ON SPAN AND INVERSE LIMITS

By

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1. Introduction.

A compact metric space is called a compactum and a connected compactum is called a continuum. All maps in this paper are continuous. Let \( f : X \to Y \) be a map between continua. Ingram [2] and Lelek [11] defined the span, semispan, surjective span, and surjective semispan of \( f \) by the following formulas (the map \( p_i : X \times X \to X \) denotes the projection to the \( i \)-th factor, \( i = 1, 2 \)).

\[
\tau = \sigma, \sigma_*, \sigma^*_e.
\]

\[
\tau(f) = \sup \left\{ c \geq 0 \mid \begin{array}{l}
\text{there exists a continuum } Z \subseteq X \times X \text{ such that } Z \text{ satisfies the condition } \tau \\
\text{and } d(f(x), f(y)) \geq c \text{ for each } (x, y) \in Z
\end{array} \right\},
\]

where the condition \( \tau \) is:

\[
p_i(Z) = p_3(Z) \text{ if } \tau = \sigma, \quad p_i(Z) \supset p_3(Z) \text{ if } \tau = \sigma_*, \quad p_i(Z) = X \text{ if } \tau = \sigma^*_e.
\]

The span of a continuum \( X \) is defined by \( \sigma(id_X) \). The other cases are similar. In the same way, we can define the symmetric span of \( f \) by the formula

\[
s(f) = \sup \begin{cases} 
\{ c \geq 0 \mid \begin{array}{l}
\text{there exists a continuum } Z \subseteq X \times X \text{ such that } Z \text{ is symmetric (i.e. } (x, y) \in Z \iff (y, x) \in Z) \\
\text{and } d(f(x), f(y)) \geq c \text{ for each } (x, y) \in Z
\end{array} \} \end{cases}.
\]

It is a mapping version of symmetric span of a continuum due to J. F. Davis [1].

Let \( X = \lim(X_n, p_{n,n+1}) \) be a continuum, where \( p_{n,n+1} : X_{n+1} \to X_n \). Ingram [2] and [4] showed that \( \sigma(X) = 0 \) if and only if there exists a cofinal subsequence \((n_i)_{i \in \omega}\) such that \( \lim_j \sigma(p_{n_i,n_j}) = 0 \) for each \( i \geq 1 \). In section 2 of this paper, we will prove a mapping version of this theorem. H. Cook proved essentially that the symmetric span of the dyadic solenoid is zero ([1], p. 134), while its span is positive. The author wishes to thank to the referee for pointing out this fact. In section 3, we generalize this to the poly-adic solenoid. Let \( f \) and \( g : X \to Y \) be maps. \( d(f, g) \) denotes \( \sup \{ d(f(x), g(x)) \mid x \in X \} \).

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2. Span and a limit of maps.

Let \( X = \lim (X_n, p_{n, n+1}) \) and \( Y = \lim (Y_n, q_{n, n+1}) \) be compacta, where all \( X_n \) and \( Y_n \) are polyhedra and both of \( p_{n, n+1} : X_{n+1} \to X_n \) and \( q_{n, n+1} : Y_{n+1} \to Y_n \) are surjective for each \( n \geq 0 \). The maps \( p_n : X \to X_n \) and \( q_n : Y \to Y_n \) denote the projection maps. Under these notations, Mioduszewski showed the following [15].

**Theorem 1.** 1) For every sequence \( (\varepsilon_n) \) of positive numbers with \( \lim \varepsilon_n = 0 \), there exist cofinal increasing subsequences \( (m_k) \) and \( (n_k) \) and maps \( f_k : X_{m_k} \to Y_{n_k} \) such that diagrams (A) and (B) are \( \varepsilon_k \)-commutative for each \( s \leq k \leq l \).

\[
\begin{array}{cccc}
X_{m_k} & \leftrightarrow & X & \leftrightarrow & X_{m_l} \\
\downarrow f_k & & \downarrow f & & \downarrow f_l \\
Y_{n_s} & \leftrightarrow & Y_{n_k} & \leftrightarrow & Y_{n_l}
\end{array}
\quad
\begin{array}{cccc}
X_{m_k} & \leftrightarrow & Y_{n_k} & \leftrightarrow & Y_{m_l} \\
\downarrow f_k & & \downarrow f_l & & \\
Y_{n_s} & \leftrightarrow & Y_{n_l}
\end{array}
\]

(A) \quad (B)

2) Conversely, if we are given diagram (B), then we can find a map \( f : X \to Y \) which satisfies diagram (A) for each \( k \). If all \( f_k \)'s are surjective, \( f \) can be constructed so as to be surjective.

Notice that the map \( f \) is defined by \( q_{n_k} f = \lim k p_{n_k} f_k p_{m_k} \).

We say that \( f \) is weakly induced by the sequence \( (f_k) \). This terminology is due to Oversteegen and Tymchatyn [13].

**Theorem 2.** Let \( f : X \to Y \) be a map between continua which is weakly induced by a sequence \( (f_k : X_{m_k} \to Y_{n_k}) \). Then, \( \tau(f) = 0 \) if and only if there exists a cofinal subsequence \( (n_k) \) of \( (n_k) \) such that \( \lim_j \tau(q_{n_k} f_k p_{m_j}) = 0 \) for each \( i \). Where, \( \tau = \sigma, \sigma^*, \sigma_0, \sigma^*_0, \) and \( s \).

The basic idea of the proof is in [2] and [3]. But we need some preparations. Throughout this section, \( \tau \) denotes \( \sigma, \sigma_0, \sigma^*, \sigma^*_0, \) and \( s \) unless otherwise stated.

**Proposition 3.** Let \( f : X \to Y \) and \( g : Y \to Z \) be maps.
1) \( \tau(gf) \leq \tau(g) \). 2) If \( \tau(f) = 0 \), then \( \tau(gf) = 0 \).

**Proposition 4.** Let \( (f_n : X \to Y) \) be a sequence of maps which converges uniformly to a map \( f : X \to Y \). Then \( \tau(f) = \lim_n \tau(f_n) \).

The proof of the above two propositions are easy and will be omitted.
Proposition 5. 1) Let $X_n$'s and $X$ be continua in a metric space $M$ and let $Y_n$'s and $Y$ be continua in a metric space $N$. Suppose that $f : X \to Y$, $f_n : X_n \to Y_n$, $p_n : X \to X_n$, and $q_n : Y \to Y_n$ satisfy the following conditions.

a) $\lim X_n = X$, $\lim Y_n = Y$. Both of $X \cup \bigcup_{n=1}^{\infty} X_n$ and $Y \cup \bigcup_{n=1}^{\infty} Y_n$ are compact.

b) Both of the maps $p_n$ and $q_n$ are $1/2^n$-translation (that is, $d(x, p_n(x)) < 1/2^n$ for each $x \in X$ etc.).

c) There exists a decreasing sequence of positive numbers $\varepsilon_n$'s with $\lim \varepsilon_n = 0$, such that $d(q_n f, f_n p_n) < \varepsilon_n$.

d) Define $F : X \cup \bigcup_{n \geq 1} X_n \to Y \cup \bigcup_{n \geq 1} Y_n$ by $F\arrowvert X = f$, $F\arrowvert X_n = f_n$. Then $F$ is well-defined and continuous.

Then $\tau(f) = \lim \tau(f_n)$.

2) We can replace condition d) by

e) Each $p_n$ is surjective.

Reasoning the same way as in [10, 3.1] and [5, 2.1], we can show two inequalities: $\limsup \tau(f_n) \leq \tau(f) \leq \liminf \tau(f_n)$, which imply the conclusion.

Proof of Theorem 2. To simplify the notations, a cofinal subsequence of $(n_i)$ is also denoted by $(n_i)$. First we assume that $\tau(f) = 0$. Take any subsequence $(n_i)$ and an integer $j > 0$. It suffices to prove that $\lim_i \tau(q_{n_i, j}, f_i) = 0$. Let $A$ be a compactum satisfying the following conditions.

1) $A = X \cup \bigcup_{m} X_m$, where $X$ and $X_m$ are homeomorphic to $X$ and $X_m$ respectively. $X \cap X_m = \emptyset = X_m \cap X_m$ for each $m \neq l$.

2) Let $h : X \to X$ and $h_k : X_m \to X_m$ be homeomorphisms. There exists an $\varepsilon_k$-translation $\bar{p}_m : X \to X_m$ satisfying $h_k p_m = \bar{p}_m h$.

3) $\lim X_m = X$.

That such space $A$ exists is well known. As each bonding map is surjective, we can take each $\bar{p}_m$ to be surjective. Consider the following diagram.

Where, $a = q_{n_j} f h^{-1}$ and $b_i = q_{n_i, n_j} f_i h_i^{-1}$. Then,
4) 
\[ d(a, b) = d(q_{n_j} f h^{-1}, q_{n_j} n_j f h^{-1} h \cdot p_{n_i} h^{-1}) \]
\[ = d(q_{n_j} n_j f i, q_{n_j} n_j f i p_{n_i} < \varepsilon_i \]

by the \( \varepsilon_i \)-commutativity of \( A \). It is easy to see that \( \tau(a) = \tau(q_{n_j} f) \) and \( \tau(b) = \tau(q_{n_j} n_j f i) \). Applying Proposition 3.2, Proposition 5 and by condition 4), we have

\[ \lim_{i} \tau(q_{n_j} n_j f i) = \lim_{i} \tau(b) = \tau(a) = \tau(q_{n_j} f) = 0. \]

Next we assume that a cofinal subsequence satisfies the hypothesis. By Proposition 4 and Proposition 3.1,

\[ \tau(q_{n_j} f) = \lim_{i} \tau(q_{n_j} n_j f i p_{n_i} \varepsilon_i) \]
\[ \leq \lim_{i} \tau(q_{n_j} n_j f i) = 0. \]

To show that \( \tau(f) = 0 \), we take any continuum \( Z \) in \( X \times X \) satisfying condition \( \tau \). There exists a point \( (x^i, y^i) \in Z \) such that \( q_{n_j} f(x^i) = q_{n_j} f(y^i) \), because \( \tau(q_{n_j} f) = 0 \) for each \( j \). We can assume that \( (x^i, y^i) \to (x, y) \) as \( j \to \infty \). If \( j < i \),

\[ q_{n_j} f(x^i) = q_{n_j} n_j q_{n_i} f(x^i) \]
\[ = q_{n_j} n_j q_{n_j} f(y^i) = q_{n_j} f(y^i). \]

Tending \( i \) to infinity, we have

\[ q_{n_j} f(x) = q_{n_j} f(y) \quad \text{for each } j \text{ and hence } f(x) = f(y). \]

This completes the proof.

**Theorem 6.** Suppose that \( X, Y, f \) and \( f_n \) satisfy the hypothesis of Theorem 2. If there exists a cofinal subsequence \( (n_i) \) such that \( \lim_{i} \tau(f_n p_{n_i} n_j) = 0 \), then \( \tau(f) = 0 \).

**Proof.** For each \( s < i \), \( \tau(q_{n_i} n_j f_n p_{n_i} n_j) = 0 \), because by Proposition 3,

\[ \tau(f_n p_{n_i} n_j) = \lim_{j} \tau(f_n p_{n_i} n_j p_{n_j}) \]
\[ \leq \lim_{j} \tau(f_n p_{n_i} n_j) = 0. \]

Using the \( \varepsilon_j \)-commutativity of the diagram \( A \) and \( B \), we have \( \tau(q_{n_i} f) \leq \tau(q_{n_i} n_j f j p_{n_j}) + 2\varepsilon_j = 2\varepsilon_j \) for each \( j > i \). Therefore \( \tau(q_{n_i} f) = 0 \) for each \( i \) and \( \tau(f) = 0 \).

**Corollary 7 [8 and 10].** Let \( X = \lim(X_n, p_{n_n}) \) be a continuum represented as the inverse limit of continua and onto bonding maps. Then the followings are equivalent.

1) \( \tau(X) = 0 \).

2) There exists a cofinal subsequence \( (n_i) \) such that \( \lim_{j} \tau(p_{n_i n_j}) = 0 \) for each \( i \).
3) For each \( n \), \( \tau(p_n)=0 \).

In Theorem 2 and 6, no conditions on \( p_n \)'s and \( q_n \)'s, on \( X_n \)'s and \( Y_n \)'s are required. If we add some conditions, the followings are obtained.

**Proposition 8.** Suppose \( X, Y, f, f_n, p_n \) and \( q_n \) satisfy the hypothesis of Theorem 2. Moreover assume that:

1) All \( p_n \)'s are monotone, or
2) \( X \) is tree-like and each \( X_n \) is a finite tree. Each \( p_n \) is an open onto map.

\( \tau=\sigma, \sigma_0, \) and \( s \).

If \( \tau(f) = 0 \), then \( \lim_n \tau(f_n) = 0 \).

**Proof.** 1) For each \( n \geq 0 \) and for each continuum \( Z \subseteq X_n \times X_n \) satisfying \( \tau \), \( (p_n \times p_n)^{-1}(Z) \) is a continuum in \( X \times X \) satisfying \( \tau \). There exists a \( (x, y) \in (p_n \times p_n)^{-1}(Z) \) such that \( f(x) = f(y) \). Then

\[
\begin{align*}
d(f_n p_n(x), f_n p_n(y)) & \leq d(f_n p_n(x), q_n f(x)) + d(q_n f(y), f_n p_n(y)) \\
& \leq 2\varepsilon_n.
\end{align*}
\]

Hence \( \tau(f_n) \leq 2\varepsilon_n \) and this completes the proof.

2) We need the following theorem for the proof.

**Theorem 9** [14, p. 189]. Let \( X \) and \( Y \) be compacta and \( f: X \rightarrow Y \) be a light open map from \( X \) onto \( Y \). For each dendrite \( D \) in \( Y \), there exists a dendrite \( D_1 \) in \( X \) such that \( f(D_1) = D \) and \( f \upharpoonright D_1 \) is a homeomorphism on \( D \).

Using this Theorem, 2) is shown as follows.

Let \( n \) be a positive integer. There exists a continuum \( W_n \) and maps \( r_n: X \rightarrow W_n, s_n: W_n \rightarrow X_n \) such that \( r_n \) is monotone and \( s_n \) is light open and \( s_n r_n = p_n \).

As \( X_n \) is a tree, there exists a dendrite \( T_n \) in \( W_n \) such that \( s_n(T_n) = X_n \) and \( s_n \upharpoonright T_n \) is a homeomorphism by Theorem 9. For each continuum \( Z \subseteq X_n \times X_n \) satisfying the condition \( \tau \) (\( \tau=\sigma, \sigma_0, \) and \( s \)), the set \( (s_n \circ (r_n \upharpoonright r_n^{-1}(T_n)) \times s_n \circ (r_n \upharpoonright r_n^{-1}(T_n))^{-1}(Z) \) is a continuum in \( X \times X \) which also satisfies the condition \( \tau \).

Arguing the same way as in 1), we obtain the conclusion.

An easy example shows that the converse of Proposition 8 does not hold. But by Theorem 6 and Proposition 3, we can prove:

If \( \tau(f_n) = 0 \) for each \( n \), then \( \tau(f) = 0 \).

Monotone maps preserve span zero ([3], theorem 2). The author recently proved that open maps also preserve span zero [7]. Hence,
Corollary 10. Let \( X = \lim(X_n, p_{n+1}) \) be a continuum as the inverse limit of continua and onto bonding maps. Suppose that all \( p_{n+1} : X_{n+1} \rightarrow X_n \)'s are monotone or all \( p_{n+1} \)'s are open. Then \( \sigma(X) = 0 \) if and only if \( \sigma(X_n) = 0 \) for each \( n \).

3. Some examples.

In this section, we are concerned with circle-like continua.

Proposition 11. Let \( X = \lim(X_n, p_{n+1}) \), \( Y = \lim(Y_n, q_{n+1}) \) be circle-like continua and \( f : X \rightarrow Y \) be a map which is weakly-induced by a sequence of maps \( (f_n : X_n \rightarrow Y_n) \). If all \( X_n \)'s and \( Y_n \)'s are simple closed curves and all \( q_{n+1} \) are essential, then the followings are equivalent.

a) \( \sigma(f) = 0 \).

b) There exists a subsequence \( (n_j) \) such that \( f_{n_j} = 0 \) for each \( j \).

As was shown in [5, 2.2], a map \( f : X \rightarrow S^1 \) from a continuum \( X \) to the unit circle \( S^1 \) is essential if and only if \( \sigma(f) = \text{diam } S^1 > 0 \). Using this result, this proposition is easily proved. (See also [16]).

H. Cook has essentially proved that the symmetric span of the dyadic solenoid is zero ([1], p.134). Here we consider general \( p \)-adic solenoid. Let \( p = (p_1, p_2, \ldots) \) be a sequence of positive integers. The \( p \)-adic solenoid \( S_p \) is defined by the inverse limit of the unit circles \( X_n = S^1 = \{ z \in C | |z| = 1 \} \), whose bonding maps \( f_n : X_{n+1} \rightarrow X_n \) are defined by the formulas; \( f_n(z) = z^{p_n} \). We show the following result.

Proposition 12. Let \( S_p \) be the \( p \)-adic solenoid, \( p = (p_1, p_2, \ldots) \). Then \( s(S_p) > 0 \) if and only if there exists a positive integer \( N \) such that for each \( n > N \), \( p_n \) is odd.

First we calculate the symmetric span of maps between the unit circles.

Lemma 13. Let \( f : S^1 \rightarrow S^1 \) be the map between the unit circles defined by \( f(z) = z^n \), where \( n \) is a positive integer. Then \( s(f) = 0 \) or \( \text{diam } S^1 \) (\( = 2 \)). Also, \( s(f) = 0 \) if and only if \( n \) is even.

Proof. \( S^1 \times S^1 \) is obtained from the rectangle \([0, 2\pi] \times [0, 2\pi] \) by identifying \((x, 0)\) and \((x, 2\pi)\), \((0, y)\) and \((2\pi, y)\) \((0 \leq x, y \leq 2\pi)\). Let \( F = \{(x, y) \in S^1 \times S^1 | f(x) = f(y)\} \). Then \( F \) contains diagonal set. Let

\[
A_i = [2\pi \cdot (i-1)/n, 2\pi \cdot i/n] \times 0,
\]

\[
B_i = 0 \times [2\pi \cdot (i-1)/n, 2\pi \cdot i/n],
\]
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\[ C_i = [2\pi \cdot (i-1)/n, 2\pi \cdot i/n] \times 2\pi, \]
\[ D_i = 2\pi \times [2\pi \cdot (i-1)/n, 2\pi \cdot i/n], \quad i=1, \ldots, n. \]

\( A_i \) and \( C_i \), \( B_i \) and \( D_i \) are identified in \( S^1 \times S^1 \) respectively. Let \( X_i \) be the tetragon bounded by \( F \) and \( A_i \) and \( D_{n+i} \) in \([0, 2\pi] \times [0, 2\pi]\), and \( \tilde{X}_i \) be the set in \( S^1 \times S^1 \) obtained from \( X_i \) by the identification. Notice that \( s(f) > 0 \) if and only if there exists a continuum \( Z \) in \( S^1 \times S^1 \) such that \( Z \) is symmetric and \( Z \cap F = \emptyset \).

First we assume that \( n \) is odd. Then \((\pi, 0) = (\pi, 2\pi)\) in \( S^1 \times S^1 \) and \((0, \pi) = (2\pi, \pi)\) in \( S^1 \times S^1 \) do not belong to \( F \). So we can join \((\pi, 0)\) and \((0, \pi)\) by the symmetric arc \( A = \{(x, y) \in S^1 \times S^1 \mid \arg x - \arg y = \pi\} \). It is easy to see that \( d(f(x), f(y)) = \text{diam } S^1 = 2 \) for each \((x, y) \in A\). Hence \( s(f) = 2 \).

Next we assume that \( n \) is even. Suppose that \( s(f) > 0 \). Then by the above remark, there exists a continuum \( Z \) in \( S^1 \times S^1 \) such that \( Z \) is symmetric and \( Z \cap F = \emptyset \). For each \( i = 1, \ldots, n \), let \( Z_i = Z \cap \tilde{X}_i \). Then \( Z_i = Z \cap \tilde{X}_i \). Let \( j \) be the first integer such that \( Z_j \neq \emptyset \).

We claim that \( Z_j \cap Z_j^1 = \emptyset \). If \( j = 1 \), \( \tilde{X}_j \cap \tilde{X}_j^1 \subset (\text{diagonal}) \subset F \). Since \( Z \cap F = \emptyset \), \( Z \cap Z_j^1 = \emptyset \). Assume \( j > 1 \). As \( n \) is even, \( i \neq n+1-i \) for each integer. Hence \( B_i \cap D_{n+i} \subset F \), \( A_i \cap C_{n+i} \subset F \), and we have \( \tilde{X}_j \cap \tilde{X}_j^1 \subset F \). As \( Z \cap F = \emptyset \), we have the claim.

As \( Z \) is connected, \( Z_j \cup Z_j^1 \neq Z \). If \( Z \) does not intersect \( \text{Int}_{S^1 \times S^1} (\tilde{X}_{n+i}^1) \), then \( Z_j \cup Z_j^1 \) is a clopen set in \( Z \), because \( \tilde{X}_{n+i}^1 \) is the only one of the \( \tilde{X}_i \)'s which meets \( \tilde{X}_j \) in \( S^1 \times S^1 - F \). So \( Z \cap \text{Int}_{S^1 \times S^1} \tilde{X}_{n+i}^1 \neq \emptyset \). By the similar argument, we see that \( \tilde{X}_j \cup \tilde{X}_{n+i} \) does not intersect any other \( \tilde{X}_i \)s and \( \tilde{X}_i^1 \)s in \( S^1 \times S^1 - F \) and \( \tilde{X}_j \neq \tilde{X}_{n+i} \). Therefore \( Z \cap \text{Int}_{S^1 \times S^1} \tilde{X}_{n+i}^1 \) is a clopen proper subset of \( Z \). This is a contradiction which completes the proof.

**Proof of Proposition 12.**

First we assume \( s(S_p) > 0 \). If there exists a cofinal subsequence \((n_i)\) such that \( p_{n_i} \) is even, \( s(f_{n+1} n_{i+1}) = 0 \) by Lemma 13. By Corollary 7, \( s(S_p) = 0 \), a contradiction.

Next suppose that there exists a positive integer \( N \) satisfying the hypothesis. Then for each \( m > n > N \), \( s(f_n m) = 2 \). Therefore \( \lim_{m \to n} s(f_n m) > 0 \) and \( s(S_p) > 0 \), as desired.

This completes the proof.

**References**


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