A CLASSIFICATION OF ORTHOGONAL TRANSFORMATION GROUPS OF LOW COHOMOGENEITY
Dedicated to Professor Ichiro Yokota on his 60th birthday

By

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1. Introduction

A Lie transformation group on a smooth manifold M is a pair (G, M) of a Lie group G which acts smoothly on M. This paper is concerned with the cohomogeneity (abbrev. coh) of (G, M), which is defined by

\[ \text{coh}(G, M) = \dim M - \dim G + \min \{ \dim G_x; x \in M \}, \]

where \( G_x \) is the isotropy subgroup of G at x. Then

\[ \text{coh}(G, M) \geq \dim M - \dim G \quad (=: \text{doh}(G, M)), \]

\( \{ x \in M; \text{coh}(G, M) = \text{doh}(G, M) + \dim G_x \} \) is an open subset of M, and

\[ \text{coh}(G^*, M) = \text{coh}(G, M) \]

where \( G^* \) is the identity connected component of G.

An orthogonal transformation group (abbrev. o.t.g.) on an N dimensional Euclidean space \( E^N \) is defined as a pair \((G, E^N)\) of a connected Lie subgroup G of the full orthogonal group \( O(N) \) on \( E^N \). \((G, E^N)\) is said to be contained in another o.t.g. \((G', E^N)\) on \( E^N \) if there is a real linear isometry \( r: E^N \rightarrow E^N \) and a Lie group monomorphism \( \tau: G \rightarrow G' \) such that

\[ \tau(g)r = rg \]

for all \( g \in G \).

If moreover \( \tau \) is a Lie group isomorphism, \((G, E^N)\) is said to be equivalent to \((G', E^N)\).

Let \( \rho \) be a linear representation on \( R^N \) over the field \( R \) of all real numbers of a Lie group G. We say \((G, \rho, R^N)\) an orthogonal linear triple and \( \rho \) an orthogonal representation of G if there is a positive definite inner product on \( R^N \) which is invariant under the action of G.

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\( \rho(G) \). Suppose \( \rho' \) is another orthogonal representation of \( G \). We call \((G, \rho', \mathbb{R}^N)\) and \((G, \rho, \mathbb{R}^N)\) are equivalent as real representation if \( \rho' \) and \( \rho \) are equivalent as real representations of \( G \).

An orthogonal linear triple \((G, \rho, \mathbb{R}^N)\) naturally induces an o.t.g. \((\rho(G'), \mathbb{E}^N)\) which is well defined up to equivalences and denoted by \(O(G, \rho, \mathbb{R}^N)\). We denote

\[
\text{coh} (G, \rho, \mathbb{R}^N) = \text{coh} (O(G, \rho, \mathbb{R}^N))
\]

\[
\text{doh} (G, \rho, \mathbb{R}^N) = \text{doh} (O(G, \rho, \mathbb{R}^N))
\]

If \( G \) is compact, then any real representation of \( G \) is an orthogonal linear representation, and the corresponding o.t.g. is called a compact linear group.

An o.t.g. is called maximal if it is not properly contained in an o.t.g. of the same cohomogeneity. Suppose \((G, \mathbb{E}^N)\) is a maximal o.t.g. If it contains a compact linear group \((K, \mathbb{E}^N)\) of the same cohomogeneity, then itself is a compact linear group. In fact, the closure \( \bar{G} \) of \( G \) in \( O(N) \) is compact and

\[
\text{coh} (\bar{G}, \mathbb{E}^N) = \text{coh} (G, \mathbb{E}^N)
\]

since \( \{x \in \mathbb{E}^N; G(x) \text{ is closed (i.e., } \bar{G}(x) = G(x) \} \) contains an open dense subset \( \{x \in \mathbb{E}^N; \text{coh } (K, \mathbb{E}^N) = N - \text{dim } K + \text{dim } K_x \} \) of \( \mathbb{E}^N \).

Hsiang-Lawson [11] studied a classification of all compact linear groups of cohomogeneity 2 or 3 and maximal by means of the classification of compact linear groups which has a non trivial isotropy subgroup at a point of a principal orbit (cf. Kramer [15], Hsiang [10] and Hsiang-Hsiang [9]). As a result, most of them can be induced from the linear isotropy representations of Riemannian symmetric pairs.

Conversely, the linear isotropy representation of each Riemannian symmetric pair of rank \( r \) induces a compact linear group of cohomogeneity \( r \) (cf. Takagi-Takahashi [19]). Any of its orbit in the representation space is an \( R \)-space in the meaning of Takeuchi [20] (cf. Takeuchi-Kobayashi [21]). A principal \( R \)-space denotes an \( R \)-space of the highest dimension among all \( R \)-spaces associated with a given Riemannian symmetric pair.

From tables of Takagi-Takahashi [19, Table I and II], it appears that two principal \( R \)-spaces associated with two distinct Riemannian symmetric pairs of rank 2 are not equivalent as Riemannian manifolds nor Riemannian submanifolds of a hypersphere of the representation space. Especially if two maximal o.t.g.’s of cohomogeneity 2 contain o.t.g.’s from two distinct Riemannian symmetric pairs of rank 2 respectively, then they are not equivalent (cf. Ozeki-Takeuchi [17; Theorem 1, Theorem 2]).

However it is well known that the o.t.g. from the Riemannian symmetric pair \((G_2, SO(4))\) of rank 2 is missed in a theorem of Hsiang-Lawson [11; Theorem 5] (cf. Takagi-Takahashi [19], Uchida [23]). More than before, Uchida [23] pointed out many examples of real reducible (i.e., non irreducible) compact linear groups of cohomogeneity 3 which shows that another theorem of Hsiang-Lawson[11; Theorem 6] should be properly
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modified. Uchida [23; Theorem] also gave a classification theorem of real reducible compact linear groups of cohomogeneity 3 and maximal in a correct form by the use of a classification of compact Lie groups which act transitively on spheres (cf. Montgomery-Samelson [16], Borel [3], [4]).

In this paper, we study the classification of real irreducible o.t.g.'s of cohomogeneity at most 3 by a direct method (cf. Sato-Kimura [18], Yokota [25]). We have the list of them in Section 4, which shows that the other theorem of Hsiang-Lawson [11; Theorem 7] should be properly modified and also gives a classification of real irreducible compact linear groups of cohomogeneity 3 in a correct form (cf. Theorem 4.8, Remark 4.10).

Our results also give a proof of the fact that a compact linear group of cohomogeneity 2 and maximal is equivalent to an o.t.g. which is induced from the linear isotropy representation of a Riemannian symmetric pair of rank 2. Topologically, Asoh [2] has already completed the classification of compact Lie groups acting on spheres with an orbit of codimension one, which properly modified the result of H.C. Wang [26] (cf. Hsiang-Hsiang [8]). Recently, Dadok [5] classified real irreducible compact linear groups with certain property, so-called 'polar', which is satisfied by each compact linear group of cohomogeneity 2.

2. Preliminaries

For each type of compact simple Lie algebra of dimension \( g \) and rank \( k \), we shall investigate (cf. Adams [1], Goto-Grosshans [6])

(1) 'Real' complex irreducible representations of degree \( m \) such that

\[
d_0 := m - g \leq 3,
\]

(2) Complex irreducible representations of degree \( m \) such that

\[
d_1 = 2m - g \leq 4,
\]

(3) 'Quaternion' complex irreducible representations of degree \( 2m \) such that

\[
d_2 = 4m - g \leq 6.
\]

We denote a compact simple Lie algebra of type \( X_k \) by \( X_k \) (\( X = A, B, C, D, E, F, \) or \( G \)) and the corresponding compact simply connected Lie group also by \( X_k \). A complex irreducible representation of the highest weight \( \Lambda \) is denoted by \( \Lambda \). Especially the trivial representation is denoted by 0. The fundamental weights with respect to the simple roots \( \alpha_1, \alpha_2, \ldots, \alpha_k \) are denoted by

\[
\Lambda_1, \Lambda_2, \ldots, \Lambda_k.
\]

(A)

The simple roots of \( A_k \) are given by a Dynkin diagram:
(1) ‘Real’ complex irreducible representations of $A_k$ are given by
\[ A = 2\lambda_1 A_1 (\text{if } k = 1), \sum_{i=1}^{k+1} \lambda_i (A_i + A_{k-i+1}) (\text{if } k = 2h + 2), \]
\[ \lambda_{2k+2} A_{2k+2} + \sum_{i=1}^{2h+1} \lambda_i (A_i + A_{k-i+1}) (\text{if } k = 4h + 3), \]
or
\[ 2\lambda_{2k+3} A_{2k+3} + \sum_{i=1}^{2k+2} \lambda_i (A_i + A_{k-i+1}) (\text{if } k = 4h + 5), \]
where $h$ and $\lambda_i (i = 1, \ldots, [(k+1)/2])$ are non-negative integers, and $[p]$ denotes the maximal integer at most $p$.

**PROPOSITION 2.1** If $d_0 := \deg A - k^2 - 2k \leq 3$, then $A$ is equivalent as a complex representation of $A_k (k \geq 1)$ to one of the followings:

\[ d_0 < 0: \quad 0(k \geq 1), \quad A_2(k = 3), \quad A_1 + A_k(k \geq 2), \]
\[ d_0 = 0: \quad 2A_1(k = 1), \quad A_1 + A_k(k \geq 2), \]
\[ d_0 = 2: \quad 4A_1(k = 1). \]

**PROOF:** If $\lambda_i \geq 1$ for some $i = 4, \ldots$, or $[(k+1)/2]$, then $k \geq 7$ and $d_0 \geq \deg A - k^2 - 2k \geq k + C_k - k^2 - 2k \geq 7'. If [(k+1)/2] \geq 3 and $\lambda_2 \geq 1$, then $k \geq 5$ and $d_0 \geq \deg (A_3 + A_{k-3}) - k^2 - 2k = (k + 2)(k + 1)^2(k^2 - 2k) - 36 - 2k^2 - 2k \leq 140$. If $\lambda_2 \geq 1$ and $k \geq 4$, then $d_0 \geq \deg (A_2 + A_{k-1}) - k^2 - 2k = (k + 1)^2(k^2 - 4)/36 - 2k^2 - 2k \geq 51$. Therefore $A = 0(k \geq 1)$, $2\lambda_1 A_1(k = 1), \lambda_1 (A_1 + A_2)(k \geq 2), \quad \lambda_2 A_2 + \lambda_1 (A_1 + A_3)(k = 3)$. If $k = 1$ and $\lambda_1 \geq 3$, then $d_0 \geq \deg 6A_1 - 3 = 4$. If $k \geq 2$ and $\lambda_1 \geq 2$, then $d_0 \geq \deg 2(A_1 + A_1) - k^2 - 2k = (k + 1)^2(k^2 + 4)/4 - 2k^2 - 2k \geq 19$. Therefore $A = 0(k \geq 1), A_1 + A_2(k = 2), \lambda_1 A_1 + A_2(k \geq 2), \lambda_2 A_2 + \lambda_1 (A_1 + A_3)(k = 3)$. If $k = 1$ and $\lambda_1 \geq 2$, then $d_0 \geq \deg 3A_1 - 2 = 5$. If $k \geq 2$ and $\lambda_1 \geq 1$, then $d_0 \geq \deg (A_k + A_{k-1})(k \geq 4) - 15 = 49$. Q.E.D.

(2) Complex irreducible representations of $A_k (k \geq 1)$ are given by $A = \sum_{i=1}^{k} \lambda_i A_i$ where $\lambda_i (i = 1, \ldots, k)$ are non-negative integers.

**PROPOSITION 2.2** If $d_0 := 2\deg A - k^2 - 2k \leq 4$, then $A$ is equivalent as a complex representation of $A_k (k \geq 1)$ to one of the followings:

\[ 0(k \geq 1), \quad A_1(k \geq 1), \quad A_2(k = 1), \quad A_2(k = 2), \quad A_2(k = 2), \quad A_4(k \geq 1), \quad A_4(k \geq 3). \]

**PROOF:** If $k = 1$ and $\lambda_1 \geq 3$, then $\deg A \geq \deg 3A_1 = 4$ and $d_1 \geq 5$. If $k = 2$ and $\lambda_1 (or \lambda_2) \geq 3$, then $\deg A \geq \deg 3A_1 + 10$ and $d_1 \geq 12$. If $k \geq 2$, $\lambda_1 \geq 1$ and $\lambda_2 \leq 1$, then $\deg A \geq \deg (A_1 + A_2) = k(k + 2)$ and $d_1 \geq 8$. If $k \geq 3$ and $\lambda_1 (or \lambda_2) \geq 2$, then $\deg A \geq \deg 2A_1 = (k + 1)(k + 2)/2$ and $d_1 \geq 5$. If $\lambda_1 (or \lambda_k) \geq 1$ for some $i = 3, \ldots, k - 2$, then $\deg A \geq \deg A_i = k(k^2 - 1)/6$, $k \leq 5$ and $d_1 \geq 5$. If $\lambda_2 (or \lambda_{k-1}) \geq 2$ and $2 \leq k - 1$, then $\deg A \geq \deg 2A_2 = (k + 1)^2(k + 2)/12, k \geq 3$ and $d_1 \geq 25$. If $\lambda_2 \geq 1, \lambda_{k-1} \geq 1$ and $2 < k - 1$, then $\deg A \geq \deg (A_2 + A_{k-1}) = (k + 1)^2(k^2 - 4)/4$, $k \geq 3$.
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If $\lambda_1 \geq 1$, $\lambda_{k-1} \geq 1$ and $1 < k - 1$, then $\deg \Lambda \geq \deg (\Lambda_1 + \Lambda_{k-1}) = (k + 2) (k^2 - 1)/2$, $k \geq 3$ and $d_i \geq 15$. If $\lambda_2 \geq 1$, $\lambda_k \geq 1$ and $2 < k$, then $d_i \geq 15$. If $\lambda_1 \geq 1$, $\lambda_2 \geq 1$ (or $\lambda_{k-1} \geq 1$, $\lambda_k \geq 1$) and $2 < k - 1$, then $\deg \Lambda \geq \deg (\Lambda_1 + \Lambda_2) = 2(k(k + 1)(k + 2)/3, d_i \geq 56.$

REMARK 2.3 2$A_1(k=1)$, $A_2(k=3)$ are 'real'. $A_1(k=1)$ is 'quaternion'. $A_1, A_k(k=2)$ (resp. $A_2, A_{k-1}(k=4)$, resp. $2A_1, 2A_2(k=2)$) are conjugate from each other.

(3) 'Quaternion' complex irreducible representations of $A_k(k \geq 1)$ are given as $A = (2 \lambda_{2h+1} + 1) A_{2h+1} + \sum_{i=1}^{2h} \lambda_i (A_i + A_{k-i+1})$ where $k = 4h + 1$, $\lambda_i$ and $h$ are non-negative integers.

PROPOSITION 2.4 If $d_2 := 2\deg \Lambda - k^2 - 2k \leq 8$, then $\Lambda$ is equivalent as a complex representation of $A_k(k \geq 1)$ to one of the followings:

- $d_2 = 1$: $A_1(k=1)$,
- $d_2 = 5$: $3A_1(k=1)$, $A_3(k=5)$.

PROOF: If $k = 4h + 1 \geq 6$, then $k \geq 9$ and $d_2 \geq 2\deg A_{2h+1} - k^2 - 2k \geq 2\deg A_3 - k^2 - 2k \geq 405$. So $k = 1$ or $5$. Suppose $k = 1$. If $\lambda_1 \geq 2$, then $d_2 = 2\deg (2\lambda_1 + 1) A_1 - 3 \geq 2\deg 5A_1 - 3 = 9$. So $\Lambda = A_1$ or $3A_1$. Next suppose $k = 5$. If $\lambda_2 \geq 1$, then $d_2 \geq 2\deg (A_2 + A_4) - 35 = 343$. If $\lambda_1 \geq 1$, then $d_2 \geq 2\deg (A_1 + A_3) - 35 = 35$. If $\lambda_3 \geq 1$, then $d_2 \geq 2\deg 3A_3 - 35 = 1925$. So $\Lambda = A_3$.

Q.E.D.

(C)

The simple roots of $C_k$ are given by a Dynkin diagram:

$$\alpha_1 \rightarrow \alpha_2 \rightarrow \cdots \rightarrow \alpha_{k-1} \rightarrow \alpha_k \ (k \geq 2).$$

(1) 'Real' complex irreducible representations of $C_k(k \geq 2)$ are given by $A = \Sigma_{i=1}^{k} \lambda_i A_i$ where $\Sigma_{\text{odd}} \lambda_i$ is even and $\lambda_i (i = 1, \cdots, k)$ are non-negative integers.

PROPOSITION 2.5 If $d_0 := \deg \Lambda - k(2k+1) \leq 3$, then $\Lambda$ is equivalent as a complex representation of $C_k(k \geq 2)$ to one of the followings:

- $d_0 < 0$: $0(k \geq 2)$, $A_2(k \geq 2)$,
- $d_0 = 0$: $2A_1(k \geq 2)$.

PROOF: Suppose $k \geq 5$. Then $\deg A_3 < \deg A_i$ for $i = 4, \cdots, k$ and $\deg A_3 - \dim C_k = 4k(k^2 - 3k - 7) \geq 20$. $\deg 3A_1 - \dim C_5 = k(2k + 1)(4k - 1)/3 \geq 165$. $\deg (A_1 + A_2) - \dim C_4 = k(k^2 - 6k - 11)/3 \geq 265$. $\deg 2A_2 - \dim C_4 = k^2(4k^2 - 13)/3 \geq 725$. So $\Lambda = 0$, $A_2$ or $2A_1$. Suppose $k = 4$. Then the assertion holds since $\deg A_3 - \dim C_4 = 12$, $\deg A_4 - \dim C_1 = 6$, $\deg 2A_2 - \dim C_4 = 272$, $\deg 3A_1 - \dim C_4 = 84$ and $\deg (A_1 + A_2) - \dim C_4 = 124$. Suppose
$k=3$. Then the assertion holds since $\deg 3A_1 - \dim C_3 = 35$, $\deg (A_1 + A_3) - \dim C_3 = 43$, $\deg (A_1 + A_3) - \dim C_3 = 49$, $\deg 2A_1 - \dim C_3 = 63$ and $\deg 2A_2 - \dim C_3 = 69$. Suppose $k=2$. Then the assertion holds since $\deg 4A_1 - \dim C_2 = 25$, $\deg 2A_2 - \dim C_2 = 4$ and $\deg (2A_1 + A_3) - \dim C_2 = 25$. Q.E.D.

(2) Complex irreducible representations of $C_k(k \geq 2)$ are given by $\lambda = \sum_{i=1}^k \lambda_i A_i$, where $\lambda_i(i=1, \cdots, k)$ are non-negative integers.

**Proposition 2.6** If $d_1 = 2\deg A - k(2k+1) \leq 6$, then $A$ is equivalent as a complex representation of $C_k(k \geq 2)$ to one of the followings:

$$0(k \geq 2), \ A_1(k \geq 2), \ A_2(k=2).$$

**Proof:** Suppose $k \geq 3$. If $A$ is not equivalent to $0$ nor $A_1$, then $\deg A \geq \deg A_2$, so $d_1 \geq 2\deg A_2 - \dim C_2 = 2k^2 - 3k - 2 \geq 7$. Suppose $k=2$. The assertion holds since $2\deg 2A_1 - \dim C_2 = 10$, $2\deg (A_1 + A_2) - \dim C_2 = 22$ and $2\deg 2A_2 - \dim C_2 = 18$. Q.E.D.

(3) 'Quaternion' complex irreducible representations of $C_k(k \geq 2)$ are given by $A = \sum_{i=1}^k \lambda_i A_i$, where $\lambda_i(i=1, \cdots, k)$ are non-negative integers.

**Proposition 2.7** If $d_2 = 2\deg A - k(2k+1) \leq 6$, then $A$ is equivalent as a complex representation of $C_k(k \geq 2)$ to one of the followings:

$$A_1(k \geq 2).$$

**Proof:** Suppose $k \geq 3$. If $A$ is not equivalent to $A_1$, then $\deg A \geq \deg A_2$, so $d_2 \geq 2\deg A_2 - \dim C_2 = 2k^2 - 3k - 2 \geq 7$. Suppose $k=2$. If $A$ is not equivalent to $A_1$, then $\deg A \geq \deg (A_1 + A_2) = 16$, so $d_2 \geq 22$. Q.E.D.

(B) The simple roots of $B_k$ are given by a Dynkin diagram:

$$\alpha_1 \longrightarrow \alpha_2 \longrightarrow \cdots \longrightarrow \alpha_{k-1} \longrightarrow \alpha_k \ (k \geq 3).$$

(1) 'Real' complex irreducible representations of $B_k(k \geq 3)$ are given by $A = \sum_{i=1}^k \lambda_i A_i$ (if $k=4h+3$ or $4h+4$), $2\lambda_1 A_1 + \sum_{i=1}^{k-1} \lambda_i A_i$ (otherwise) where $h$ and $\lambda_i(i=1, \cdots, k)$ are non-negative integers.

**Proposition 2.8** If $d_0 = \deg A - k(2k+1) \leq 5$, then $A$ is equivalent as a complex representation of $B_k(k \geq 3)$ to one of the followings:

$$d_0 < 0: \ A_1(k \geq 3), \ A_4(k=3 \text{ or } 4), \ 0(k \geq 3),$$

$$d_0 = 0: \ A_2(k \geq 3).$$

**Proof:** If $\lambda_i \geq 1$ for some $i=3, \cdots, k-1$, then $k \geq 4$ and $d_0 \geq \deg A_3 - \dim B_k = k(2k+1)(2k-4)/3 \geq 48$. If $\lambda_i \geq 2$, then $d_0 \geq \deg 2A_1 - \dim B_k = 2k-6$. If $\lambda_2 \geq 2$, then $d_0 \geq \deg 2A_2 - \dim B_k = (2k+3)(2k+1)(k-1)/3 - k(2k+1) \geq 147$. If $\lambda_1 \geq 1$ and $\lambda_2 \geq 1$, then $d_0 \geq \deg (A_1 + A_2) - \dim B_k = (2k+1)(k+1)(4k-3) \geq 84$. Then $A = A_1, A_2, A_4,$ or
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\[\Lambda_2 + \Lambda_4 (\text{if } k=4h+3 \text{ or } 4h+4), \ \Lambda_1 \text{ or } \Lambda_2 \text{ (otherwise)} \text{ since } \deg 2\Lambda_k - \dim B_k = 2k+1 \geq 14 \text{ and } \deg (\Lambda_1 + \Lambda_4) - \dim B_k = k2^{k+1} - k(2k+1) \geq 27. \] If \( k=4h+3 \) or \( 4h+4 \), \( k \geq 5 \) and \( \lambda_k \geq 1 \), then \( k \geq 8 \) and \( d_0 = \deg \Lambda_k - \dim B_k = 2^k - k(2k+1) \geq 120. \) If \( k=3 \) (resp. \( 4 \)), then \( \deg (\Lambda_1 + \Lambda_2) - \dim B_k = 91 \) (resp. \( 396 \)). Q.E.D.

(2) Complex irreducible representations of \( B_k (k \geq 3) \) are given by

\[ \Lambda = \sum_{i=1}^k \lambda_i \Lambda_i \] where \( \lambda_i (i=1, \ldots, k) \) are non-negative integers.

**Proposition 2.9** If \( d_1 := 2\deg \Lambda - k(2k+1) \leq 8 \), then \( \Lambda \) is equivalent as a complex representation of \( B_k (k \geq 3) \) to one of the followings:

\[ d_1 < 0: \ \Lambda_1 (k \geq 3), \ \Lambda_4 (k=3 \text{ or } 4), \ 0 (k \geq 3). \]

**Proof:** If \( \lambda_i \geq 1 \) for some \( i=2, \ldots, k-1 \), then \( d_1 \geq 2\deg \Lambda_2 - k(2k+1) = 2k+1 \geq 21. \) If \( \lambda_i \geq 2 \), then \( d_1 \geq 2\deg \Lambda_1 - k(2k+1) = 2(2k+5) \geq 33. \) If \( \lambda_i \geq 2 \), then \( d_1 \geq 2\deg \Lambda_2 - k(2k+1) = 2(2k+5) \geq 49. \) If \( \lambda_1 \geq 1 \) and \( \lambda_i \geq 1 \), then \( d_1 \geq 2\deg (\Lambda_1 + \Lambda_k) - k(2k+1) = k2^{k+1} - k(2k+1) \geq 75. \) If \( k \geq 5 \), then \( 2\deg \Lambda_4 - k(2k+1) = 2^k - k(2k+1) \geq 9. \) Q.E.D.

(3) ‘Quaternion’ complex irreducible representations of \( B_k (k \geq 3) \) are given by

\[ \Lambda = \sum_{i=1}^k \lambda_i \Lambda_i + (2\lambda_k + 1) \Lambda_k \] where \( k=4h+5 \) or \( 4h+6 \), \( h \) and \( \lambda_i (i=1, \ldots, k) \) are non-negative integers. Then \( k \geq 5. \)

**Proposition 2.10** There is no ‘quaternion’ complex irreducible representation of \( B_k \) such that \( d_2 := 2\deg \Lambda - k(2k+1) \leq 8. \)

**Proof:** Since \( k \geq 5 \), \( d_2 \geq 2\deg \Lambda_k - k(2k+1) = 2^k - k(2k+1) \geq 9. \) Q.E.D.

\[ \alpha_k \]

The simple roots of \( D_k \) are given by a Dynkin diagram:

\[ \alpha_1 \rightarrow \alpha_2 \rightarrow \cdots \rightarrow \alpha_{k-2} \rightarrow \alpha_{k-1} \rightarrow \alpha_k \ (k \geq 4). \]

(D)

(1) ‘Real’ complex irreducible representations of \( D_k (k \geq 4) \) are given by

\[ \Lambda = \sum_{i=1}^{k-2} \lambda_i \Lambda_i + \lambda_{k-1} (\Lambda_{k-1} + \Lambda_k) \] (if \( k=2h+5 \)), \( \sum_{i=1}^{k-2} \lambda_i \Lambda_i \) (if \( k=4h+4 \)), or \( \sum_{i=1}^{k-2} \lambda_i \Lambda_i + \lambda_{k-1} \Lambda_{k-1} + \lambda_k \Lambda_k \) (if \( k=4h+6 \)), where \( \lambda_{k-1} + \lambda_k \) is even, \( h \) and \( \lambda_i (i=1, \ldots, k) \) are non-negative integers.

**Proposition 2.11** If \( d_0 := \deg \Lambda - k(2k-1) \leq 6 \), then \( \Lambda \) is equivalent as a complex representation of \( D_k (k \geq 4) \) to one of the followings:

\[ d_0 < 0: \ 0 (k \geq 4), \ \Lambda_1 (k \geq 4), \ \Lambda_4 (k=4), \ \Lambda_3 (k=4) \]

\[ d_0 = 0: \ \Lambda_2 (k \geq 4). \]
Proof: If $\lambda_i \geq 1$ for some $i = 3, \ldots, k - 2$, then $k \geq 5$ and $d_0 \geq \deg \Lambda_3 - k(2k - 1) = k(2k - 1)(2k - 5)/3 \geq 75$. So $\lambda_i = 0$ for $i = 3, \ldots, k - 2$. Since $\deg 2\Lambda_1 - k(2k - 1) = 2k - 1 \geq 7$, $\deg 2\Lambda_2 - k(2k - 1) = k^2(4k^2 - 13) \geq 272$ and $\deg (\Lambda_1 + \Lambda_2) - k(2k - 1) = k(4k - 5)(2k + 1)/3 \geq 132$, we have $\lambda_1 + \lambda_2 \leq 1$. Suppose $\Lambda_1^{(*)}$ or $\Lambda_2^{(*)} \neq 1$. If $k \geq 8$, then $d_0 \geq k^4 - k(2k - 1) \geq 8$. If $k = 7$, then $d_0 \geq \deg (\Lambda_3 + \Lambda_2) - 91 = 2912$. If $k = 6$, then $d_0 \geq \deg (\Lambda_3 + \Lambda_2) - 66 = \deg (2\Lambda_4) - 66 = 6 = 66 = 396$. If $k = 5$, then $d_0 \geq \deg (\Lambda_3 + \Lambda_2) - 45 = 164.5$. If $k = 4$ and $\lambda_1, \lambda_2 \geq 1$, then $d_0 \geq \deg (\Lambda_4 + \Lambda_3) + 28 = \deg (\Lambda_4 + \Lambda_3) - 28 = 132$. So $k = 4$ and $\Lambda = \Lambda_4$ or $\Lambda_3$. Q.E.D.

(2) Complex irreducible representations of $D_k(k \geq 4)$ are given by $\Lambda = \sum_{i=1}^k \lambda_i \Lambda_i$ where $\lambda_i (i = 1, \ldots, k)$ are non-negative integers.

Proposition 2.12 If $d_1 := 2\deg \Lambda - k(2k - 1) \leq 36$, then $\Lambda$ is equivalent as a complex representation of $D_k(k \geq 4)$ to one of the followings:

- $d_1 < 0$: $0(k \geq 4), \ A_1(k \geq 4), \ A_3(k = 4), \ A_4(k = 4)$,
- $d_1(k = 5), \ A_5(k = 5), \ A_6(k = 6)$. 

Proof: If $\lambda_i \geq 1$ for some $i = 2, \ldots, k - 2$, then $d_1 \geq 2\deg \Lambda_2 - k(2k - 1) = k(2k - 1) \geq 28$. So that $\lambda_i = 0$ for $i = 2, \ldots, k - 2$. Since $2\deg 2\Lambda_1 - k(2k - 1) = (k + 2)(2k - 1) \geq 42$, we have $\lambda_1 \leq 1$. Suppose $\lambda_k \geq 1, \lambda_1 \leq 1$. Then $k \geq 6$ since $d_1 \geq 2\deg \Lambda_4 - k(2k - 1) = 2\deg \Lambda_4 - k(2k - 1) = 2^{k - 1} - k(2k - 1) \geq 37$ if $k \geq 7$. We have that $\lambda_1 + \lambda_k + \lambda_k \leq 1$. Since $2\deg (\Lambda_1 + \Lambda_3) - k(2k - 1) = 2\deg (\Lambda_1 + \Lambda_1) - k(2k - 1) = 2^{k - 1} - k(2k - 1) \geq 34$, $2\deg (\Lambda_{k - 1} + \Lambda_1) - k(2k - 1) = 2\deg 2\Lambda_k - k(2k - 1) = 2\deg 2\Lambda_{k - 1} = k(2k - 1) - 1 \geq 34$ and $2\deg 2\Lambda_k - k(2k - 1) = 2\deg 2\Lambda_k - k(2k - 1) = (2k - 2)/k! - 1 \geq 12$. Q.E.D.

Remark 2.13 $A_4(k = 5)$ and $A_5(k = 5)$ are conjugate. $A_3(k = 4)$ and $A_4(k = 4)$ are 'real', and there are outer automorphisms $\tau_i (i = 1, 2)$ of $D_k$ such that $A_3 \circ \tau_1$ and $A_4 \circ \tau_2$ are equivalent as complex representations of $D_k$ to $A_1$. There is also an outer automorphism $\tau_3$ of $D_k$ such that $A_3 \circ \tau_3$ are equivalent as complex representations of $D_k$.

(3) 'Quaternion' complex irreducible representations of $D_k(k \geq 4)$ are given by $\Lambda = \sum_{i=1}^k \lambda_i \Lambda_i$ where $\lambda_{k + 1} + \lambda_k$ is odd, $k = 4h + 6$, and $h, \lambda_i (i = 1, \ldots, k)$ are non-negative integers.

Proposition 2.14 If $d_2 := 2\deg \Lambda - k(2k - 1) \leq 36$, then $\Lambda$ is equivalent as a complex representation of $D_k(k \geq 4)$ to one of the followings:

- $d_2 < 0$: $A_5(k = 6), \ A_6(k = 6)$. 

Proof: The assertion follows from Proposition 2.12 and Remark 2.13. Q.E.D.
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The simple roots of exceptional Lie algebras are given by Dynkin diagrams:

\[ G_2: \quad \alpha_1 \rightarrow \alpha_2 \]
\[ F_4: \quad \alpha_1 \rightarrow \alpha_2 \rightarrow \alpha_3 \rightarrow \alpha_4 \]
\[ E_6: \quad \alpha_1 \rightarrow \alpha_2 \rightarrow \alpha_3 \rightarrow \alpha_4 \rightarrow \alpha_5 \rightarrow \alpha_6, \\alpha_7 \]
\[ E_7: \quad \alpha_1 \rightarrow \alpha_2 \rightarrow \alpha_3 \rightarrow \alpha_4 \rightarrow \alpha_5 \rightarrow \alpha_6 \rightarrow \alpha_7 \]
\[ E_8: \quad \alpha_1 \rightarrow \alpha_2 \rightarrow \alpha_3 \rightarrow \alpha_4 \rightarrow \alpha_5 \rightarrow \alpha_6 \rightarrow \alpha_7 \rightarrow \alpha_8 \]

**Proposition 2.15** Suppose \( \Lambda \) is a complex irreducible representation of an exceptional Lie algebra of dimension \( g \). If \( d_0 = \deg \Lambda = g \geq 12 \), then \( \Lambda \) is equivalent as a complex representation to one of the followings:

- \( d_0 < 0: \quad A_2(G_2), \quad A_4(F_4), \quad A_1(E_6), \quad A_5(E_6), \quad A_6(E_7), \quad A_7(E_8). \)
- \( d_0 = 0: \quad A_1(G_2), \quad A_2(F_4), \quad A_6(E_6), \quad A_1(E_7), \quad A_1(E_8). \)

**Proof:**

**Case \( G_2 \)** If \( \Lambda \) is not equivalent to \( A_1 \) nor \( A_2 \), then \( d_0 \geq 13 \) since \( \deg 2A_1 = 77, \) \( \deg 2A_2 = 27 \) and \( \deg (A_1 + A_2) = 64. \) **Case \( F_4 \)** If \( \Lambda \) is not equivalent to \( A_1 \) nor \( A_2 \), then \( d_0 \geq 273 \) since \( \deg 2A_1 = \deg (A_1 + A_2) = 1053, \) \( \deg 2A_4 = 324, \) \( \deg A_2 = 1274 \) and \( \deg A_3 = 273. \) **Case \( E_6 \)** If \( \Lambda \) is not equivalent to \( A_1, A_5, \) nor \( A_6, \) then \( d_0 \geq 273 \) since \( \deg 2A_1 = \deg 2A_5 = \deg A_2 = \deg A_4 = 351, \) \( \deg A_3 = 2925, \) \( \deg 2A_6 = 2430, \) \( \deg (A_1 + A_3) = 650 \) and \( \deg (A_1 + A_5) = \deg (A_3 + A_6) = 1728. \) **Case \( E_7 \)** If \( \Lambda \) is not equivalent to \( A_1, A_6, \) nor \( A_7, \) then \( d_0 \geq 779 \) since \( \deg A_2 = 8645, \) \( \deg A_3 = 365750, \) \( \deg A_4 = 27664, \) \( \deg A_5 = 1539, \) \( \deg A_7 = 912, \) \( \deg 2A_1 = 7371, \) \( \deg 2A_6 = 1463 \) and \( \deg (A_1 + A_6) = 3920. \) **Case \( E_8 \)** If \( \Lambda \) is not equivalent to \( A_1, A_7, \) nor \( A_8, \) then \( d_0 \geq 3627 \) since \( \deg A_2 = 3825, \) \( \deg A_3 = 6696000, \) \( \deg A_3 = 6899079264, \) \( \deg A_4 = 146325270, \) \( \deg A_5 = 2450240, \) \( \deg A_6 = 30380, \) \( \deg A_8 = 147250, \) and \( \deg 2A_7 = 27000. \) Q.E.D.

**Remark 2.16** \( A_2(G_2) \) is ‘real’ of degree 7. \( A_4(F_4) \) is ‘real’ of degree 26. \( A_1(E_6) \) and \( A_5(E_6) \) are conjugate from each other and of degree 27. \( A_6(E_7) \) is ‘quaternion’ of degree 56. \( A_1(G_2), A_1(F_4), A_6(E_6), A_1(E_7) \) and \( A_7(E_8) \) are the adjoint representations, especially ‘real’, of degree 14, 52, 78, 144, 248 respectively. Any \( \Lambda \) of \( d_0 \) or \( d_2 \leq 12 \) is contained in the above list since \( d_1 = d_0 > d_0. \)

Next propositions are also useful in sections 3 and 4.

**Proposition 2.17** Each non trivial ‘real’ complex irreducible representation of degree at most 3 of a compact simple Lie algebra is equivalent as a complex representation to one of the followings:
degree 3: $2A_1(A_1)$.

PROOF: The assertion follows from Prop. 2.1, 2.5, 2.8, 2.11 and 2.15 since $d_0$ is less than the degree which is at most 3. Q.E.D.

PROPOSITION 2.18 Each non trivial complex irreducible representation of degree at most 3 of a compact simple Lie algebra is equivalent as a complex representation to one of the followings:

degree 2: $A_1(A_1)$,
degree 3: $2A_1(A_1), A_1(A_2), A_2(A_2)$.

PROOF: The assertion follows from Prop.'s 2.2, 2.6, 2.9, 2.12 and 2.15 since $d_1=2$ degree$-g \leq 2.3-3=3$. Q.E.D.

REMARK 2.19 $A_2(A_2)$ is conjugate to $A_1(A_2)$.

PROPOSITION 2.20 Each non trivial 'quaternion' complex irreducible representation of degree at most 6 of a compact simple Lie algebra is equivalent as a complex representation to one of the followings:

degree 2: $A_1(A_1)$,
degree 4: $3A_1(A_1), A_1(C_3)$,
degree 6: $5A_1(A_1), A_1(C_3)$.

PROOF: The assertion is trivial in the case of $A_1$. Otherwise, it follows from Prop.'s 2.4, 2.7, 2.10, 2.14 and 2.15 since $d_2=2$ degree$-g \leq 2.6-8=4$. Q.E.D.

3. Basic Classification by cohomogeneity

Let $(G, M)$ be a Lie transformation group. For $x \in M$, we denote $G(x)$ the orbit of $G$ through $x$, and $G_x$ the isotropy subgroup of $G$ at $x$.

LEMMA 3.1 Let $(G, M), (G, N)$ be Lie transformation groups and $f$ be a $G$-equivariant submersion from $M$ onto $N$ with the property:

$$f^{-1}(f(x))=G_{f(x)}x$$

at a fixed $x \in M$. Then we have that

$$\dim M - \dim G + \dim G_x = \dim N - \dim G + \dim G_{f(x)}$$

PROOF: $\dim M = \dim N + \dim f^{-1}(f(x)) = \dim N + \dim G_{f(x)}(x) = \dim N + \dim G_{f(x)} - \dim G_x$ since $(G_{f(x)})_x = G_x$. Q.E.D.

Let $R$, $C$ and $H$ be the set of real numbers, complex numbers and quaternions respec-
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The conjugate $u + jv$ of $u + jv \in H(u, v \in C)$ is defined by

$$u + jv = \overline{u} - jv$$

where $\overline{u}$ is the complex conjugate of $u$. For $u + jv, u' + jv' \in H$, the product of them is defined by

$$(u + jv)(u' + jv') = (uu' - vv') + j(vu' + uv').$$

Let $F$ be $R$, $C$, or $H$. The set of all $(n_1, n_2)$-matrices with coefficients $F$ is denoted by $F(n_1, n_2)$. For $X \in F(n_1, n_2)$, we denote the conjugate of $X$ with respect to the coefficients by $\overline{X}$, and the transposed matrix of $X$ by $X'$. We write $F^* = F(n, 1)$, $F(n) = F(n, n)$, and denote the identity matrix of $F(n)$ by $I_n$. We denote $hF(n) = \{X \in F(n); 'X = X\}, pF(n) = \{X \in hF(n); X$ is positive definite$, \}$, and use the following notations for classical groups:

$$GF(n) = \{X \in F(n); 'XX = X_\cdot X = I_n\}.$$  

If $F = R$ or $C$, denote

$$SF(n) = \{X \in GF(n); \det X = 1\}.$$  

Then $GR(n) = O(n), GC(n) = U(n), GH(n) = Sp(n), SR(n) = SO(n)$ and $SC(n) = SU(n)$ in usual notations. Any subgroup of $GF(n)$ acts on $F^*$ linearly over right multiplications of $F$ by usual manner and acts on $hF(n)$ (resp. $pF(n)$) by

$$A \cdot X = AX^*A \quad (3.1)$$

for $A \in GF(n), X \in hF(n)$ (resp. $pF(n)$). Each matrix of $hF(n)$ can be transformed to a diagonal form by the action of $GF(n)$ (resp. $SF(n)$). Similarly any subgroup of $GF(n_1) \times GF(n_2)$ acts on $F(n_1, n_2)$ by

$$(A, B) \cdot X = AX^*B \quad (3.2)$$

for $(A, B) \in GF(n_1) \times GF(n_2), X \in F(n_1, n_2)$.

We use mappings $k, k': H(n_1, n_2) \to C(2n_1, 2n_2)$,

$$h: H(n_1, n_2) \to C(2n_1, n_2) \text{ and } h': H(n_1, n_2) \to C(n_1, 2n_2)$$

such that

$$k(U + jV) = \begin{pmatrix} U & -\bar{V} \\ V & \bar{U} \end{pmatrix}, \quad k'(U + Vj) = \begin{pmatrix} U & V \\ -\bar{V} & \bar{U} \end{pmatrix}, \quad h(U + jV) = \begin{pmatrix} U \\ V \end{pmatrix},$$

$$h'(U + Vj) = (U, V)$$

for $U, V \in C(n_1, n_2)$.

Then $k, k'$ are real linear injections such that

$$k(P) = k(P), \quad k'(P) = k'(P), \quad k(PQ) = k(P)k(Q), \quad k'(PQ) = k'(P)k'(Q)$$

for $P \in H(n_1, n_2), Q \in H(n_2, n_3)$,

and $h$ (resp. $h'$) is a linear bijection over right (resp. left) multiplications of $C$ such that
\(h(PQ) = k(P)h(Q)\) (resp. \(h'(PQ) = h'(P)k(Q)\)).

For \(P \in \mathbb{H}(n_1, n_2)\), we see that column-rank \(\mathbb{H}(P) := n_2 - \text{dim}_E \{Q \in \mathbb{H}^n; PQ = 0\} = (2n_2 - \text{dim}_C \{Q \in \mathbb{H}^n; PQ = 0\})/2 = (\text{rank}_E k(P))'=2= (\text{rank}_E k'(P))'=2=(2n_1 - \text{dim}_C \{Q \in \mathbb{H}(1, n_1); QP=0\})/2=n_1-\text{dim}_H(\mathbb{H}(1, n_1); QP=0) =: \text{row-rank}_H(P)\). Note that the linear independence in \(\mathbb{H}^n, H(1, n_1)\) over right multiplications of \(H\) is equivalent to one over left multiplications of \(\mathbb{H}\) respectively owing to \(p \cdot q = q \cdot p\) (\(p, q\) in \(H\)). Therefore \(\text{rank}_H(P) := \text{column-rank}_H(P) = \text{row-rank}_H(P)\) is well-defined. Denote \(\text{MF}(n_1, n_2) = \{X \in F(n_1, n_2); \text{rank}_F(X) = \text{max}(n_1, n_2)\}\). Then \(k(\text{MF}(n_1, n_2)) = \text{MC}(2n_1, 2n_2) \cap k(\mathbb{H}(n_1, n_2))\).

Assume \(n_1 \geq n_2\). Denote \(f: \text{MF}(n_1, n_2) \rightarrow pF(n_2)\)
such that \(f(X) = \tilde{X}X\) for \(X \in \text{MF}(n_1, n_2)\). Then \(f\) is \(G(F(n_1) \times F(n_2))\)-equivariant with respect to the action (3.2) on \(\text{MF}(n_1, n_2)\) and the following action on \(pF(n_2)\):

\[(A, B) \cdot Y = BY^t \bar{B} \quad (3.3)\]

for \((A, B) \in GF(n_1) \times GF(n_2), Y \in pF(n_2)\).

**Lemma 3.2** (1) \(f\) is a submersion.

(2) \(f^{-1}(f(X)) = (GF(n_1) \times \{I_{n_1}\}) \cdot X\) for \(X \in \text{MF}(n_1, n_2)\).

(3) If \(n_1 > n_2\), then \(f^{-1}(f(X)) = (SF(n_1) \times \{I_{n_1}\}) \cdot X\) for \(X \in \text{MF}(n_1, n_2)\) where \(F = R\) or \(C\).

**Proof:** (1) Since any diagonal matrix in \(pF(n_2)\) is in the image of \(f\), it follows that \(f\) is onto from the diagonalizability by the action (3.3). To prove \(df_{x_0}: F(n_1, n_2) \rightarrow hF(n_2)\); \(X \rightarrow \tilde{X}X_0 + \tilde{X}_0X\) is onto at \(X_0 \in \text{MF}(n_1, n_2)\), if we use the action (3.2) of \(GF(n_1) \times GF(n_2)\), we may assume that \(X_0\) has the following form for some non-zero \(x_i \in R\) (\(i = 1, \ldots, n_2\)):

\[X_0 = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ x_{n_2} \end{bmatrix}.\]

In fact, the action (3.3) of \(\{I_{n_1}\} \times GF(n_2)\) transforms \(\tilde{X}_0X_0\) to a diagonal form and the action (3.2) of \(GF(n_1) \times \{I_{n_1}\}\) gives a required form. Then it is easy to show that \(df_{x_0}\) is onto. (2) Suppose \(f(X) = f(Y)\). Denote \(X = [x_1, \ldots, x_{n_2}], Y = [y_1, \ldots, y_{n_2}]\) where \(x_i, y_i \in F^{n_1}\), then \(t_i x_i = t_i y_i\) (\(i = 1, \ldots, n_2\)). We can choose \(x_k, y_k\) (\(h = n_2 + 1, \ldots, n_1\)) such that \(t_i x_k = t_i y_k = 0\) and \(t_k x_k = t_k y_k = \delta_{hk}\). Then \(X' = [x_1, \ldots, x_{n_2}], Y' = [y_1, \ldots, y_{n_2}]\) have the inverse matrices. For \(A = Y'X'^{-1}\), \(A\) is in \(GF(n_1)\) since \(Y'X'^{-1} = \bar{Y}' Y\). We have \((A, I_{n_2}) \cdot X = Y\). (3) If \(F = R\) or \(C\), then \(X' = X' \cdot \text{diag}[1, \ldots, 1, \text{det} X'^{-1}]\) and \(Y' = Y' \cdot \text{diag}[1, \ldots, 1, \text{det} Y'^{-1}]\) are in \(SL(n_1, F)\). Then \(B = Y'X'^{-1}\) is in \(SF(n_1)\) and \((B, I_{n_2}) \cdot X = Y\) if \(n_1 > n_2\). Q.E.D.

The tensor product \(F^m \otimes \cdots \otimes F^m\) over \(F\) of \(F^m, \ldots, F^m\) is defined if \(F = R\) or \(C\). Naturally \(R^n \otimes \cdots \otimes R^n = [z \in C^n \otimes \cdots \otimes C^n; \bar{z} = z]\) where denotes the complex conjugation extended naturally on \(C^n \otimes \cdots \otimes C^n\). If \(F = H\), then we consider the real linear map \(\tilde{f}: C^{2n} \otimes \cdots \otimes C^{2n} \rightarrow C^{2n} \otimes \cdots \otimes C^{2n}; \Sigma_i z_i h(P_i) \otimes \cdots \otimes h(P_i)) \rightarrow \Sigma_i z_i h(P_i) \otimes \cdots \otimes h(P_i))\), where \(z_i \in C\) and \(P_i \in H^s (t = 1, \ldots, s)\). Then \(\tilde{f}^2 = id\) (if \(s\) is even), or \(-id\) (if \(s\) is odd). The ten-
sor product $H^{n_1} \otimes \cdots \otimes H^{n_i}$ over right $H$ of $H^{n_1}, \cdots, H^{n_i}$ is defined by $H^{n_1} \otimes \cdots \otimes H^{n_i} := [Z \in C^{n_1} \otimes \cdots \otimes C^{n_i}, JZ = Z]$ (if $s$ is even), or $C^{n_1} \otimes \cdots \otimes C^{n_i}$ with the quaternion structure $J$ (if $s$ is odd). If $s = 1$, then $J$ is the standard quaternion structure on $C^{n_1}$.

For $s = 2$, then $H^{m} \otimes H^{n}$ is a real form of $C^{n_{1}} \otimes C^{n_{2}}$ with respect to the real structure $J$ on $C^{n_{1}} \otimes C^{n_{2}}$. For an even $s$, $H^{m}\otimes \cdots \otimes H^{n}$ is equivalent as real spaces to

$$(H^{m_1} \otimes H^{m_2}) \otimes \cdots \otimes (H^{m_{s-1}} \otimes H^{m_s})$$

since the complexifications are isomorphic over $C$.

Let $\rho_1, \cdots, \rho_i$ be linear representations of Lie groups $G_1, \cdots, G_i$ on $F^{m_i}$, $\cdots$, $F^{n_i}$ over $F$ respectively. If $F = R$ or $C$, then the exterior tensor product $\rho_1 \otimes \cdots \otimes \rho_i$ over $F$ is defined as the representation of the direct product group $G_1 \times \cdots \times G_i$ on the tensor product space $F^{m_1} \otimes \cdots \otimes F^{n_i}$ over $F$ such that

$$(\rho_1 \otimes \cdots \otimes \rho_i)(g_1, \cdots, g_i) := \rho_1(g_1) \otimes \cdots \otimes \rho_i(g_i)$$

for $(g_1, \cdots, g_i)$ in $G_1 \times \cdots \times G_i$, where the right hand side is the usual tensor product of linear transformations. If $F = H$, then note that $J$ commutes with the representation $(k \circ \rho_i)$ $\otimes \cdots \otimes (k \circ \rho_i)$ of $G_1 \times \cdots \times G_i$ on $h(H^{m_i}) \otimes \cdots \otimes h(H^{n_i})$. The exterior tensor product $\rho_1 \otimes \cdots \otimes \rho_i$ over right $H$ is defined as the representation of $G_1 \times \cdots \times G_i$ on $H^{m_1} \otimes \cdots \otimes H^{n_i}$ such that

$$(\rho_1 \otimes \cdots \otimes \rho_i)(g_1, \cdots, g_i) := ((k \circ \rho_1) \otimes \cdots \otimes (k \circ \rho_i))(g_1, \cdots, g_i)|_{H^{m_1} \otimes \cdots \otimes H^{n_i}}.$$
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\[ \iota(H^n \otimes \mathbb{H}^e) = k(H(n_1, n_2)) \]

since \( \tilde{Z}_1 = \tilde{J}Z_i \), \( Z_i \in \mathbb{C}^{2n_1} \), \( \tilde{J}(Z) = \tilde{J}_i(Z) = \tilde{J}(Z_i) \), \( Z \in \mathbb{C}^{2n_1} \otimes \mathbb{C}^{2n_2} \) and \( k(H(n_1, n_2)) = \{ X \in \mathbb{C}(2n_1, 2n_2); J_1 \tilde{X} J_2 = X \} \) where

\[
J_i = \begin{pmatrix}
0_{n_i} & -I_{n_i} \\
I_{n_i} & 0_{n_i}
\end{pmatrix} \quad (i = 1, 2).
\]

Through \( \iota \), the action of \( Sp(n_1) \times K \) on \( k(H(n_1, n_2)) \) is induced from the representation \( id \otimes id \) on \( H^n \otimes \mathbb{H}^e \) by \( (A, B) \cdot k(X) = k(A) k(X) k(B) \) for \( X \in H(n_1, n_2) \), \( (A, B) \in Sp(n_1) \times K \). The o.t.g. induced from this action is equivalent to the one which is induced from the action (3.2) of \( Sp(n_1) \times K \) on \( H(n_1, n_2) \), since \( k(B) = k(B) \) and \( k(A) k(X) k(B) = k(A X B) \).

Then (1) follows from Lemma 3.1 and Lemma 3.2(0), (1), (2), since \( MF(n_1, n_2) \) is open and dense in \( F(n_1, n_2) \). (2) follows from (1) since \( GR(n_1)^0 = SO(n_1) \). (3) follows from Lemma 3.1 and Lemma 3.2(0), (1), (3), (4) follows from that \( GF(n_2)(\text{resp. } SF(n_2) \text{ if } F = R \text{ or } C) \) transforms any matrix in \( pB(n_2) \) to a diagonal form. Q.E.D.

Denote \( r(n_1, n_2, n_3) = \text{coh}(SO(n_1) \times SO(n_2) \times SO(n_3), id \otimes id \otimes id, R^n_1 \otimes R^n_2 \otimes R^n_3) \),
\[
c(n_1, n_2, n_3) = \text{coh}(U(n_1) \times SU(n_2) \times SU(n_3), id \otimes id \otimes id, C^n_1 \otimes C^n_2 \otimes C^n_3),
\]
\[
q(n_1, n_2, n_3) = \text{coh}(Sp(n_1) \times Sp(n_2) \times SO(n_3), id \otimes id \otimes id, (H^n_1 \otimes H^n_2) \otimes R^n_3).
\]

**Proposition 3.4**

\[
(1) \quad r(n_1, n_2, n_3) \geq 18 \quad \text{if} \quad n_1 \geq n_2 \geq n_3 \geq 3.
\]
\[
(2) \quad c(n_1, n_2, n_3) \geq 6 \quad \text{if} \quad n_1 \geq n_2 \geq n_3 \geq 2.
\]
\[
(3) \quad q(n_1, n_2, n_3) \geq 3 \quad \text{if} \quad n_3 \geq 3, \quad n_1 \geq n_2 \geq 1.
\]
\[
(4) \quad q(n_1, n_2, n_3) \geq 8 \quad \text{if} \quad n_3 \geq 3, \quad n_1 \geq 2, \quad n_2 \geq 1.
\]

**Proof:** Denote \( \lambda(n_1, n_2, n_3) = \text{dim } pB(n_2, n_3) - \text{dim } SO(n_1) \times SO(n_3) \) (if \( n_1 \geq n_2 \geq n_3 \)) or \( \text{dim } R^n_1 \otimes R^n_2 \otimes R^n_3 - \text{dim } SO(n_1) \times SO(n_2) \times SO(n_3) \) (otherwise), \( \kappa(n_1, n_2, n_3) = \text{dim } pC(n_2, n_3) = \text{dim } SU(n_2) \times SU(n_3) \) (if \( n_1 \geq n_2 \geq n_3 \)) or \( \text{dim } C^n_1 \otimes C^n_2 \otimes C^n_3 - \text{dim } U(n_1) \times SU(n_2) \times SU(n_3) \) (otherwise), and \( \mu(n_1, n_2, n_3) = \text{dim } pR(4n, n_2) - \text{dim } Sp(n_1) \times Sp(n_2) \) (if \( n_3 \geq 4n_1, n_3 \)) or \( \text{dim } pR(n_2, n_3) - \text{dim } Sp(n_2) \times SO(n_3) \) (if \( n_3 \leq 4n_1, n_2 \leq n_3 \)) or \( \text{dim } (H^n_1 \otimes H^n_2) \otimes R^n_3 - \text{dim } Sp(n_1) \times Sp(n_2) \times SO(n_3) \) (otherwise). Then \( \lambda(n_1, n_2, n_3) \leq r(n_1, n_2, n_3) \leq c(n_1, n_2, n_3) \leq s(n_1, n_2, n_3) \leq q(n_1, n_2, n_3) \) by Prop. 3.3 since \( (H^n_1 \otimes H^n_2) \otimes R^n_3 \) is equivalent to \( H^n_1 \otimes (H^n_2 \otimes R^n_3) \) as \( Sp(n_1) \times Sp(n_2) \otimes SO(n_3) \)-spaces over \( R \). Since \( \lambda(x_1, x_2, x_3) = x_1^2 x_2^3 + x_1 x_2 x_3 - x_1 x_3^2 + x_2 x_3^2 \) (if \( x_1 \geq x_2 \geq x_3 \)) or \( x_1^2 x_2^3 + (x_1^2 + x_2^2 - x_3^2 - x_1 x_2^2) \) (otherwise), \( \kappa(x_1, x_2, x_3) = x_1^3 x_2^2 - x_1^2 x_3^2 - x_1 x_2 x_3^2 + 2 \) (if \( x_1 \geq x_2 \geq x_3 \)) or \( 2x_1 x_2 x_3 - x_1^2 x_2 + x_2^2 x_3 - x_3^3 \) (otherwise), and \( \mu(x_1, x_2, x_3) = 8x_1^2 x_2^2 + 2x_1 x_2 x_3 - 2x_1 x_3^2 - 2x_2 x_3^2 - x_1 x_2^2 \) (if \( x_2 \geq x_1 \geq x_3 \)) or \( 2x_1 x_2 - x_1 x_3^2 - 2x_2 x_3 - x_1 x_3^2 - x_2 x_3^2 + x_3^3 \) (if \( x_3 \geq x_1 \geq x_2 \)) or \( 4x_1 x_2 x_3 - (2x_1 + x_2 + x_3) x_2 x_3^2 + x_3^2 x_1 x_2^2 \) (otherwise), they define continuous piecewise polynomial functions on \( R^3 \) if we take \( x = (1, 2, 3) \) as real numbers. (1) Since \( \partial r/\partial x_i(x_1, x_2, x_3) \geq 0 \) for \( x_1 \geq x_2 \geq x_3 \geq 1 \) (if \( i = 1, 2, 3 \)), we have \( \lambda(n_1, n_2, n_3) \geq \lambda(n_1, n_2, 3) \geq \lambda(n_1, n_2, 3) \geq \lambda(n_1, n_2, 3) \).
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3, 3) ≦ λ(3, 3, 3) = 18. (2) Similar to (1), κ(n1, n2, n3) ≦ κ(2, 2, 2) = 6. (3) Since θμ/θx(x1, x2, x3) ≥ 0 for i = 1, 2, 3; x1, x2, x3 ≥ 1 (if x ≥ 4x1x2 or x3x1x2 ≤ 1), and θμ/θx1(x1, x2, x3) = (4x1x2 − x3) + 1/2 > 1/2, θμ/θx1(x1, x2, x3) = 4(x1x2x3 − 1) ≥ 4x1(x3−1) − 1 ≥ 3, θμ/θx1(x1, x2, x3) = 4(x1x2x3 − 1) − 1 > −1 for x1 ≥ x2 ≥ 1, x3 ≥ 2 (if x3 < 4x1x2 and x3x2 > x1), we have μ(n1, n2, n3) ≧ μ(n1, n2, n3) = μ(n1, n1, 1, 3) = μ(n1, n1, 1, 3) + θμ/θx1(x1, x2, x3) = μ(n1, n1, 1, 3) (0 < θ < 1) ≥ μ(n1, 1, 1, 3) (since μ(n1, 1, 3) and μ(n1, 1, 3) are integers, and −1 < θμ/θx1 is also an integer, especially θμ/θx1 ≥ 0) ≥ μ(1, 1, 3) = 3. (4) Similar to (3), μ(n1, n2, n3) ≥ μ(n1, 1, 3) = μ(2, 1, 3) = 8. Q.E.D.

Let L be the Lie algebra of a connected Lie group G. We write the same letter for a linear representation of L and the corresponding representation of G. According to Iwahori [12], there is the following relation between real irreducible representations of L (resp. G) and complex irreducible representations of L (resp. G) (cf. Goto-Grosshans [6]). For a complex irreducible representation ρ on a complex vector space V, we denote the real restriction of ρ on the real restricted vector space VR (abbrev. V since V = VR as a set) by ρR (abbrev. ρ), which is not real irreducible if and only if ρ is ‘real’, and so we attach to ρ a real irreducible representation ρ′ as follows. ρ′ = ρ (if ρ is the complexification σ of a real representation σ on a real form W of V, i.e., ρ is ‘real’.) or ρR (otherwise). Note that ρ′ and ρR are equivalent as real representations if and only if ρ1 and ρ2 are conjugate or equivalent as complex representations of L (resp. G). Conversely the complexification σ of W of a real irreducible representation σ on a real vector space W is not complex irreducible if and only if W has a L (resp. G)-invariant complex structure (then it is unique), and so we attach to σ a complex irreducible representation σ′ as follows. σ′ = σ (if W has a L (resp. G)-invariant complex structure) or σR (otherwise). Note that σR and σ (resp. σ′ and σ) are equivalent as complex (resp. real) representations.

Let (G, E^N) be an o.t.g. Then the Lie algebra L of G is a real reductive Lie algebra and has a form:

\[ L = L_0 \oplus L_1 \oplus \cdots \oplus L_s \]  

(3.4)

where L0 is the center of L, and L_i (i = 1, · · · , s) are simple ideals of L. Let Go, Gi be connected Lie subgroups of G corresponding to L0, L_i respectively and G0, Gi be the universal covering groups of Go, Gi respectively, then Gi (i = 1, · · · , s) are compact. Let id:G→SO(N) be the identity representation and i id be the corresponding representation of G := G0 × G1 × · · · × Gs.

In this paper, we consider (G, E^N) in case that id is a real irreducible representation of G. Then G is compact (cf. Kobayashi-Nomizu [14]), and so Go ≈ U(1) or the trivial group 1. For \( t \in R^* := R - \{0\} \), we denote \( i_t \) the complex irreducible representation of R such that \( i_t(x) = e^{xt} \) for \( x \in R \). We shall decompose \( \tilde{i} \) of \( G \) into an exterior tensor product of complex irreducible representations of \( \tilde{G}_i (i = 0, \cdots, s) \).

Case i) \( \tilde{i} \) is equivalent to \( i \) of \( G \): Then Go is trivial, and \( (\tilde{G}_i, \tilde{i}, C^N) \) is equivalent as complex representations to some
where $\rho_i$ is a self-conjugate complex irreducible representation of $\tilde{G}_i$ on $C_n^\ast$, $n_i \geq 2$ ($i = 1, \cdots, s$), $\Pi_i = n_i - N$, and $\# |i; \rho_i$ is 'quaternion' $|$ is even. We may assume $\rho_i (j = 1, \cdots, 2r)$ are 'quaternion' and $\rho_k (k = 2r + 1, \cdots, 2r + q)$ are 'real', and $\sigma$, denotes a real representation of $\tilde{G}_i$ on $R_n^\ast$ whose complexification is $\rho_{2r+i}$ ($i = 1, \cdots, q$), where $r$ and $q$ are non-negative integers. Then $w_{2r+i}^3 (f = 1, \cdots, q)$, and $(\tilde{G}_i, \text{id}_R, R^N)$ is equivalent as real representation to

$$
(\tilde{G}_1 \times \cdots \times \tilde{G}_i \times \tilde{G}_{i+1} \times \cdots \times \tilde{G}_{2r+q}, (\rho_{1R} \cdots \rho_{iR} \sigma_{1R} \cdots \sigma_{qR})(H^{n_{i+1/2}} R_{n_{i+1/2}}) \cdots (H^{n_{2r+i/2}} R_{n_{2r+i/2}}) \cdots (H^{n_{2r+q/2}} R_{n_{2r+q/2}}) \cdots (C^m \otimes C^m) (3.5)
$$

**Case ii** $\tilde{id} = \tilde{id} \cdot G_0 \approx U(1)$: Then $(\tilde{G}_i, \tilde{id} \cdot C^{N/2})$ is equivalent as complex representations to some

$$(R \times \tilde{G}_1 \times \cdots \times \tilde{G}_i \times \tilde{G}_{i+1} \times \cdots \times \tilde{G}_{2r+q}, (\rho_{1C} \cdots \rho_{iC} \sigma_{1C} \cdots \sigma_{qC})(C^m \otimes C^m \otimes C_m) (3.6)$$

where $t \in R$, $\rho_i$ is a complex irreducible representation of $\tilde{G}_i$ on $C_m^\ast$, $n_i \geq 2$ ($i = 1, \cdots, s$) and $\Pi_i = n_i - N/2$. So $(\tilde{G}_i, \tilde{id} \cdot C^{N/2})$ is equivalent as real representation to

$$(R \times \tilde{G}_1 \times \cdots \times \tilde{G}_i \times \tilde{G}_{i+1} \times \cdots \times \tilde{G}_{2r+q}, (\rho_{1C} \cdots \rho_{iC} \sigma_{1C} \cdots \sigma_{qC})(C^m \otimes C^m \otimes C_m) (3.6)$$

**Case iii** $\tilde{id} = \tilde{id} \cdot G_0 \approx U(1)$: Then $(\tilde{G}_i, \tilde{id} \cdot C^{N/2})$ is equivalent as complex representations to some

$$(\tilde{G}_1 \times \cdots \times \tilde{G}_i \times \tilde{G}_{i+1} \times \cdots \times \tilde{G}_{2r+q}, (\rho_{1C} \cdots \rho_{iC} \sigma_{1C} \cdots \sigma_{qC})(C^m \otimes C^m \otimes C_m) (3.7)$$

where $\rho_i$ is a complex irreducible representation of $\tilde{G}_i$ on $C_m^\ast$, $n_i \geq 2$ ($i = 1, \cdots, s$) and $\Pi_i = n_i - N/2$. So $(\tilde{G}_i, \tilde{id} \cdot C^{N/2})$ is equivalent as real representation to

$$(\tilde{G}_1 \times \cdots \times \tilde{G}_i \times \tilde{G}_{i+1} \times \cdots \times \tilde{G}_{2r+q}, (\rho_{1C} \cdots \rho_{iC} \sigma_{1C} \cdots \sigma_{qC})(C^m \otimes C^m \otimes C_m) (3.7)$$

where $\rho_{iC} \cdots \rho_{iC}$ is not 'real' since $(\rho_{iC} \cdots \rho_{iC} \sigma_{1C} \cdots \sigma_{qC})(C^m \otimes C^m \otimes C_m)$ is real irreducible.

**Theorem 3.5** Let $(G, E^N)$ be an o.t.g. of cohomogeneity at most 3. If $\text{id}: G \rightarrow SO(N)$ is real irreducible and $s \geq 3$ (cf. (3.4)), then $(\tilde{G}_i, \tilde{id} \cdot R^N)$ is equivalent as real representation to

$$(\tilde{A}_1 \times \tilde{A}_1 \times \tilde{A}_1, (\tilde{A}_1 \tilde{A}_1 \tilde{A}_1 \tilde{A}_1)^h(2\tilde{A}_1)^h, (H^R \otimes H^R) \otimes R^3) (3.8)$$

Especially cohomogeneity $(G, E^N) = 3$.

**Proof:** Suppose $\text{id}$ is real irreducible and $s \geq 3$. Then $O(G, \text{id}, R^N)$ is contained in (1) $O((Sp(n_1/2) \times Sp(n_2/2) \times SO(n_3)) \times \text{id}(\tilde{id}) \cdot \text{id}, (H^{n_{1/2}} \otimes H^{n_{2/2}}) \otimes R^m)$ for some $n_1, \ n_2 \geq 2, \ n_3 \geq 3; \ N = n_1 n_2 n_3$, (2) $O(SO(n_1) \times SO(n_2) \times SO(n_3)) \times \text{id}(\tilde{id}) \cdot \text{id}, (R^m \otimes R^m \otimes R^m)$ for some $n_1,$
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\[ n_2, n_3 \geq 3; \ N = n_1 n_2 n_3, \text{ or (3) } O(U(n_1) \times SU(n_2) \times SU(n_3), (id \otimes \text{id} \otimes \text{id})_R, (C^{n_1} \otimes C^{n_2} \otimes C^{n_3})_R \]

for some \( n_1, n_2, n_3 \geq 2; \ N = 2n_1 n_2 n_3 \text{ owing to (3.5), (3.6) and (3.7). \ On the other hand, coh (2) } \geq 18, \text{ coh (3) } \geq 6, \text{ coh (1)(max} (n_1, n_3) \geq 4) \geq 8 \text{ by Prop. 3.4(1)(2)(4). \ There } G_0 \text{ is trivial, and } O(G, id, R^N) \text{ is contained in } O((Sp(1) \times Sp(1) \times SO(n_3), (id \otimes \text{id} \otimes \text{id}, (H \otimes H) \otimes R^m) \text{ which is equivalent to } O(SO(4) \times SO(n_3), id \otimes id, R^4 \otimes R^m). \text{ Then } n_3 = 3 \text{ since coh } (G, E^N) \leq 3. \text{ So } O(G, id, R^N) \text{ is contained in } O(A_1 \times A_1 \times A_1, (A_1 \otimes A_1) \otimes (2A_1)'(2A_1)', (H \otimes H) \otimes R^3). \text{ Since } s \geq 3, \ G \text{ is isomorphic to } A_1 \times A_1 \times A_1, \text{ and } O(G, id, R^N) = O(A_1 \times A_1 \times A_1, (A_1 \otimes A_1) \otimes (2A_1)', (H \otimes H) \otimes R^3). \text{ Then } (G, id, R^N) \text{ and (3.8) are equivalent as real representation since } A_1, 2A_1 \text{ are characterized by degrees of complex irreducible representations of } A_1, \text{ and } 12 = 2^3 \text{ (cf. Section 2). \ And coh } (G, E^N) = 3 \text{ by Prop. 3.3. Q.E.D.}

Suppose \( s = 2; \ L = L_0 + L_1 + L_2 \) (cf. (3.4)). Then \( (\tilde{G}, \tilde{\mu}, R^N) \) is equivalent as real representation to one of the followings:

**TYPE I** \( (\tilde{G}_1 \times \tilde{G}_2, \rho_1 \otimes \rho_2, R^t \otimes R^m); n_1 \geq n_2 \geq 3, \ N = n_1 n_2, \rho_1 \) is a 'real' complex irreducible representation of \( \tilde{G}_1 \text{ on } C^{n_1}, R^m \text{ is a } \tilde{G}_2 \text{-invariant real form of } C^{n_2}(i = 1, 2) \).

**TYPE II** \( (\tilde{G}_1 \times \tilde{G}_2, \rho_1 \otimes \rho_2, H_1 \otimes H_2); n_1 \geq n_2 \geq 2, \ N = 4 n_1 n_2, \rho_1 \) is a 'quaternion' complex irreducible representation of \( \tilde{G}_1 \text{ on } C^{2n_1}, \text{ and } H_2 \text{ with the } \tilde{G}_1 \text{-invariant quaternionic structure (i.e., the right multiplication of } j(i = 1, 2) \).

**TYPE III** \( (G \times \tilde{G}_1 \times \tilde{G}_2, (t \otimes \rho_1 \otimes \rho_2)_{R}, (C^{n_1} \otimes C^{n_2} \otimes C^{n_3})_{R}); n_1 \geq n_2 \geq 2, \ N = 2 n_1 n_2, \rho_1 \) is a complex irreducible representation of \( G, \text{ on } C^n(i = 1, 2), \text{ and } \rho_1 \otimes \rho_2 \) is not 'real'.

**LEMMA 3.6** Let \( \rho_i \) be a linear representation on \( F^m \) of a compact Lie group \( K_i, \) and denote \( d_i = 2m_i - \dim K_i \) where \( i = 0 \) (if \( F = R \)), \( 1 \) (if \( F = C \)), or \( 2 \) (if \( F = H \)). Then

1. If \( 1 \leq n < m_i, \) then \( \text{doh } (K_i \times GF(n), \rho_i \otimes \text{id}, F^m \otimes F^n) \geq d_i + n \left( \frac{1}{2}(n-3) \right) + 1 \geq d_i + 3 \) (if \( n \geq 3 \)).

2. If \( 1 \leq n < m_i, \) then \( \text{doh } (K_i \times GF(n), \rho_i \otimes \text{id}, F^m \otimes F^n) \geq d_i + \frac{1}{2}(n-1) + n \geq d_i + 2 \) (if \( n \geq 2 \) and \( i = 1 \)).

**PROOF:** \( \text{doh } (K_i \times GF(n), \rho_i \otimes \text{id}, F^m \otimes F^n) \geq \dim F^m \otimes F^n - \dim K_i \times GF(n) = d_i + 2(n-1) - m_i - \frac{(2i-1)}{2} \times \frac{n}{2} - 1 \times (n-1) \). \text{Replacing } m_i \text{ by } n \text{ resp. } n + 1, \text{ we have (1) (resp. (2)). Q.E.D.}

Suppose \( s = 1; L = L_0 + L_1 \) (cf. (3.4)). Then \( (\tilde{G}, \tilde{\mu}, R^N) \) is equivalent as real representation to one of the followings:

**TYPE V** \( (\tilde{G}_1, \rho_1, R^N); n_1 \geq 3, \ N = n_1, \rho_1 \) is a 'real' complex irreducible representation of \( \tilde{G}_1 \text{ on } C^{n_1}, \text{ and } R^N \text{ is a } \tilde{G}_1 \text{-invariant real form of } C^{n_1}. \)

**TYPE VI** \( (R \times \tilde{G}_1, (t \otimes \rho_1)_{R}, (C^{n_1} \otimes C^{n_2})_{R}); n_1 \geq 2, \ N = 2 n_1, \rho_1 \) is a complex irreducible representation of \( \tilde{G}_1 \text{ on } C^{n_1}. \)

**TYPE VII** \( (\tilde{G}_1, \rho_1, C^{n_1}); n_1 \geq 2, \ N = 2 n_1, \rho_1 \) is a complex irreducible representation of \( G_1 \text{ on } C^{n_1}, \text{ and } \rho_1 \text{ is not 'real'.} \)
Lemma 3.7 If $n_1 \leq n_2$, then $GF(n_1)(=GF(n_1) \times \{I_{n_1}\}$ in $GF(n_1) \times GF(n_2)$) transforms any matrix $X=[x_{ij}] \in F(n_1, n_2)$ into $F^{n_1 \times n_2}$ for $i=1, \ldots, n_1$ to a form $Y=[y_{ij}] \in F(n_1, n_2)$ for $i=1, \ldots, n_1$ such that $y_{ij} = c_i \delta_{ij}$ for some $c_i \in R$ $(i, j = 1, \ldots, n_1)$ by the action (3.2).

Proof: There is $A \in GF(n_1)$ such that $A$ transforms $X' \in pF(n_1)$ to a diagonal form

$$
\begin{bmatrix}
\epsilon_1 \\
\vdots \\
\epsilon_n
\end{bmatrix}

$$

by the action (3.1).

Then $Y=AX$ satisfied the desired property. Q.E.D.

Suppose $s=0: L=L_0$ (cf. (3.4)). Then $(G, id, R^s)$ is equivalent as real representation to one of the followings:

Type VIII $(R, \tilde{1}, C_R); t \in R^x$.

Type IX $(1, 0, R); 1$ is the trivial group, and 0 is the trivial representation on $R$.

Note that the o.t.g. of Type VIII is equivalent to $O(SO(2), id, R^2)$.

For general $s \geq 0$, the estimate of $coh (G, E^\lambda)$ is given in each cases i), ii), iii), if $id: G \to SO(N)$ is real irreducible, by the following theorem. If moreover $s \geq 3$, especially we have $coh (G, E^\lambda) \geq s$.

Theorem 3.8

(1) In case i), $coh (G, E^\lambda) = coh$ of $(3.5) \geq 4^r(3^s-6r-3q)$.

(2) In case ii), $coh (G, E^\lambda) = coh$ of $(3.6) \geq 2^{s+1}-3s-1$.

(3) In case iii), $coh (G, E^\lambda) = coh$ of $(3.7) \geq 2^{s+1}-3s-1$.

Proof: (3) follows from (2). For (2), we may assume $n_1 \geq \cdots \geq n_s \geq 2$. If $s < 3$, then (2) is trivial. Suppose $s \geq 3$. If $n_1 \geq n_2 \cdots n_s$, then we denote $f(n_1, \ldots, n_s) = \dim pC(n_2 \cdots n_s) - \dim SU(n_2) \times \cdots \times SU(n_s) = n_2^2 \cdots n_s^2 - n_2^2 - \cdots - n_s^2 + s - 1$. Then $\partial f/\partial n_s = 2n_1(n_2 \cdots \hat{n}_s \cdots n_s^2 - 1)$ of $0 \geq 0$. If $n_1 \leq n_2 \leq \cdots n_s$, then we denote $f(n_1, \ldots, n_s) = \dim C^s \cdots \phi C^s \cdots U(n_1) \times SU(n_2) \times \cdots \times SU(n_s) = 2n_1 \cdots n_s - n_1^2 - \cdots - n_s^2 + s - 1$. Then $\partial f/\partial n_s = 2(n_1 \cdots \hat{n}_s \cdots n_s - n_s) = 2(n_1 \cdots n_s - n_s) = 0$. Therefore $coh (3.6) \geq f(n_1, \ldots, n_s) \geq f(2, \ldots, 2) = 2^{s+1}-3s-1$.

(1) Suppose $s = 2r + q \leq 2$. If $r, q \geq 1$, then (1) is trivial. If $r=0$, $q=s=2$, then (1) follows from Prop. 3.3. If $s=3$, then (1) follows from Prop. 3.4. Assume $s \geq 4$. Suppose $s=0$: Then we may assume $n_1 \geq \cdots \geq n_s \geq 3$. If $n_1 \geq n_2 \cdots n_s$, then denote $f(n_1, \ldots, n_s) = \dim pR(n_2 \cdots n_s) - \dim SO(n_2) \times \cdots \times SO(n_s) = (n_2^2 \cdots n_s^2 + n_2 \cdots n_s - n_2^2 - \cdots - n_s^2 + n_2 + \cdots + n_s)/2$. Then $\partial f/\partial n_s = n_1(n_2^2 \cdots \hat{n}_s \cdots n_s^2 - 1) + (n_2 \cdots \hat{n}_s \cdots n_s + n_s)/2$. Therefore $coh (3.5) \geq f(n_1, \ldots, n_s) \geq f(3, \ldots, 3) = 3^s - 3s - 3q$. Suppose $q=0$: Then we may assume $n_1 \leq \cdots \leq n_s \leq 2$. If $n_1 n_2 \geq n_3 \cdots n_s$, then denote $g(n_1, \ldots, n_s) = \dim$...
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\[ pR(n_3 \cdots n_r) - \text{dim } Sp(n_2/2) \times \cdots \times Sp(n_r/2) = (n_3^3 \cdots n_r^3 - n_3 \cdots n_r^3 - \cdots - n_3 - n_r^3 - \cdots - n_r) / 2. \]
Since \( \partial g / \partial n_i \geq 0 \) (i = 1, \cdots, s), \( \text{coh} (3.5) \geq g(n_1, n_2, \cdots, n_r) \geq g(n_1, n_2, 2, \cdots, 2) = 2^{2r-5} + 2^{r-3} - 3(s-2) = 2^{2r-5} + 2^{r-3} - 6r + 6 \geq 4^r - 6r. \)
If \( n_1 n_2 \leq n_3 \cdots n_r \), then denote \( h(n_1, \cdots, n_r) = \text{dim } H^{n/r} / \cdots / H^{n/2} - \text{dim } Sp(n_1/2) \times \cdots \times Sp(n_r/2) = n_1 \cdots n_r - (n_1^2 + \cdots + n_r^2 + n_1 + \cdots + n_r) / 2. \)
Since \( \partial h / \partial n_i = n_1 \cdots n_r - n_i - 1/2 \leq n_2 \cdots n_r - n_i - 1/2 \leq n_3^2 - n_1 - 1/2 \leq n_2 - n_i - 1/2 \geq 0 \) (i = 1, \cdots, s), \( \text{coh} (3.5) \geq h(n_1, \cdots, n_r) \geq h(n_3, n_3, n_4, \cdots, n_r) \geq h(n_4, n_4, n_5, \cdots, n_r) \geq h(n_2, \cdots, 2) = 2^r - 3s = 4^r - 6r. \)
Finally suppose \( r, q \geq 1 \). Then we may assume \( n_1 \geq \cdots \geq n_2 \geq 2 \) and \( n_2r+1 \geq \cdots \geq n_2r+q \geq 3 \). If \( n_1 n_2 \leq n_3 \cdots n_r \), then denote \( g(n_1, \cdots, n_r) = \text{dim } pR(n_3 \cdots n_r) - \text{dim } Sp(n_1/2) \times \cdots \times Sp(n_r/2) \times SO(n_2r+1) \times \cdots \times SO(n_2r+q) \)
\[ = (n_3^3 \cdots n_r^3 - n_3 \cdots n_r^3 - \cdots - n_3 - n_r^3 - \cdots - n_r) / 2. \]
Since \( \partial g / \partial n_i \geq 0 \) (i = 1, \cdots, s), \( \text{coh} (3.5) \geq g(n_1, \cdots, n_r) \geq g(n_1, n_2, 2, \cdots, 2, 3, 3, 3, 3) = 2^r \cdot 3^t (2^{2r-5} - 3^t + 2^{-5}) + 6 - 6r - 3q \geq 4^r - 3^t - 6r - 3q. \)
If \( n_1 n_2 \leq n_3 \cdots n_r \), then denote \( h(n_1, \cdots, n_r) = \text{dim } H^{n/r} / \cdots / H^{n/2} - \text{dim } Sp(n_1/2) \times \cdots \times Sp(n_r/2) \times SO(n_2r+1) \times \cdots \times SO(n_2r+q) = n_1 \cdots n_r - (n_1^2 + \cdots + n_r^2 + n_1 + \cdots + n_r) / 2. \)
Since \( \partial h / \partial n_i = n_1 \cdots n_r - n_i - 1/2 \leq n_2 \cdots n_r - n_i - 1/2 \geq 2(2^r - 1)/2 = 2(2^r - 1) > 0, \) \( \text{coh} (3.5) \geq h(n_1, \cdots, n_r) \geq h(n_3, n_3, n_4, \cdots, n_r) \geq h(n_4, n_4, n_5, \cdots, n_r) \geq h(n_2, \cdots, 2) = 4^r - 3^t - 6r - 3q. \) Q.E.D.

4. Orthogonal transformation groups of cohomogeneity at most 3

(I) Let \((G, E^N)\) be a real irreducible o.t.g. of type I.

**PROPOSITION 4.1** \( \text{coh } (G, E^N) \leq 3 \) if and only if \((G, \bar{G}, R^N)\) is equivalent as real representation to one of the following:

- \( \text{coh} = 1 \): none,
- \( \text{coh} = 2 \): none,
- \( \text{coh} = 3 \):
  - (1) \((A_1 \times A_1, (2A_1)^r, R^3 \otimes R^3)\),
  - (2) \((A_3 \times A_1, A_2^r \otimes (2A_1)^r, R^6 \otimes R^3)\),
  - (3) \((C_2 \times A_1, A_2^r \otimes (2A_1)^r, R^6 \otimes R^3)\),
  - (4) \((B_2 \times A_1, A_2^r \otimes (2A_1)^r, R^{2k+1} \otimes R^3)\), \(k \geq 3\),
  - (5) \((D_2 \times A_1, A_2^r \otimes (2A_1)^r, R^{2k} \otimes R^3)\), \(k \geq 4\),
  - (6) \((B_2 \times A_1, A_2^r \otimes (2A_1)^r, R^8 \otimes R^3)\),
  - (7) \((D_2 \times A_1, A_2^r \otimes (2A_1)^r, R^8 \otimes R^3)\), \(i = 3, 4\).

**PROOF:** Suppose \( \text{coh } (G, E^N) \leq 3 \). Then \((G, \bar{G}, R^N)\) is equivalent as real representation to (1), \cdots, (6), or (7) owing to Prop. 3(3)(2)(4) Prop. 2.17, Lemma 3.6(1)\((F = R, i = 0, n = 3)\), \(3 \geq \text{doh } (G, E^N) \geq d_0 + 3\), Prop. 2.1 \((d_0 \leq 3)\), \(\text{doh } (A_3 \times A_1, (A_1 + A_1)^r \otimes (2A_1)^r, R^{\dim A_3} \otimes R^3) = 2 \dim A_3 - 3 \geq 13\) \((k \geq 2)\), Prop. 2.5, \(\text{doh } (C_2 \times A_1, (2A_1)^r \otimes (2A_1)^r, R^{\dim C_2} \otimes R^3) = 2 \dim C_2 - 3 \geq 13\) \((k \geq 2)\).
2\dim C_1 - 3 \geq 17 \ (k \geq 2), \ \doh (C_1 \times A_1, A_1^R(2A_1)^R, R^{k(2k-11-1)}_R) = 4k(k-1) - 6 \geq 18 (k \geq 3), \ \text{Prop. 2.8}, \ \doh (B_1 \times A_1, A_1^R(2A_1)^R, R^{\dim B_1}_R \otimes R^3) = 2\dim B_1 - 3 \geq 39 (k \geq 3), \ \doh (B_1 \times A_1, A_1^R(2A_1)^R, R^{\dim B_1}_R \otimes R^3) = 2\dim B_1 - 3 \geq 53 (k \geq 4), \ \text{the equivalence of o.t.g.'s} \ \O(D_1 \times A_1, A_1^R(2A_1)^R, R^{k(2k-11-1)}_R) = 4k(k-1) - 6 \geq 18 (ib^3), \ \text{Prop. 2.11}, \ \doh (D_k \times A_1, A_1^R(2A_1)^R, R^{k(2k-11-1)}_R) = 2\dim D_k - 3 \geq 25, \ \doh (F_4 \times A_1, A_1^R(2A_1)^R, R^{k(2k-11-1)}_R) = 23, \ \text{doh} (G_2 \times A_1, A_1^R(2A_1)^R, R^{k(2k-11-1)}_R) = 4.

Conversely if \ (G, E^N) \ is induced from (1), \ \cdots, \ (5), \ or \ (7), \ then \ (G, E^N) \ can also be induced from \ (SO(n_1) \times SO(3), id \otimes id, R^n \otimes R^3) \ for \ some \ n_1 \neq 4. \ So \ \coh (G, E^N) = 3 (cf. Prop. 3.3(2)(4)). \ An \ o.t.g. \ induced \ from \ (6) \ is of \ \coh 3. \ In \ fact \ Spin \ (7) \times SO(3) \ acts \ on \ R(8, 3) \ through \ \i \ \text{by the action (3.2)(cf. Prop. 3.3 Proof)}, \ and \ the isotropy subgroup at \}

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix}
\]

where \(|x_i| \ (i=1, 2, 3)\) are non-zero distinct real numbers, is locally isomorphic to \(SU(2)(cf. \ Yokota \ [24, \ Theorem 5.27, \ Theorem 5.2]). \ Q.E.D.

\[\text{(II)} \quad \text{Let} \ (G, E^N) \ \text{be \ a \ real \ irreducible o.t.g. \ of \ type II.}\]

\begin{proposition}
\coh (G, E^N) \leq 3 \ if \ and \ only \ if \ \widetilde{G}, \widetilde{id}, R^N \ \text{is equivalent as real representation to one of the followings:}
\end{proposition}

\begin{itemize}
  \item \coh = 1: \ (8) \ (A_1 \times A_1, A_1^R(2A_1)^R, H \otimes H),
  \item \coh = 2: \ (9) \ (C_4 \times A_1, A_1^R(2A_1)^R, H^2 \otimes H^2), \ k \geq 2,
  \item \coh = 3: \ (10) \ (C_4 \times C_2, A_1^R(2A_1)^R, H^4 \otimes H^4), \ k \geq 2,
  \item \coh = 4: \ (11) \ (A_1 \times A_1, 3A_1^R(2A_1)^R, H^6 \otimes H^6), \ k \geq 2,
  \item \coh = 5: \ (12) \ (A_1 \times A_1, 3A_1^R(2A_1)^R, H^8 \otimes H^8), \ k \geq 2.
\end{itemize}

\begin{proof}
\text{Suppose} \ \coh (G, E^N) \leq 3. \ \text{Then} \ n_2 \leq 3 (cf. Prop. 3.3(1)(4)). \ \text{Assume} \ n_2 = 3. \ \text{Then} \ \widetilde{G}, \rho_2, H^N) \ \text{is equivalent as complex representation to} \ (C_3, A_1, H^3) \ \text{owing to Prop. 2.20 and} \ \coh (Sp(n_3) \times A_1, A_1^R(2A_1)^R, H^6 \otimes H^6) \ \text{doh} (A_1, pH(3)) = 12 (cf. Prop. 3.3(1)). \ \text{So} \ (\widetilde{G}, \widetilde{id}, R^N) \ \text{is equivalent as real representation to} \ (12) \ \text{owing to Lemma 3.6(1)(F=H, i=2, m_2=n_1, n=n_2=3, d_1+k \lfloor 2^{-1}(k-3)+1 \rfloor = d_2+3), 3 \geq \doh (G, E^N) \geq d_2+3, Prop.'s 2.4, 2.7, 2.10, 2.14, \ \doh (D_5 \times C_2, A_1^R(2A_1)^R, H^{16+6+H^6}) = 105 (i=5, 6), Prop. 2.15, Remark 2.16, \ \doh (E_7 \times C_2, A_6^R(2A_1)^R, H^{28+6+H^6}) = 171. \ \text{Assume} \ n_2 = 2. \ \text{Then} \ (\widetilde{G}, \rho_2, H^N) \ \text{is equivalent as complex representation to} \ (C_2, A_1, H^2) \ \text{or} \ (A_1, 3A_1, H^2) \ \text{owing to Prop. 2.20,} \ \deg p_1 = 2n_1 > 4 (cf. Prop. 2.20 and} \ \doh (A_1 \times A_1, \widetilde{id}, R^N).
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\[ 3A_5 \otimes 3A_1, H^2 \otimes H^2 = 10. \] So \( (\tilde{G}, \tilde{d}, R^N) \) is equivalent as real representation to \( (10) \) or \( (13) \) owing to Lemma 3.6(2) \( (F=H, i=2, m_2=n_1 > n = n_2 = 2) \), \( 3 \geq \text{doh} \ (G, EN) \geq 3d_2 + 2 \), deg \( \rho_1 > 4 \), Prop.'s 2.4, 2.7, 2.10, 2.14, doh \( (D_6 \times A_1, A_1 \otimes 3A_1, H^{16} \otimes H^2) \geq \text{doh} \ (D_6 \times C_2, A_1 \otimes A_1, H^{16} \otimes H^2) = 52 \ (i=5, 6) \), Prop. 2.15, Remark 2.16, doh \( (E_7 \times A_1, A_6 \otimes 3A_1, H^{26} \otimes H^2) \geq \text{doh} \ (E_7 \times C_2, A_6 \otimes A_1, H^{26} \otimes H^2) = 59 \).

Assume \( n_2 = 1 \). Then \( (\tilde{G}_2, \rho_2, H^{n_2}) \) is equivalent as complex representation to \( (A_1, A_1, H) \) by Prop. 2.20. So \( (\tilde{G}_2, \tilde{d}, R^N) \) is equivalent as real representation to \( (8), (9) \) or \( (11) \) owing to Lemma 3.6(1) \( (F=H, i=2, m_2=n_1, n=1, d_1 + n \ 2^{i-1}(n-3) + 1 = d_2 - 3) \), \( 3 \geq \text{doh} \ (G, EN) \geq d_2 - 3 \), Prop. 2.4, \( \text{coh} \ (A_2 \times A_1, A_2 \otimes A_1, H^{10} \otimes H) = 4 \) (cf. The linear isotropy representation of the symmetric pair \( (E_6, SU(6) \cdot SU(1)) \) of rank 4 is characterized as a real 40 dimensional irreducible almost faithful representation of \( A_5 \times A_1 \) owing to Section 2), Prop.'s 2.7, 2.10, 2.14, Remark 2.13, \( \text{coh} \ (D_6 \times A_1, A_2 \otimes A_1, H^{16} \otimes H) = 4 \) (cf. The linear isotropy representation of the symmetric pair \( (E_7, \text{Spin}(12) \cdot Sp(1)) \) of rank 4 is characterized as a real 64 dimensional irreducible almost faithful representation of \( D_6 \times A_1 \) owing to Section 2), Prop. 2.15, Remark 2.16, \( \text{coh} \ (E_7 \times A_1, A_6 \otimes A_1, H^{26} \otimes H) = 4 \) (cf. The linear isotropy representation of the symmetric pair \( (E_8, E_7 \cdot Sp(1)) \) of rank 4 is characterized as a real 112 dimensional irreducible almost faithful representation of \( E_7 \times A_1 \) owing to Section 2).

The linear isotropy representation of the symmetric pair \( (E_7, \text{Spin}(12) \cdot Sp(1)) \) of rank 4 is characterized as a real 64 dimensional irreducible almost faithful representation of \( D_6 \times A_1 \) owing to Section 2), Prop. 2.15, Remark 2.16, \( \text{coh} \ (E_7 \times A_1, A_6 \otimes A_1, H^{26} \otimes H) = 4 \) (cf. The linear isotropy representation of the symmetric pair \( (E_7, E_7 \cdot Sp(1)) \) of rank 4 is characterized as a real 112 dimensional irreducible almost faithful representation of \( E_7 \times A_1 \) owing to Section 2).

Conversely an o.t.g. induced from \( (8) \) or \( (9) \) is of \( \text{coh} \ 1 \) by Prof. 3.3(1) \( (F=H, n_2 = 1, K=Sp(1)) \). An o.t.g. induced from \( (10) \) is of \( \text{coh} \ 2 \) by Prof. 3.3 \( (F=H, n_2 = 2, K=Sp(2)) \). An o.t.g. induced from \( (12) \) is of \( \text{coh} \ 3 \) by Prof. 3.3 \( (F=H, n_2 = 3, K=Sp(3)) \). An o.t.g. induced from \( (11) \) is of \( \text{coh} \ 2 \) (cf. The linear isotropy representation of the symmetric pair \( (G_2, SO(4)) \) of rank 2 is characterized as a real 8 dimensional irreducible almost faithful representation of \( A_1 \times A_1 \) owing to Prop.'s 2.1, 2.2, 2.4). If \( (G, EN) \) is induced from \( (13) \), then \( \text{coh} \ (G, EN) = \text{coh} \ (A_1, pH(2)) \geq \text{doh} \ (A_1, pH(2)) = 3 \) (cf. Prop. 3.3) and \( \text{coh} \ (G, EN) \leq \text{coh} \ (A_1, pH(2)) = \text{coh} \ (A_1, 0' \oplus (4A_1)' , R \oplus R^3) = 1 + \text{coh} \ (A_1, (4A_1)', R^3) = 3 \) (cf. The linear isotropy representation of the symmetric pair \( (SU(3), SO(3)) \) of rank 2 is characterized as a real 5 dimensional irreducible representation of \( A_1 \) owing to Prop.'s 2.1, 2.2, 2.4), where the action of \( A_1 \) on \( pH(2) \) is given as Prop. 3.3 and Lemma 3.2. Q.E.D.
(III) Let $(G, E^N)$ be a real irreducible o.t.g. of type III.

**Proposition 4.3** coh $(G, E^N) \leq 3$ if and only if $(G, \tilde{i}, R^N)$ is equivalent as real representation to one of the following:

coh = 1: none,

coh = 2: (14) $(R \times A_3 \times A_1, \tilde{i} \otimes A_1 \otimes A_1, C \otimes C^{+1} \otimes C^0)$; $k \geq 1, t \in R^x$.

coh = 3: (15) $(R \times A_2 \times A_2, \tilde{i} \otimes A_1 \otimes A_1, C \otimes C^{+1} \otimes C^0)$; $k \geq 2, t \in R^x$.

(16) $(R \times C_2 \times A_1, \tilde{i} \otimes A_1 \otimes A_1, C \otimes C^{2k} \otimes C^0)$; $k \geq 2, t \in R^x$.

**Proof:** Suppose coh $(G, E^N) \leq 3$. Then $n_3 \leq 3$ (cf. Prop. 3.3(1)(4)).

Assume $n_3 = 3$. Then $(\tilde{G}_2, \rho_2, C^{n_3})$ is equivalent as complex representation to $(A_2, A_1, C^n)$ owing to Prop. 2.18, Remark 2.19 and coh $(U(n_1) \times A_1, id \otimes 2A_1, C^{n_2} \otimes C^n) \cong \text{doh}(A_1, pC(3)) = 6$. If $\rho_1$ is ‘real’ and $n_1 \geq 6$, then coh $(G, E^N) = \text{coh}(U(1) \times \tilde{G}_1 \times A_2, id \otimes \rho \otimes A_1, C \otimes C^{n_2} \otimes C^n) \cong \text{coh}(G \times (U(1) \times A_2, \rho_1 \otimes \rho \otimes A_1, R^{n_2} \otimes (C \otimes C)^0)) \cong \text{coh}(SO(n_1) \times U(3), id \otimes id_R, R^{n_2} \otimes (C \otimes C)_k) \cong \text{coh}(U(3), pR(6)) \cong \text{doh}(U(3), pR(6)) = 12$ (cf. Prop. 3.3). So $(\tilde{G}_1, \rho_1, C^{n_3})$ is not ‘real’ or $n_1 \leq 5$. Then $(\tilde{G}, \tilde{i}, R^N)$ is equivalent as real representation to (15) owing to Lemma 3.6(1) $(F = C, \varphi = 1, m_1 = n_1, n = n_2 = 3), 3 \geq \text{coh}(G, E^N) \geq d_1 + 3$, Prop. 2.2 $(A_k(k = 3)$ is ‘real’ of degree 6), Remark 2.3, $\text{doh}(R \times A_3 \times A_2, \tilde{i} \otimes 2A_1 \otimes A_1, C \otimes C^{k+2 + (k-1)/2} \otimes C^0) = (k+1)(2k-1) - 8 \geq 2(k \geq 4)$, Prop. 2.6 $(A_2(k = 2)$ is ‘real’ of degree 11), $\text{doh}(R \times C_2 \times A_2, \tilde{i} \otimes A_1 \otimes A_1, C \otimes C^{k+2 + (k-1)/2} \otimes C^0) = 3 \cdot 2^{k+1} - 2k^2 - k - 9 \geq 18(k \geq 3)$, Prop. 2.12 $(A_1(k \geq 4)$ is ‘real’ of degree $\geq 8), \text{doh}(R \times D_4 \times A_2, \tilde{i} \otimes A_1 \otimes A_1, C \otimes C^{k+1} \otimes C^0) = 3 \cdot 2^k - k(2k-1) - 9 \geq 11$ for $i = k, k - 1$ (if $k \geq 4)$, Prop. 2.15, Remark 2.16, $\text{doh}(R \times E_6 \times A_2, \tilde{i} \otimes A_1 \otimes A_1, C \otimes C^{k+1} \otimes C^0) = 75$, $\text{doh}(R \times E_7 \times A_2, \tilde{i} \otimes A_1 \otimes A_1, C \otimes C^{k+4} \otimes C^0) = 194$.

Assume $n_3 = 2$. Then $(\tilde{G}_2, \rho_2, C^{n_3})$ is equivalent as complex representation to $(A_1, A_1, C^n)$ by Prop. 2.18. If $(\tilde{G}_1, \rho_1, C^{n_3})$ is ‘real’ of degree $n_1 \geq 4$, then coh $(G, E^N) = \text{coh}(U(1) \times \tilde{G}_1 \times A_1, id \otimes \rho \otimes A_1, C \otimes C^{n_2} \otimes C^n) = \text{coh}(\tilde{G}_1 \times (U(1) \times A_1), \rho_1 \otimes \rho \otimes A_1, R^{n_2} \otimes (C \otimes C)^0) \cong \text{coh}(SO(n_1) \times U(2), id \otimes id_R, R^{n_1} \otimes (C \otimes C)_k) \cong \text{coh}(U(2), pR(4)) \cong \text{doh}(U(2), pR(4)) = 6$. So $(\tilde{G}_1, \rho_1, C^{n_3})$ is not ‘real’ or $n_1 \leq 3$. Then $(\tilde{G}, \tilde{i}, R^N)$ is equivalent as real representation to (14) or (16) owing to Prop. 2.18, Lemma 3.6(2) $(F = C, i = 1, m_1 = n_1 > n = n_2 = 2), 3 \geq \text{coh}(G, E^N) \geq d_1 + 2$, Prop. 2.2 $(A_2(k = 3)$ is ‘real’ of degree 6), Remark 2.3, $\text{doh}(R \times A_3 \times A_1, \tilde{i} \otimes A_1 \otimes A_1, C \otimes C^{k+3 + (k-1)/2} \otimes C^0) = (k+1)(k+3) - 3 \geq 5(k \geq 1)$, Prop. 2.6 $(A_2(k = 2)$ is ‘real’ of degree 11), Prop. 2.9 $(A_1(k \geq 3)$
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is 'real' of degree $\geq 7$, $\doh(R \times B \times A_1, i_c \hat{A}_1 A_1 A_1, C \otimes C^{\otimes 3} \otimes C^3) = 2^{k+2} - k(2k + 1) - 4 \geq 7(k \geq 3)$, Prop. 2.12($A_1(k \geq 4)$ is 'real' of degree $\geq 8$, $A_1(k = 4)$ for $i = 3, 4$ are 'real' of degree $8$), $\doh(R \times D_x \times A_1, i_c \hat{A}_1 A_1 A_1, C \otimes C^{\otimes 3} \otimes C^3) = 2^{i+1} - k(2k - 1) - 4 \geq 15$ for $i = k - 1, k$(if $k \geq 5$), Prop. 2.15, Remark 2.16, $\doh(R \times E \times A_1, i_c \hat{A}_1 A_1 A_1, C \otimes C^{\otimes 3} \otimes C^3) = 26$, $\doh(R \times E \times A_1, i_c \hat{A}_1 A_1 A_1, C \otimes C^{\otimes 3} \otimes C^3) = 26$.

Conversely an o.t.g. induced from (14)(resp. (15)) is of $\coth 2$(resp. 3)(cf. Prop. 3.3(1) (4)). If $(G, E^N)$ is induced from (16), then $\coth(G, E^N) = \coth (U(1) \times C_4 \times A_1, i_c \hat{A}_1 A_1 A_1, C \otimes C^{\otimes 3} \otimes C^3) = \coth (U(1) \times (C_4 \times A_1), i_c \hat{A}_1 A_1 A_1, C \otimes \overline{C} \otimes C^3) = \coth (SO(2) \times (C_4 \times A_1), i_c \hat{A}_1 A_1 A_1, C \otimes \overline{C} \otimes C^3) = \coth (SO(2) \times A_1, \overline{C} \otimes C^3) = \coth (SO(2), \overline{C} \otimes C^3) + \coth (A_1, \overline{C} \otimes C^3) = 2 + 1 = 3$(cf. Prop. 3.3).

Q.E.D.

(IV) Let $(G, E^N)$ be a real irreducible o.t.g. of type IV.

PROPOSITION 4.4 $\coth(G, E^N) \leq 3$ if and only if $(\hat{G}, \hat{\id}, R^N)$ is equivalent as real representation to one of the followings:

$\coth = 1$: none,
$\coth = 2$: (17) $(A_1 \times A_1, A_1 \hat{A}_1 A_1, C^{\otimes 2} \otimes C^3); k \geq 2$,
$\coth = 3$: (18) $(A_1 \times A_2, A_1 \hat{A}_1 A_1, C^{\otimes 2} \otimes C^3); k \geq 3$.

PROOF: Suppose $\coth(G, E^N) \leq 3$. Then $(\hat{G}, \hat{\id}, R^N)$ is equivalent as real representation to (17) or (18) owing to Prop. 4.3. In fact, $(C_4 \times A_1, A_1 \hat{A}_1 A_1, C^{\otimes 2} \otimes C^3)(k \geq 2)$ and $(A_1 \times A_1, A_1 \hat{A}_1 A_1, C^{\otimes 2} \otimes C^3)$ are 'real', so they are not real irreducible, and $\coth(A_2 \times A_2, A_1 \hat{A}_1 A_1, C^{\otimes 2} \otimes C^3)$ is equivalent to the linear isotropy representation of the Hermitian symmetric pair $(SU(6), S(U(3) \times U(3)))$ of rank 3 whose restricted root system is of type $C$(cf. Tasaki-Yasukura[22], Helgason[7]).

Conversely an o.t.g. induced from (17)(resp. (18)) is of $\coth 2$(resp. 3) since $(U(1) \times A_1 \times A_1, i_c \hat{A}_1 A_1 A_1, C^{\otimes 2} \otimes C^3)$ of $k \geq 2$(resp. $(U(1) \times A_1 \times A_1, i_c \hat{A}_1 A_1 A_1, C^{\otimes 2} \otimes C^3)$ of $h \geq 3$) is equivalent to the linear isotropy representation of the Hermitian symmetric pair $(SU(h + 3), S(U(h + 1) \times U(2)))$ of rank 2(resp. $(SU(h + 4), S(U(h + 1) \times U(3)))$ of rank 3 whose restricted root system is of type $BC$(cf. [22], [7]). Q.E.D.

(V) Let $(G, E^N)$ be a real irreducible o.t.g. of type V.

PROPOSITION 4.5 $\coth(G, E^N) \leq 3$ if and only if $(\hat{G}, \hat{\id}, R^N)$ is equivalent as real representation to one of the followings:

$\coth = 1$: (19) $(A_1, (2A_1)^{\gamma}, R^3)$,
(20) $(A_3, A_2, R^5)$,
(21) $(C_2, A_2, R^3)$,
(22) $(B_4, A_2, R^{2k+1}); k \geq 3$.
(23) \((D_n, A_1, R^{2k}); k \geq 4\), (24) \((D_i, A_i, R^8); i = 3, 4, 5\),
(25) \((B_n, A_2, R^3)\),
(26) \((B_4, A_4, R^{16})\),
(27) \((G_2, A_2, R^3)\),
(28) \((A_2, (A_1 + A_2)^r, R^6)\),
(29) \((A_1, (4A_1)^r, R^6)\),
(30) \((C_3, A_3, R^{14})\),
(31) \((C_3, (2A_1)^r, R^{10})\),
(32) \((G_2, A_2, R^{14})\),
(33) \((F_4, A_4, R^{26})\),
(34) \((A_3, (A_1 + A_3)^r, R^{15})\),
(35) \((C_3, (2A_1)^r, R^{21})\),
(36) \((C_4, A_4, R^{27})\),
(37) \((B_3, A_2, R^{21})\).

**Proof:** Suppose \(\text{coh}(G, E^N) \leq 3\). Then \((G, \tilde{id}, R^N)\) is equivalent as real representation to one of (19)~(37) owing to Prop. 2.1, \(\text{coh}(A_n, (A_1 + A_2)^r, R^{2n}A_1) = k\), Prop. 2.5, \(\text{coh}(C_n, (2A_1)^r, R^{2n}A_1) = k\), \(\text{coh}(C_n, A_2, R^{2n-1}A_1 + 1) = k - 1\) (cf. \(O(C_n, A_2, R^{2n-1}A_1 + 1)\)) is equivalent to the linear isotropy representation of the symmetric pair \((SU(2k), Sp(k))\) of rank \(k - 1\), Prop. 2.8, \(\text{coh}(B_n, A_2, R^{2n}A_1) = k\), Prop. 2.11, \(\text{coh}(D_n, A_2, R^{2n}A_1) = k\), Prop. 2.15, \(\text{coh}(F_4, A_2, R^{2n}A_1) = 4\), \(\text{coh}(E_6, A_4, R^{27}) = 6\), \(\text{coh}(E_7, A_4, R^{24}) = 7\), \(\text{coh}(E_8, A_4, R^{24}) = 8\).

Conversely an o.t.g. induced from one of (19)~(24) is equivalent to \((SO(n), id, R^n)\) for some \(n \neq 4\), which is of \(\text{coh} 1\). An o.t.g. induced from (25), (26) or (27) is of \(\text{coh} 1\) (cf. Yokota [24, Theorems 5.27, 5.50, 5.3]). O.t.g.'s (28)~(33) are equivalent to the linear isotropy representation of the symmetric pairs \((SU(3) \times SU(3), SU(3)), (SU(3), SU(2)), (SU(6), Sp(3)), (Sp(2) \times Sp(2), Sp(2)), (G_2 \times G_2, G_2), (E_6, F_4)\) of rank 2 respectively (cf. Prop.'s 2.1, 2.5, 2.15). O.t.g.'s induced from (34)~(37) are equivalent to the linear isotropy representations of the symmetric pairs \((SU(4) \times SU(4), SU(4)), (Sp(3) \times Sp(3), Sp(3)), (SU(8), Sp(4)), (SO(7) \times SO(7), SO(7))\) of rank 3 respectively (cf. Prop.'s 2.1, 2.5, 2.8). They are also characterized by their degrees among 'real' complex irreducible representations. Q.E.D.

(VI) Let \((G, E^N)\) be a real irreducible o.t.g. of type VI.

**Proposition 4.6** \(\text{coh}(G, E^N) \leq 3\) if and only if \((\tilde{G}, \tilde{id}, R^N)\) is equivalent as real representation to one of the followings:

\(\text{coh} = 1:\) 
(38) \((R \times A_1, R \otimes A_1, C \otimes C^{k+1}); k \geq 1, t \in R^x\),
(39) \((R \times C_1, A_1, C \otimes C^{2k+1}); k \geq 2, t \in R^x\),
\(\text{coh} = 2:\) 
(40) \((R \times B_2, R \otimes A_1, C \otimes C^{2k+1}); k \geq 3, t \in R^x\),
(41) \((R \times D_2, R \otimes A_1, C \otimes C^{k+1}); k \geq 4, t \in R^x\),
(42) \((R \times D_2, R \otimes A_1, C \otimes C^{k+1}); i = 3, 4, t \in R^x\),
(43) \((R \times A_1, R \otimes A_1, C \otimes C^k); t \in R^x\),
(44) \((R \times A_3, R \otimes A_2, C \otimes C^k); t \in R^x\),
(45) \((R \times C_3, R \otimes A_2, C \otimes C^k); t \in R^x\).
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(46) \((R \times G_2, \mathbb{R}^5 \otimes A_2, C \otimes C^7); t \in R^*\),
(47) \((R \times B_3, \mathbb{R}^5 \otimes A_3, C \otimes C^9); t \in R^*\),
(48) \((R \times D_5, \mathbb{R}^5 \otimes A_5, C \otimes C^{16}); t \in R^*\),
(49) \((R \times D_6, \mathbb{R}^5 \otimes A_6, C \otimes C^{20}); t \in R^*\),
co = 3:
(50) \((R \times A_2, \mathbb{R}^5 \otimes 2A_1, C \otimes C^5); t \in R^*\),
(51) \((R \times A_5, \mathbb{R}^5 \otimes A_2, C \otimes C^{15}); t \in R^*\),
(52) \((R \times A_6, \mathbb{R}^5 \otimes A_2, C \otimes C^{21}); t \in R^*\),
(53) \((R \times B_4, \mathbb{R}^5 \otimes A_4, C \otimes C^{16}); t \in R^*\),
(54) \((R \times E_6, \mathbb{R}^5 \otimes A_1, C \otimes C^{27}); t \in R^*\).

**Proof:** Suppose coh \((G, \mathbb{E}^n) \leq 3\). Then \((G, id, R^k)\) is equivalent as real representation to one of (38)~(54) owing to Lemma 3.6(1)(F = C, i = 1, n = 1), Prop. 2.2, Remark 2.3, coh \((U(1) \times A_k, id \otimes A_2, C \otimes C^{(k+1)l}) = [(k+1)/2]\) (cf. \((U(1) \times A_k, id \otimes A_2, C \otimes C^{(k+1)l})\) is equivalent to the linear isotropy representation of the symmetric pair \((SO(2l+2), U(l+1))\) of rank \([(k+1)/2]\), \([(k+1)/2]\) \(\geq 4\) (cf. Prop. 2.6, Prop. 2.9, Prop. 2.12, Remark 2.13, coh \((U(1) \times D_6, id \otimes A_6, C \otimes C^{56}) \geq 4\) (cf. \((U(1) \times D_6, id \otimes A_6, C \otimes C^{56})\) is contained in the linear isotropy representation of the symmetric pair \((SU(4), Sp(1) \cdot Spin(12))\) of rank 4), Prop. 2.15, Prop. 2.16, coh \((U(1) \times F_4, id \otimes A_4, C \otimes C^{56}) \geq 7\) (cf. Each isotropy subgroup contains a group which is isomorphic to \(SU(3) \subset G_2 \subset Spin(7) \subset Spin(8) \subset F_4\) by Yokota[24, Prop.'s 5.45, 5.48, Thm's 5.33, 5.27, 5.2]), coh \((U(1) \times E_7, id \otimes A_6, C \otimes C^{56}) \geq 4\) (cf. \((U(1) \times E_7, id \otimes A_6, C \otimes C^{56})\) is contained in the linear isotropy representation of the symmetric pair \((E_8, Sp(1) \cdot E_2)\) of rank 4), doh \((U(1) \times G_2, id \otimes A_1, C \otimes C^{14}) = 13\), doh \((U(1) \times F_4, id \otimes A_1, C \otimes C^{20}) = 51\), doh \((U(1) \times E_6, id \otimes A_6, C \otimes C^{78}) = 77\), doh \((U(1) \times E_7, id \otimes A_1, C \otimes C^{133}) = 132\), doh \((U(1) \times E_8, id \otimes A_1, C \otimes C^{247}) = 247\).

Conversely coh \((38) = coh \((39) = 1\) since \(SU(k+1)\) and \(Sp(k)\) are transitive on hyperspheres in the representation spaces. (40)~(45) are equivalent to \((SO(2) \times SO(4), id \otimes id, R^2 \otimes R^4)\) for some \(n \neq 4\) of coh 2. The o.t.g. induced from (46) is equivalent to \(O(SO(2) \times G_2, id \otimes A_2^2, R^2 \otimes R^7)\) and the isotropy subgroup at \(\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\) in \(R(2, 7) \simeq R^2 \otimes R^7\) \((\alpha > \beta > 0)\) is isomorphic to \(SU(2)\) by Yokota [24, Example 5.1], so coh \((46) = 2\) (cf. Prop. 3.3(1)(4)). The o.t.g. induced from (48) is equivalent to the linear isotropy representation of the symmetric pair \((E_6, U(1) \cdot Spin(10))\) of rank 2 by Prop. 2.12 and Remark 2.13 since it is characterized by its degree up to equivalence. Since \([(k+1)/2] = 2\) for \(k = 4\), coh \((49) = 2\). The o.t.g. induced from (50) is equivalent to the linear isotropy representation of the symmetric pair \((Sp(3), U(3))\) of rank 3 by Prop. 2.2 and Remark 2.3. Since
\[(k+1)/2\] = 3 for \(k = 5\) or \(6\), \(\cohn{51} = \cohn{52} = 3\). The o.t.g. induced from (53) is equivalent to \(O(SO(2) \times Spin(9), id) \otimes A_1, R^2 \otimes R^{16}\). Any element of \(R(2, 16) = R^2 \otimes R^{16}\) to the form \(\begin{pmatrix} \alpha & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \beta & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}\). and the isotropy subgroup is isomorphic to \(SU(3)\) if \(\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \epsilon^2\) owing to the use of the mapping \(f\) in Lemma 3.2 and Yokota [24, Theorems 5.51, 5.27, 5.2]. So \(\cohn{53} = 3\). The o.t.g. induced from (54) is equivalent to the linear isotropy representation of the symmetric pair \((E_7, U(1) \cdot E_6)\) of rank 3 by Prop. 2.15 and Remark 2.16. So \(\cohn{54} = 3\). Q.E.D.

(VII) Let \((G, E^N)\) be a real irreducible o.t.g. of type VII.

**PROPOSITION 4.7** \(\cohn{G, E^N} \leq 3\) if and only if \((\tilde{G}, \tilde{id}, R^N)\) is equivalent as real representation to one of the followings:

- \(\cohn{1} = (55) (A\sb{k}, A\sb{1}, C\sb{k+1}); k \geq 1, \)
- \((56) (C\sb{k}, A\sb{1}, C\sb{2k}); k \geq 2, \)
- \(\cohn{2} = (57) (D\sb{5}, A\sb{5}, C\sb{10}), \)
- \((58) (A\sb{4}, A\sb{2}, C\sb{10}), \)
- \(\cohn{3} = (59) (A\sb{6}, A\sb{2}, C\sb{21}). \)

**PROOF:** Suppose \(\cohn{G, E^N} \leq 3\). Then \((\tilde{G}, \tilde{id}, R^N)\) is equivalent as real representation to (55)\(\sim\)(58) or (59) by Prop. 4.6. In fact, \((B\sb{k}, A\sb{1}, C\sb{2k+1}), (D\sb{k}, A\sb{1}, C\sb{2k}), (A\sb{1}, 2A\sb{1}, C\sb{9}), (A\sb{3}, A\sb{2}, C\sb{6}), (C\sb{2}, A\sb{2}, C\sb{9}), (G\sb{2}, A\sb{2}, C\sb{7}), (B\sb{3}, A\sb{3}, C\sb{6}), (B\sb{4}, A\sb{4}, C\sb{16})\) are 'real' and not real irreducible, so they are not of type VII, and \(\cohn{A\sb{2}, 2A\sb{1}, C\sb{9}} = \cohn{A\sb{5}, A\sb{2}, C\sb{15}} = \cohn{E\sb{6}, A\sb{1}, C\sb{27}} = 4\) since the restricted root systems of \((Sp(3), U(3)), (SO(12), U(6)), (E\sb{7}, U(1), U(1) \cdot E\sb{6})\) are of type BC (cf. [7], [22]).

Conversely \(\cohn{55} = \cohn{56} = 1\) is evident. O.t.g.'s induced from (57), (58) are of coh 2 since the restricted root systems of \((E\sb{6}, U(1) \cdot Spin(10))\) and \((SO(10), U(5))\) are of type BC. The o.t.g. induced from (59) is of coh 3 since the restricted root system of \((SO(14), U(7))\) is of type BC (cf. [7] and [22]). Q.E.D.

Now we have the following result.

**THEOREM 4.8** Let \((G, E^N)\) be an o.t.g. such that the identity representation \(id: G \rightarrow SO(N)\) is real irreducible. Then \(\cohn{G, E^N} \leq 3\) if and only if \((\tilde{G}, \tilde{id}, R^N)\) is equivalent as real representation to one of the followings:

- \(\cohn{1} = (IX), (VIII), (8), (9), (19), (20), (21), (22), (23), (24), (25), (26), (27), (38), (39), (55), (56). \)
- \(\cohn{2} = (10), (11), (14), (17), (28), (29), (30), (31), (32), (33), (40), (41), (42), (43), (44), (45), (46), (47), (48), (49), (57), (58). \)
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\[ \text{coh} = 3: (3, 7), (1, 2), (3, 4), (5, 6), (7, 12), (13), (15, 16), (18), (34), (35), (36), (37), (50), (51), (52), (53), (54), (59). \]

**PROOF:** Unifying (3.7) of Theorem 3.5, Propositions 4.1~4.7 and type VIII, IX in Section 3, we have the result. Q.E.D.

**REMARK 4.9** O.t.g.'s induced from (25), (26), (27), (39), (55), (56), (17), (46), (47), (57), (58), (6), (18), or (59) are not maximal. O.t.g.'s induced from (13), (16), or (53) are not obtained from the linear isotropy representations of any Riemannian symmetric pairs. Others are equivalent to the linear isotropy representations of some Riemannian symmetric pairs of rank at most 3 if they are maximal. (26) is obtained from the linear isotropy representation of \( (F_4, \text{Spin}(9)) \). The o.t.g. induced from (24) (resp. (42), (7)) is equivalent to one from (23)(resp. (41), (5)) of \( k = 4 \).

**REMARK 4.10** O.t.g.'s induced from (13) or (16) are missed in the Theorem 7 of Hsiang-Lawson [11] if \( k \) and 3 are relatively prime and \( k \geq 4 \), since the dimension of the representation spaces of (13) or (16) is \( 8k \) and the others of cohomogeneity 3 are of dimension \( 3m \) for some integer \( m \) except (53) of dimension 16.

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