THE CONVERGENCE OF MOMENTS IN THE CENTRAL LIMIT THEOREM FOR WEAKLY DEPENDENT RANDOM VARIABLES

By

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1. Introduction.

Let \((\Omega, \mathcal{F}, P)\) be a probability space. For any two \(\sigma\)-fields \(\mathcal{A}\) and \(\mathcal{B}\) define the mixing coefficients \(\phi\) and \(\alpha\) and the maximal correlation coefficient \(\rho\) by

\[
\phi(\mathcal{A}, \mathcal{B}) = \sup_{A \in \mathcal{A}, B \in \mathcal{B}} |P(B \mid A) - P(B)| \quad A \in \mathcal{A}, B \in \mathcal{B}, P(A) > 0;
\]

\[
\alpha(\mathcal{A}, \mathcal{B}) = \sup_{A \in \mathcal{A}, B \in \mathcal{B}} |P(A \cap B) - P(A)P(B)| \quad A \in \mathcal{A}, B \in \mathcal{B};
\]

\[
\rho(\mathcal{A}, \mathcal{B}) = \sup_{\xi \in L^p(\mathcal{A}), \eta \in L^q(\mathcal{B})} |\text{Corr}(\xi, \eta)|.
\]

Let \(\{X_j: -\infty < j < \infty\}\) be a strictly stationary sequence of random variables on \((\Omega, \mathcal{F}, P)\). For integers \(n\) let \(\mathcal{F}_n\) be the \(\sigma\)-field generated by \(\{X_j: j \leq n\}\) and \(\mathcal{F}_n\) the \(\sigma\)-field generated by \(\{X_j: j \geq n\}\). The sequence \(\{X_j\}\) is said to be \(\phi\)-mixing (or uniformly mixing) if

\[
\phi(n) \equiv \phi(\mathcal{F}_n, \mathcal{F}_n) \to 0 \quad \text{as} \quad n \to \infty
\]

(see Ibragimov [9]), strongly mixing if

\[
\alpha(n) \equiv \alpha(\mathcal{F}_n, \mathcal{F}_n) \to 0 \quad \text{as} \quad n \to \infty
\]

(see Rosenblatt [15]) and completely regular if

\[
\rho(n) \equiv \rho(\mathcal{F}_n, \mathcal{F}_n) \to 0 \quad \text{as} \quad n \to \infty
\]

(see Kolmogorov-Rozanov [13]).

Among these coefficients, the following inequalities always hold:

\[
4\alpha(n) \leq \rho(n) \leq 2\phi^{1/2}(n).
\]

The left-hand inequality is an easy consequence of the definitions of the coefficients \(\alpha(n)\) and \(\rho(n)\), and the right-hand inequality is a consequence of the Ibragimov fundamental inequality for \(\phi\)-mixing sequences (see [11, Theorem 17.2.3, p. 309]). Thus a \(\phi\)-mixing sequence is completely regular (the converse...
is false; see [11, pp. 310-314]), and a completely regular sequence is strongly mixing (the converse is false; see [16, pp. 206-209], but for Gaussian sequences complete regularity is equivalent to strong mixing; see [13]). Formulations of various mixing conditions are given by Ibragimov-Rozanov [12] for stationary Gaussian sequences in terms of the spectral density, and by Rosenblatt [16] for stationary Markov sequences in terms of the transition operator.

Let \( \{X_j\} \) be a strictly stationary sequence with \( EX_j=0 \) and \( EX_j^2<\infty \). Set

\[
S_n = \sum_{j=1}^n X_j, \quad \sigma_n^2 = ES_n^2.
\]

In numerous papers conditions are investigated which guarantee asymptotic normality of the distribution of the normed sum \( \sigma_n^{-1}S_n \) (see, for example, [2, Chap. 4], [9], [10] and [11, Chap. 18]).

We are interested in knowing when the \( r \)th absolute moment of \( \sigma_n^{-1}S_n \) \((r>2)\) converges to that of the normal distribution. When \( X_j \) are independent (but not necessarily identically distributed) random variables, Bernstein [1] presented a necessary and sufficient condition (the \( r \)th Lindeberg condition) for the convergence of absolute moments in the central limit theorem. Brown [4, 5] gave an alternative proof of Bernstein's result. Hall [8] extended Bernstein's theorem in both the independence and the martingale cases. For stationary \( \phi \)-mixing and strongly mixing sequences the author [17, 18] obtained some results on the convergence of moments. Recently, in the \( \phi \)-mixing case, the following much broader result was proved; the proof is completely different from those in [17] and [18].

**Theorem A ([19]).** Let \( \{X_j\} \) be a strictly stationary sequence with \( EX_j=0 \) and \( E|X_j|^r<\infty \) for some \( r>2 \). If \( \phi(n)\to0 \) and \( \sigma_n^2\to\infty \) as \( n\to\infty \), then

\[
\lim_{n\to\infty} E|S_n/\sigma_n|^r = \int_{-\infty}^{\infty} (2\pi)^{-1/2} |u|^r \exp(-u^2/2) du.
\]

In Theorem A it is not assumed that \( \phi(n)\to0 \) at a specific rate, while the series-type conditions on the mixing coefficients were imposed in all the theorems of [18] (cf. [9, Theorem 1.4]). The purpose of this paper is to generalize the above \( \phi \)-mixing result to the complete regularity case. The basic idea, which was used in [19], is a martingale representation of the sum \( S_n \), and the proof is based on Ibragimov's moment inequality (Lemma 2 below) and a martingale result of Hall [8].
2. Statement of a result.

First we state a result of Ibragimov [10, Theorem 2.1], which generalizes an earlier result of his own [9, Theorem 1.4].

**Theorem B (Ibragimov).** Let \( \{X_j\} \) be a strictly stationary sequence with \( EX_j=0 \) and \( EX_j^2<\infty \). (i) If \( \lim \rho(n)=0 \) and \( \limsup_{n \to \infty} \sigma_n^2/n = \infty \), where \( \rho(n) \) is a slowly varying function in the sense of Karamata. (ii) If in addition \( EX_j^r<\infty \) for some \( r>2 \), then

\[
\lim_{n \to \infty} P\{S_n/\sigma_n<x\} = \int_{-\infty}^{x} (2\pi)^{-1/2} \exp(-u^2/2) \, du.
\]

**Remarks.** Theorem B (ii) fails if its hypothesis \( EX_j^r<\infty \) \( (r>2) \) is omitted; a counterexample is constructed by Bradley [3]. Lifshits [14] proved some central limit theorems on Markov chains under \( \rho(n) \to 0 \) and other slightly weaker conditions.

In this article the conditions of Theorem B, without any additional conditions, will be shown to imply the convergence of the \( r \)th absolute moments in the central limit theorem. More precisely, we shall prove

**Theorem C.** Let \( \{X_j\} \) be a strictly stationary sequence with \( EX_j=0 \) and \( EX_j^2<\infty \) for some \( r>2 \). If \( \rho(n) \to 0 \) and \( \limsup_{n \to \infty} \sigma_n^2/n \to \infty \) as \( n \to \infty \), then

\[
(1) \quad \lim_{n \to \infty} E|S_n/\sigma_n|^r = \int_{-\infty}^{\infty} (2\pi)^{-1/2} |u|^r \exp(-u^2/2) \, du.
\]

As we have remarked in Sect. 1, the \( \phi \)-mixing condition implies the complete regularity condition, thus Theorem C contains Theorem A as a special case. For strongly mixing sequences the relation (1) holds under the conditions \( EX_j=0 \), \( EX_j^{r+\delta}<\infty \) for some \( r>2 \) and \( \delta>0 \), \( EX_j^2+2 \sum_{j=2}^{\infty} E(X_jX_{j+1})>0 \) and \( \sum_{n=1}^{\infty} n^{r/2-1}(\alpha(n))^{\delta/(r+\delta)} < \infty \) (see [18]).

3. The proof.

In the proof, limits will be taken as \( n \to \infty \). The symbol \( K \) denotes a generic constant, not necessarily the same at each appearance. \( \beta_r \) denotes the \( r \)th absolute moment of the standard normal distribution. \( I(A) \) denotes the indicator function of the event \( A \).

For the proof of Theorem C we need a few well-known inequalities.
Lemma 1. Suppose that the random variables $\xi$ and $\eta$, respectively, are measurable with respect to $\mathcal{F}_k$ and $\mathcal{F}_{k+1}$;

1. If $E\xi^s < \infty$ and $E\eta^s < \infty$, then

$$|E(\xi\eta) - E\xi \cdot E\eta| \lesssim (E\xi^s)^{1/\alpha}(E\eta^s)^{1/\alpha} \rho(n);$$

2. If $|\xi| \leq B$ a.s. and $E|\eta|^s < \infty$ for some $s > 1$, then

$$|E(\xi\eta) - E\xi \cdot E\eta| \lesssim 6B(E|\eta|^s)^{1/\alpha} \alpha(n)^{1-1/\alpha}.$$

The inequality (2) is an immediate consequence of the definition of the coefficient $\rho(n)$. The inequality (3) is due to Davydov [7]. The following inequality, due to Ibragimov [10], is fundamental to our proof.

Lemma 2. Under the assumptions of Theorem C, there exists a constant $C$ such that

$$E|S_n|^r \leq C\sigma_n^r \quad \text{for all } n \geq 1.$$

We shall divide the sum $S_n$ into three parts:

$$S_n = S'_n + S''_n = \sigma_n T_n + \sigma_n T'_n + S''_n,$$

and show that $\sigma_n^{-1} S''_n$ and $T'_n$ are asymptotically negligible, while the $r$th absolute moment of $T_n$ converges to $\beta_n$, where the variable $T_n$ will be chosen to be a martingale.

The first step is to represent the sum $S_n$ in the form

$$S_n = \sum_{j=1}^k y_j + \sum_{j=1}^{k+1} z_j = S'_n + S''_n,$$

where

$$y_j = \sum_{i=0}^{(j-1)q} X_i, \quad 1 \leq j \leq k;$$

$$z_j = \sum_{i=0}^{(j+q)q} X_i, \quad 1 \leq j \leq k;$$

$$z_{k+1} = \sum_{i=0}^{n} X_i;$$

$p = p(n)$ and $q = q(n) \in \{1, 2, \ldots, n\}$ and satisfy the following conditions:

1. $p \to \infty$, $q \to \infty$, $n^{-1} p \to 0$, $p^{-1} q \to 0$,
2. $n^{1+\beta} q^{-\beta} p^{-1} \to 0$ for some $\beta > 0$,
3. $n^{-1} p^{1/\beta} q^{-1} \to 0$,

and $k = k(n) = \lceil n/(p+q) \rceil$. Here $\lceil a \rceil$ denotes the greatest integer $\leq a$. Such systems of $p$ and $q$ actually exist. In fact, if we set
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\[ \lambda(n) = \max \{ \rho^{1/r}(n^{1/4}), (\log n)^{-1} \}, \]
\[ p = \max \left\{ n\rho^{3/4}(n^{1/4}), \lambda(n)(n^{-1})^{1/4}, n^{3/4}(\lambda(n))^{-1} \right\}, \]
\[ q = [n^{1/4}] , \]

then all the conditions in (5) are satisfied:

a) \( p \to \infty, \quad q \to \infty, \quad n^{-1}p \to 0, \quad p^{-1}q \to 0, \)

b) \( n^{1+\beta}q^{-1-\beta}p^{-1} \leq n^{-(1-\beta)1/4}\lambda(n) \to 0, \quad \text{if} \quad \beta \leq 1/3, \)

c) \( np^{-1}q^{2/r}(q) \leq \lambda(n) \to 0. \)

Now we break the sum \( S_n \) into two parts. We denote by \( \mathcal{L}_{nj} \) the \( \sigma \)-fields \( \mathcal{F}_{j,p+1} \) and define the random variables

\[ Y_{nj} = y_{nj} - E\{ y_{nj} | \mathcal{L}_{nj} \}, \quad 1 \leq j \leq k, \]

where \( y_{nj} = y_j / \sigma_n. \) Then \( \{ Y_{nj}, \mathcal{L}_{nj} : 1 \leq j \leq k \} \) is trivially a martingale difference sequence for each \( n \geq 1. \) Let

\[ T_n = \sum_{j=1}^k Y_{nj}, \quad T'_n = S_n / \sigma_n - T_n = \sum_{j=1}^k E\{ y_{nj} | \mathcal{L}_{nj} \}. \]

Then \( S_n = \sigma_n T_n + \sigma_n T'_n + S''_n. \)

The theorem will be proved in three stages:

(i) \( E|S'_n / \sigma_n|^r \to 0, \)

(ii) \( E|T'_n|^r \to 0, \)

(iii) \( E|T_n|^r \to 0. \)

In view of (i), (ii), (iii) and the inequality:

\[ |(E|S'_n / \sigma_n|^r - (E|T'_n|^r)|^r \leq 2^{r-1}(E|T'_n|^r + E|S''_n / \sigma_n|^r), \]

the assertion of the theorem follows.

Proof of (i). Since \( \sigma_n^2 = nh(n) \), where \( h(n) \) is a slowly varying function (Theorem B), using Lemma 2, Minkowski's inequality and stationarity, and arguing as in [11, p. 337], we obtain

\[ E|S'_n / \sigma_n|^r \leq \sigma_n^r (k(E|z_1|^r)^{1/r} + (E|z_{k+1}|^r)^{1/r})^r \]

\[ \leq K(k^q q h(q') / nh(n) + q' h(q') / nh(n))^{1/r} \to 0, \]

where \( q' = n - k(p+q) \) is the number of terms in \( z_{k+1} \), and (i) is proved.

Before proving (ii) and (iii), we note that under the requirements imposed on \( p, q \) and \( k, \)
In fact,
\[
E(S_n^2/\sigma_n^2) = k\sigma_p^2/\sigma_n^2 + 2 \sum_{j=1}^{k} (k-j+1)E(y_{n1} y_{nj})
\]
by stationarity. Since \(y_{n1}\) is measurable with respect to \(\mathcal{F}_p\) and \(y_{nj}, 2 \leq j \leq k\), are measurable with respect to \(\mathcal{F}_{p+q}\), applying the inequality (2),
\[
\sum_{j=1}^{k} (k-j+1)\left|E(y_{n1} y_{nj})\right| \leq (k\sigma_p/\sigma_n)^2 \rho(q).
\]
Moreover, by condition (5),
\[
k\rho(q) \sim n p^{-1} \rho(q) \leq n p^{-1} \rho^{2q} \to 0.
\]
Hence
\[
E(S_n^2/\sigma_n^2) = (k\sigma_p^2/\sigma_n^2)(1 + o(1)).
\]
On the other hand,
\[
E(S_n^2/\sigma_n^2) = E(S_n^2/\sigma_n^2) + E(S_n^2/\sigma_n^2) - 2E(S_n S_n^*/\sigma_n^2)
\]
\[
= 1 + o(1)
\]
by (i). The equality (6) now follows from (7) and (8).

Proof of (ii). For simplicity we put
\[
w_{nj} = E\{y_{nj} \mid \mathcal{L}_{n,j-1}\}, \quad 1 \leq j \leq k,
\]
and because of the stationarity, we put
\[
a_n = E\{|y_{nj}|^r\}, \quad 1 \leq j \leq k.
\]
By Hölder's inequality,
\[
E|w_{nj}|^{r/2} = E(E\{y_{nj} w_{nj} \mid |w_{nj}|^{r/2}\})
\]
\[
= E(E\{y_{nj} w_{nj} \mid |w_{nj}|^{r-2}\} \mid \mathcal{L}_{n,j-1})
\]
\[
= E(y_{nj} w_{nj} \mid |w_{nj}|^{r/2})
\]
\[
\leq (E(y_{nj} w_{nj})^{r/2})^{r/2}(E|w_{nj}|^r)^{1/2r},
\]
so that we have
\[
E|w_{nj}|^{r/2} \leq E|y_{nj} w_{nj}|^{r/2}.
\]
Since \(w_{nj}\) is measurable with respect to \(\mathcal{F}_{(j-1)(p+q)}\) and \(y_{nj}\) is measurable with respect to \(\mathcal{F}_{(j-1)(p+q)}\) for each \(1 \leq j \leq k\), using (2) and Jensen's inequality,
\[
E|y_{nj} w_{nj}|^{r/2} \leq (E|y_{nj}|^{r/2})(E|w_{nj}|^{r/2})^{1/2} \rho(q) + E|y_{nj}|^{r/2} E|w_{nj}|^{r/2}
\]
\[
\leq a_n \rho(q) + a_n^{1/2} E|w_{nj}|^{r/2}.
\]
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Using (2) and Jensen's inequality again,

\[ E |w_{nj}|^{r/2} = E(y_{nj}w_{nj} | w_{nj}|^{r/2} \cdot I(|w_{nj}| > 0)) \]
\[ \leq (E y_{nj}^2)^{1/2} (E |w_{nj}|^{r/2})^{1/2} \rho(q) \]
\[ \leq a_n^{r/2} \rho(q). \]

Combining the estimates above, we find that

(9) \[ E |w_{nj}|^r \leq 2a_n \rho(q). \]

We obtain from Minkowski's inequality, (4)-(6) and (9) that

\[ E |T_n'|^r \leq \left( \sum_{j=1}^k (E |w_{nj}|^r)^{1/r} \right)^r \]
\[ \leq 2k^r a_n \rho(q) \]
\[ \leq K(k \sigma_p^2/\sigma_h^2)^{r/2} k^{r/2} \rho(q) \rightarrow 0, \]

and hence (ii) is proved.

**Proof of (iii).** Define \( w_{nj} \) and \( a_n \) as before. For simplicity of notation we also define

\[ u_{nj} = y_{nj}^2 - E y_{nj}^2, \quad 1 \leq j \leq k, \]
\[ v_{nj} = E \{ u_{nj} | \mathcal{L}_{n,j-1} \}, \quad 1 \leq j \leq k \]

and (because of the stationarity)

\[ b_n = E |u_{nj}|^{r/2}, \quad 1 \leq j \leq k. \]

Now by stationarity,

\[ \sum_{j=1}^k E Y_{nj}^2 = k \sigma_p^2/\sigma_h^2 - \sum_{j=1}^k E(y_{nj}w_{nj}), \]

and using (2),

\[ \sum_{j=1}^k E(y_{nj}w_{nj}) \leq (k \sigma_p^2/\sigma_h^2) \rho(q). \]

Thus, taking account of (6), we see that

\[ \sum_{j=1}^k E Y_{nj}^2 = 1 + o(1). \]

Therefore, according to Hall's [8] theorem, the proof of (iii) will be complete if we can show that

(10) \[ \max_{j \leq k} E \{ Y_{nj}^2 | \mathcal{L}_{n,j-1} \} \rightarrow 0 \quad \text{in probability}, \]

(11) \[ \sum_{j=1}^k E |Y_{nj}|^r \rightarrow 0 \]

and
However, (11) immediately implies the conditional Lindeberg condition:

for all \( \varepsilon > 0 \), 
\[
\sum_{j=1}^{k} E\{Y_{nj}^2 \mid \mathcal{L}_{n,j-1}\} - 1 \rightarrow 0.
\]

Hence (10) is a consequence of (11) combined with (12) (see Brown [6, Theorem 1 and Lemma 1]). We have from Jensen’s inequality, (4) and (6) that

\[
\sum_{j=1}^{k} E|Y_{nj}|^r \leq 2^{r-1} \sum_{j=1}^{k} (E|y_{nj}|^r + E|w_{nj}|^r)
\]

\[
\leq 2^r k a_n
\]

\[
\leq K(k \sigma_p^n / \sigma_n^2)^{r/2} k^{-r/2+1} \rightarrow 0,
\]

and thus (11) holds.

Our goal is to show that (12) holds. Now,

\[
E\left| \sum_{j=1}^{k} E\{Y_{nj}^2 \mid \mathcal{L}_{n,j-1}\} - 1 \right|^{r/2}
\]

\[
= E\left| \sum_{j=1}^{k} E\{y_{nj}^2 \mid \mathcal{L}_{n,j-1}\} - \sum_{j=1}^{k} w_{nj}^2 - 1 \right|^{r/2}
\]

\[
\leq 2^{r/2-1} \left\{ E\left| \sum_{j=1}^{k} E\{y_{nj}^2 \mid \mathcal{L}_{n,j-1}\} - 1 \right|^{r/2} + E\left( \sum_{j=1}^{k} w_{nj}^2 \right)^{r/2} \right\}.
\]

Making use of the inequality (9), and arguing just as in the proof of (ii), we get

\[
E\left( \sum_{j=1}^{k} w_{nj}^2 \right)^{r/2} \leq 2k^{r/2} a_n \rho(q)
\]

\[
\leq K(k \sigma_p^n / \sigma_n^2)^{r/2} \rho(q) \rightarrow 0.
\]

Moreover, we have from (6) and Minkowski’s inequality that

\[
E\left| \sum_{j=1}^{k} E\{y_{nj}^2 \mid \mathcal{L}_{n,j-1}\} - 1 \right|^{r/2}
\]

\[
\sim E\left| \sum_{j=1}^{k} E\{y_{nj}^2 \mid \mathcal{L}_{n,j-1}\} - k \sigma_p^2 / \sigma_n^2 \right|^{r/2}
\]

\[
= E\left| \sum_{j=1}^{k} v_{nj} \right|^{r/2}
\]

\[
\leq \left( \sum_{j=1}^{k} (E|v_{nj}|^{r/2})^{2/r} \right)^{r/2}.
\]

Consequently, to prove (12) it is sufficient to show that

(13) 

\[
\sum_{j=1}^{k} (E|v_{nj}|^{r/2})^{2/r} \rightarrow 0.
\]
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We shall separate the proof of (13) in three cases; \( r > 4 \), \( r = 4 \) and \( 2 < r < 4 \).

Suppose first that \( r > 4 \). By replacing \( y_{nj} \), \( w_{nj} \), \( a_n \) and \( r \) in the proof of (9) by \( u_{nj} \), \( v_{nj} \), \( b_n \) and \( r/2 \) respectively, we deduce that

\[
E |v_{nj}|^{r/2} \leq 2b_n \rho(q).
\]

Since

\[
b_n \leq 2^{r/2-1} [E |y_{nj}|^{r} + (E y_{nj}^2)^{r/2}] \leq 2^{r/2} a_n,
\]
then, by virtue of (4) and (6), we see that

\[
\sum_{j=1}^{k} (E |v_{nj}|^{r/2})^{2/r} \leq k(2b_n \rho(q))^{2/r}
\]

\[
\leq k(2^{r/2+1} a_n \rho(q))^{2/r}
\]

\[
\leq K(k \sigma_n^2 / \sigma_n^2) \rho^{2/4}(q) \to 0,
\]

and thus (13) is proved for the case \( r > 4 \).

When \( r = 4 \), using (2) and Jensen’s inequality, we get

\[
E |v_{nj}|^{r/2} = E(u_{nj} v_{nj})
\]

\[
\leq (E u_{nj}^2)^{1/2}(E v_{nj}^2)^{1/2} \rho(q)
\]

\[
\leq E u_{nj}^2 \rho(q).
\]

Hence (13) also holds for \( r = 4 \).

Finally, we assume that \( 2 < r < 4 \). By Hölder’s inequality,

\[
E |v_{nj}|^{r/2} = E(u_{nj} v_{nj} | v_{nj})^{r/2}_+ I(|v_{nj}| > 0)
\]

\[
\leq E(|u_{nj}|^{2-r/2} | u_{nj} v_{nj})^{r/2}_+ I(|v_{nj}| > 0)
\]

\[
\leq b_n^{r_2-1} (E |u_{nj} v_{nj}|)^{2-2/r}.
\]

Using (2) and Jensen’s inequality, and noting that \( r/4 < 1 \),

\[
E |u_{nj} v_{nj}|^{r/4} \leq (E |u_{nj}|^{r/2})^{1/4}(E |v_{nj}|^{r/4})^{1/4} \rho(q) + E |u_{nj}|^{r/4} E |v_{nj}|^{r/4}
\]

\[
\leq b_n \rho(q) + b_n^{r/2}(E |v_{nj}|)^{r/4}.
\]

Since \( \alpha(n) \leq \rho(n) \), applying the inequality (3) with \( \xi = v_{nj} |v_{nj}|^{-1} I(|v_{nj}| > 0) \), \( \eta = u_{nj} \)

and \( s = r/2 \),

\[
E |v_{nj}| = E(u_{nj} v_{nj} | v_{nj})^{-1} I(|v_{nj}| > 0)
\]

\[
\leq 6(E |u_{nj}|^{r/2})^{2/4}(\rho(q))^{2-2/r}.
\]

Inserting the inequalities (16) and (17) into (15), we have

\[
E |v_{nj}|^{r/2} \leq b_n^{r_2-1} (b_n \rho(q) + 6^{r/4} b_n \rho(q))^{2-1/2} \rho^{2-2/r}
\]

\[
\leq K b_n (\rho(q))^{(r-2)/2r}.
\]
Just as in (14), we obtain that for $2 < r < 4$,

$$\sum_{j=1}^{k} (E|v_n|^{r/2})^{2/r} \leq K(k\sigma_2^p/\sigma_2^q)(\rho(q))^{(r-2)/2}r^2 \to 0,$$

and hence (13) follows as desired.

The proof of Theorem C is now complete.

References


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