, where all X_n and Y_n are polyhedra and both of $p_{n\,n+1}: X_{n+1}\to X_n$ and $q_{n\,n+1}: Y_{n+1}\to Y_n$ are surjective for each $n\geqq 0$. The maps $p_n: X\to X_n$ and $q_n Y\to Y_n$ denote the projection maps. Under these notations, Mioduszewski showed the following [15].

THEOREM 1. 1) *For every sequence* (ε_n) *of positive numbers with* $\lim \varepsilon_n=0$, *there exist cofinal increasing subsequences* (m_k) *and* (n_k) *and maps* $f_k: X_{m_k}\to Y_{n_k}$ *such that diagrams (A) and (B) are* ε_k*-commutative for each* $s\leqq k\leqq l$.

$$\begin{array}{ccccc} & & X_{m_k} & \longleftarrow & X \\ & & {\scriptstyle f_k}\downarrow & & \downarrow{\scriptstyle f} \\ Y_{n_s} & \longleftarrow & Y_{n_k} & \longleftarrow & Y \end{array} \qquad\qquad \begin{array}{ccccc} & & X_{m_k} & \longleftarrow & Y_{n_l} \\ & & {\scriptstyle f_k}\downarrow & & \downarrow{\scriptstyle f_l} \\ X_{m_s} & \longleftarrow & Y_{n_k} & \longleftarrow & Y_{m_l} \end{array}$$

(A) (B)

2) *Conversely, if we are given diagram (B), then we can find a map* $f: X\to Y$ *which satisfies diagram (A) for each k. If all* f_k*'s are surjective, f can be constructed so as to be surjective.*

Notice that the map f is defined by $q_{n_s}f=\lim_k q_{n_s n_k} f_k p_{m_k}$.

We say that f is *weakly induced* by the sequence (f_k). This terminology is due to Oversteegen and Tymchatyn [13].

THEOREM 2. *Let* $f: X\to Y$ *be a map between continua which is weakly induced by a sequence* $(f_k: X_{m_k}\to Y_{n_k})$ *Then,*

$\tau(f)=0$ *if and only if there exists a cofinal subsequence* (n_{k_j}) *of* (n_k) *such that* $\lim_j \tau(q_{n_{k_i} n_{k_j}} f_{k_j})=0$ *for each i. Where,* $\tau=\sigma, \sigma^*, \sigma_0, \sigma_0^*$, *and s.*

The basic idea of the proof is in [2] and [3]. But we need some preparations. Throughout this section, τ denotes σ, σ_0, σ^*, σ_0^*, and s unless otherwise stated.

PROPOSITION 3. *Let* $f: X\to Y$ *and* $g: Y\to Z$ *be maps.*
1) $\tau(gf)\leqq\tau(g)$. 2) *If* $\tau(f)=0$, *then* $\tau(gf)=0$.

POPOSITION 4. *Let* $(f_n: X\to Y)$ *be a sequence of maps which converges uniformly to a map* $f: X\to Y$. *Then* $\tau(f)=\lim \tau(f_n)$.

The proof of the above two propositions are easy and will be omitted.

PROPOSITION 5. 1) *Let X_n's and X be continua in a metric space M and let Y_n's and Y be continua in a metric space N. Suppose that $f: X\to Y$, $f_n: X_n\to Y_n$, $p_n: X\to X_n$, and $q_n: Y\to Y_n$ satisfy the following conditions.*

a) $\operatorname{Lim} X_n=X$, $\operatorname{Lim} Y_n=Y$. *Both of* $X\cup\bigcup_{n=1}^{\infty} X_n$ *and* $Y\cup\bigcup_{n=1}^{\infty} Y_n$ *are compact.*

b) *Both of the maps p_n and q_n are $1/2^n$-translation (that is, $d(x, p_n(x))<1/2^n$ for each $x\in X$ etc.).*

c) *There exists a decreasing sequence of prositive numbers ε_n's with* $\lim \varepsilon_n=0$, *such that $d(q_n f, f_n p_n)<\varepsilon_n$.*

d) *Define $F: X\cup\bigcup X_n\to Y\cup\bigcup Y_n$ by $F|X=f$, $f|X_n=f_n$. Then F is well defined and continuous.*

Then $\tau(f)=\lim \tau(f_n)$.

2) *We can replace condition* d) *by*

e) *Each p_n is surjective.*

Reasoning the same way as in [10, 3.1] and [5, 2.1], we can show two inequalities; $\limsup \tau(f_n)\leqq\tau(f)\leqq\liminf \tau(f_n)$, which imply the conclusion.

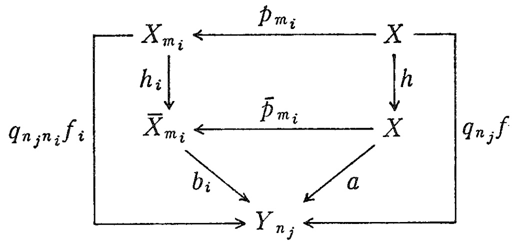
PROOF OF THEOREM 2. To simplify the notations, a cofinal subsequence of (n_i) is also denoted by (n_i). First we assume that $\tau(f)=0$. Take any subsequence (n_i) and an integer $j>0$. It suffices to prove that $\lim_i \tau(q_{n_j n_i} f_i)=0$. Let A be a compactum satisfying the following conditions.

1) $A=\bar{X}\cup\bigcup\bar{X}_{m_k}$, where $\bar{X}$ and $\bar{X}_{m_k}$ are homeomorphic to X and X_{m_k} respectively. $\bar{X}\cap\bar{X}_{m_k}=\emptyset=\bar{X}_{m_k}\cap\bar{X}_{m_l}$ for each $k\neq l$.

2) Let $h: X\to\bar{X}$ and $h_k: X_{m_k}\to\bar{X}_{m_k}$ be homeomorphisms. There exists an ε_k-translation $\bar{p}_{m_k}:\bar{X}\to\bar{X}_{m_k}$ satisfying $h_k p_{m_k}=\bar{p}_{m_k} h$.

3) $\operatorname{Lim}\bar{X}_{m_k}=\bar{X}$.

That such space A exists is well known. As each bonding map is surjective, we can take each $\bar{p}_{m_k}$ to be surjective. Consider the following diagram.



Where, $a=q_{n_j} f h^{-1}$ and $b_i=q_{n_i n_j} f_i h_i^{-1}$. Then,

4) $d(a, b_i\bar{p}_{m_i})=d(q_{n_j}fh^{-1}, q_{n_j n_i}f_i h_i^{-1}h_i p_{m_i}h^{-1})$

$$=d(q_{n_j n_i}q_{n_i}f, q_{n_j n_i}f_i p_{m_i})<\varepsilon_i$$

by the ε_i-commutativity of (A). It is easy to see that $\tau(a)=\tau(q_{n_j}f)$ and $\tau(b_i)=\tau(q_{n_j n_i}f_i)$. Applying Proposition 3.2), Proposition 5 and by condition 4), we have

$$\lim_i \tau(q_{n_j n_i}f_i)=\lim_i \tau(b_i)=\tau(a)=\tau(q_{n_j}f)=0.$$

Next we assume that a cofinal subsequence satisfies the hypothesis. By Proposition 4 and Proposition 3.1),

$$\tau(q_{n_j}f)=\lim_i \tau(q_{n_j n_i}f_i p_i)$$
$$\leqq \lim_i \tau(q_{n_j n_i}f_i)=0.$$

To show that $\tau(f)=0$, we take any continuum Z in $X\times X$ satisfying condition τ). There exists a point $(x^j, y^j)\in Z$ such that $q_{n_j}f(x^j)=q_{n_j}f(y^j)$, because $\tau(q_{n_j}f)=0$ for each j. We can assume that $(x^j, y^j)\to(x, y)$ as $j\to\infty$. If $j<i$,

$$q_{n_j}f(x^i)=q_{n_j n_i}q_{n_i}f(x^i)$$
$$=q_{n_j n_i}q_{n_i}f(y^i)=q_{n_j}f(y^i).$$

Tending i to infinity, we have

$$q_{n_j}f(x)=q_{n_j}f(y) \quad \text{for each } j \text{ and hence } f(x)=f(y).$$

This completes the proof.

THEOREM 6. *Suppose that X, Y, f and f_n satisfy the hypothesis of Theorem 2. If there exists a cofinal subsequence (n_i) such that $\lim_j \tau(f_{n_i}p_{n_i n_j})=0$, then $\tau(f)=0$.*

PROOF. For each $s<i$, $\tau(q_{n_s n_i}f_{n_i}p_{n_i})=0$, because by Proposition 3,

$$\tau(f_{n_i}p_{n_i})=\lim_j \tau(f_{n_i}p_{n_i n_j}p_{n_j})$$
$$\leqq \lim_j \tau(f_{n_i}p_{n_i n_j})=0.$$

Using the ε_j-commutativity of the diagram (A) and (B), we have $\tau(q_{n_i}f)\leqq \tau(q_{n_i n_j}f_j p_{n_j})+2\varepsilon_j=2\varepsilon_j$ for each $j>i$. Therefore $\tau(q_{n_i}f)=0$ for each i and $\tau(f)=0$.

COROLLARY 7 [8 and 10]. *Let $X=\varprojlim(X_n, p_{n\,n+1})$ be a continuum represented as the inverse limit of continua and onto bonding maps. Then the followings are equivalent.*

1) $\tau(X)=0$.

2) *There exists a cofinal subsequence (n_i) such that $\lim_j \tau(p_{n_i n_j})=0$ for each i.*

3) *For each* n, $\tau(p_n)=0$.

In Theorem 2 and 6, no conditions on p_n's and q_n's, on X_n's and Y_n's are required. If we add some conditions, the followings are obtained.

PROPOSITION 8. *Suppose* X, Y, f, f_n, p_n *and* q_n *satisfy the hypothesis of Theorem* 2. *Moreover assume that:*

1) *All* p_n*'s are monotone. or*

2) X *is tree-like and each* X_n *is a finite tree. Each* p_n *is an open onto map.* $\tau=\sigma$, σ_0, *and* s.

If $\tau(f)=0$, *then* $\lim_n \tau(f_n)=0$.

PROOF. 1) For each $n\geqq 0$ and for each continuum $Z\subset X_n\times X_n$ satisfying τ), $(p_n\times p_n)^{-1}(Z)$ is a continuum in $X\times X$ satisfying τ). There exists a $(x, y)\in (p_n\times p_n)^{-1}(Z)$ such that $f(x)=f(y)$. Then

$$\begin{aligned} d(f_n p_n(x), f_n p_n(y)) &\leqq d(f_n p_n(x), q_n f(x)) + d(q_n f(y), f_n p_n(y)) \\ &\leqq 2\varepsilon_n . \end{aligned}$$

Hence $\tau(f_n)\leqq 2\varepsilon_n$ and this completes the proof.

2) We need the following theorem for the proof.

THEOREM 9 [14, p. 189]. *Let* X *and* Y *be compacta and* $f: X\to Y$ *be a light open map from* X *onto* Y. *For each dendrite* D *in* Y, *there exists a dendrite* D_1 *in* X *such that* $f(D_1)=D$ *and* $f|D_1$ *is a homeomorphism on* D.

Using this Theorem, 2) is shown as follows.

Let n be a positive integer. There exists a continuum W_n and maps $r_n: X\to W_n$, $s_n: W_n\to X_n$ such that r_n is monotone and s_n is light open and $s_n r_n=p_n$. As X_n is a tree, there exists a dendrite T_n in W_n such that $s_n(T_n)=X_n$ and $s_n|T_n$ is a homeomorphism by Theorem 9. For each continuum $Z\subset X_n\times X_n$ satisfying the condition τ) ($\tau=\sigma, \tau_0$, and s), the set $(s_n\cdot(r_n|r_n^{-1}(T_n))\times s_n\cdot(r_n|r_n^{-1}(T_n)))^{-1}(Z)$ is a continuum in $X\times X$ which also satisfies the condition τ). Arguing the same way as in 1), we obtain the conclusion.

An easy example shows that the converse of Proposition 8 does not hold. But by Theorem 6 and Proposition 3, we can prove:

If $\tau(f_n)=0$ for each n, then $\tau(f)=0$.

Monotone maps preserve span zero ([3], theorem 2). The author recently proved that open maps also preserve span zero [7]. Hence, COROLLARY 10. *Let* $X=\varprojlim(X_n, p_{n\,n+1})$ *be a continuum as the inverse limit of continua and onto bonding maps. Suppose that all* $p_{n\,n+1}: X_{n+1}\to X_n$*'s are monotone or all* $p_{n\,n+1}$*'s are open. Then* $\sigma(X)=0$ *if and only if* $\sigma(X_n)=0$ *for each* n.

3. Some examples.

In this section, we are concerned with circle-like continua.

PROPOSITION 11. *Let* $X=\varprojlim(X_n, p_{n\,n+1})$, $Y=\varprojlim(Y_n, q_{n\,n+1})$ *be circle-like continua and* $f: X\to Y$ *be a map which is weakly-induced by a sequence of maps* $(f_n: X_n\to Y_n)$. *If all* X_n*'s and* Y_n*'s are simple closed curves and all* $q_{n\,n+1}$ *are essential, then the followings are equivalent.*

a) $\sigma(f)=0$.

b) *There exists a subsequence* (n_j) *such that* $f_{n_j}\cong 0$ *for each* j.

As was shown in [5, 2.2], a map $f: X\to S^1$ from a continuum X to the unit circle S^1 is essential if and only if $\sigma(f)=\text{diam}\,S^1>0$ Using this result, this proposition is easily proved. (See also [16]).

H. Cook has essentially proved that the symmetric span of the dyadic solenoid is zero ([1], p. 134). Here we consider general p-adic solenoid. Let $p=(p_1, p_2, \cdots)$ be a sequence of positive integers. The *p-adic solenoid* S_p is defined by the inverse limit of the unit circles $X_n=S^1=\{z\in\boldsymbol{C}\,|\,|z|=1\}$, whose bonding maps $f_n: X_{n+1}\to X_n$ are defined by the formulas; $f_n(z)=z^{p_n}$. We show the following result.

PROPOSITION 12. *Let* S_p *be the p-adic solenoid,* $p=(p_1, p_2, \cdots)$. *Then* $s(S_p)>0$ *if and only if there exists a positive integer* N *such that for each* $n>N$, p_n *is odd.*

First we calculate the symmetric span of maps between the unit circles.

LEMMA 13. *Let* $f: S^1\to S^1$ *be the map between the unit circles defined by* $f(z)=z^n$, *where* n *is a positive integer. Then* $s(f)=0$ *or* diam S^1 $(=2)$. *Also,* $s(f)=0$ *if and only if* n *is even.*

PROOF. $S^1\times S^1$ is obtained from the rectangle $[0, 2\pi]\times[0, 2\pi]$ by identifying $(x, 0)$ and $(x, 2\pi)$, $(0, y)$ and $(2\pi, y)$ $(0\leqq x, y\leqq 2\pi)$. Let $F=\{(x, y)\in S^1\times S^1\,|\, f(x)=f(y)\}$. Then F contains diagonal set. Let

$$A_i=[2\pi\cdot(i-1)/n, 2\pi\cdot i/n]\times 0,$$

$$B_i=0\times[2\pi\cdot(i-1)/n, 2\pi\cdot i/n],$$

$$C_i=[2\pi\cdot(i-1)/n,\ 2\pi\cdot i/n]\times 2\pi,$$
$$D_i=2\pi\times[2\pi\cdot(i-1)/n,\ 2\pi\cdot i/n], \qquad i=1, \cdots, n.$$

A_i and C_i, B_i and D_i are identified in $S^1\times S^1$ respectively. Let X_i be the tetragon bounded by F and A_i and D_{n+1-i} in $[0, 2\pi]\times[0, 2\pi]$, and $\tilde{X}_i$ be the set in $S^1\times S^1$ obtained from X_i by the identification. Notice that $s(f)>0$ if and only if there exists a continuum Z in $S^1\times S^1$ such that Z is symmetric and $Z\cap F=\emptyset$.

First we assume that n is odd. Then $(\pi, 0)(=(\pi, 2\pi)$ in $S^1\times S^1)$ and $(0, \pi)(=(2\pi, \pi)$ in $S^1\times S^1)$ do not belong to F. So we can join $(\pi, 0)$ and $(0, \pi)$ by the symmetric arc $A=\{(x, y)\in S^1\times S^1 \mid |\arg x-\arg y|=\pi\}$. It is easy to see that $d(f(x), f(y))=\operatorname{diam} S^1=2$ for each (x, y) of A. Hence $s(f)=2$.

Next we assume that n is even. Suppose that $s(f)>0$. Then by the above remark, there exists a continuum Z in $S^1\times S^1$ such that Z is symmetric and $Z\cap F=\emptyset$. For each $i=1, \cdots, n$, let $Z_i=Z\cap\tilde{X}_i$. Then $Z_i^{-1}=Z\cap\tilde{X}_i^{-1}$. Let j be the first integer such that $Z_j\neq\emptyset$.

We claim that $Z_j\cap Z_j^{-1}=\emptyset$. If $j=1$, $\tilde{X}_1\cap\tilde{X}_1^{-1}\subset$(diagonal)$\subset F$. Since $Z\cap F=\emptyset$, $Z_1\cap Z_1^{-1}=\emptyset$. Assume $j>1$. As n is even, $i\neq n+1-i$ for each integer. Hence $B_i\cap D_{n+1-i}\subset F$, $A_i\cap C_{n+1-i}\subset F$, and we have $\tilde{X}_j\cap\tilde{X}_j^{-1}\subset F$. As $Z\cap F=\emptyset$, we have the claim.

As Z is connected, $Z_j\cup Z_j^{-1}\neq Z$. If Z does not intersect $\operatorname{Int}_{S^1\times S^1}(\tilde{X}_{n+1-j}^{-1})$, then $Z_j\cup Z_j^{-1}$ is a clopen set in Z, because $\tilde{X}_{n+1-j}^{-1}$ is the only one of the $\tilde{X}_s$'s which meets $\tilde{X}_j$ in $S^1\times S^1-F$. So $Z\cap\operatorname{Int}\tilde{X}_{n+1-j}^{-1}\neq\emptyset$. By the similar argument, we see that $\tilde{X}_j\cup\tilde{X}_{n+1-j}$ does not intersect any other $\tilde{X}_s$'s and $\tilde{X}_s^{-1}$'s in $S^1\times S^1-F$ and $\tilde{X}_j\neq\tilde{X}_{n+1-j}$. Therefore $Z_j\cup Z_{n+1-j}^{-1}$ is a clopen proper subset of Z. This is a contradiction which completes the proof.

PROOF OF PROPOSITION 12.

First we assume $s(S_p)>0$. If there exists a cofinal subsequence (n_i) such that p_{n_i} is even, $s(f_{n_i+1\, n_{i+1}})=0$ by Lemma 13. By Corollary 7, $s(S_p)=0$, a contradiction.

Next suppose that there exists a positive integer N satisfying the hypothesis. Then for each $m>n>N$, $s(f_{nm})=2$. Therefore $\lim_m s(f_{nm})>0$ and $s(S_p)>0$, as desired.

This completes the proof.

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