COVERINGS OF GENERALIZED CHEVALLEY GROUPS
ASSOCIATED WITH AFFINE LIE ALGEBRAS

By

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R. Steinberg [21] has given a presentation of a simply connected Chevalley group (=the group of \( k \)-rational points of a split, semisimple, simply connected algebraic group defined over a field \( k \)) and has constructed the (homological) universal covering of the group. In this note, we will consider an analogy for a certain family of groups associated with affine Lie algebras.

1. Chevalley groups, Steinberg groups and the functor \( K_2(\Phi, \cdot) \).

Let \( \Phi \) be a reduced irreducible root system in a Euclidean space \( \mathbb{R}^n \) with an inner product \( (\cdot, \cdot) \) (cf. [4], [6]). We denote by \( \Phi^+ \) (resp. \( \Phi^- \)) the positive (resp. negative) root system of \( \Phi \) with respect to a fixed simple root system \( H = \{ \alpha_i, \ldots, \alpha_n \} \). We suppose that \( \alpha_i \) is a long root (for convenience' sake). Let \( \alpha_{n+1} \) be the negative highest root of \( \Phi \). Set \( \alpha_{ij} = 2(\alpha_i, \alpha_j) / (\alpha_j, \alpha_j) \) for each \( i, j = 1, 2, \ldots, n+1 \).

The matrices \( A = (a_{ij})_{i,j=1}^n \) and \( \tilde{A} = (a_{ij})_{i,j=n+1}^{n+n+1} \) are called a Cartan matrix of \( \Phi \) and the affine Cartan matrix associated with \( A \) respectively (cf. [4], [5], [6]).

Let \( G(\Phi, \cdot) \) be a Chevalley-Demazure group scheme of type \( \Phi \) (cf. [1], [20]). For a commutative ring \( R \), with 1, we call \( G(\Phi, R) \) a Chevalley group over \( R \). For each \( \alpha \in \Phi \), there is a group isomorphism—"exponential map"—of the additive group of \( R \) into \( G(\Phi, R) \). The elementary subgroup \( E(\Phi, R) \) of \( G(\Phi, R) \) is defined to be the subgroup generated by \( x_\alpha(t) \) for all \( \alpha \in \Phi \) and \( t \in R \). We use the notation \( G_i(\Phi, \cdot) \) and \( E_i(\Phi, \cdot) \) (resp. \( G_\alpha(\Phi, \cdot) \) and \( E_\alpha(\Phi, \cdot) \)) if \( G(\Phi, \cdot) \) is simply connected (resp. of adjoint type). It is well-known that \( G_i(\Phi, R) = E_i(\Phi, R) \) if \( R \) is a Euclidean domain (cf. [22, Theorem 18(Corollary 3)]).

Let \( St(\Phi, R) \) be the group generated by the symbols \( \check{x}_\alpha(t) \) for all \( \alpha \in \Phi \) and \( t \in R \) with the defining relations

\[
\begin{align*}
(A) \quad & \check{x}_\alpha(s) \check{x}_\alpha(t) = \check{x}_\alpha(s+t), \\
(B) \quad & [\check{x}_\alpha(s), \check{x}_\beta(t)] = \prod \check{x}_{\alpha+i}^j(\mathbb{N}_{\alpha, \beta, i, j, s^j} t^j), \\
(B') \quad & \check{\omega}_\alpha(u) \check{x}_\alpha(t) \check{\omega}_\alpha(-u) = \check{x}_{-\alpha}(-u^{-1} t)
\end{align*}
\]

for all \( \alpha, \beta \in \Phi(\alpha + \beta \neq 0) \), \( s, t \in R \) and \( u \in R^* \), the units of \( R \), where \( \check{\omega}_\alpha(u) = \check{x}_\alpha(u) \check{x}_{-\alpha}(-u^{-1} t) \).

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We call $St(\Phi, R)$ a Steinberg group over $R$.

Since the relations corresponding to $(A)$, $(B)$, $(B)'$ hold in $E_i(\Phi, R)$, there is a homomorphism $\theta$ of $St(\Phi, R)$ onto $E_i(\Phi, R)$ such that $\theta(\bar{x}_a(t)) = x_a(t)$ for all $a \in \Phi$ and $t \in R$. Put $K_2(\Phi, \cdot) = \ker [St(\Phi, \cdot) \longrightarrow E_i(\Phi, \cdot)]$, i.e., $1 \longrightarrow K_2(\Phi, \cdot) \longrightarrow St(\Phi, \cdot) \longrightarrow E_i(\Phi, \cdot) \longrightarrow 1$ is exact. For each $a \in \Phi$ and $u, v \in R^*$, we set $[u, v]_a = h_a(uv)h_a(u)^{-1}h_a(v)^{-1}$, called a Steinberg symbol, where $h_a(u) = \omega_a(u)\omega_a((-1)).$ Let $\tilde{K} = \langle [u, v]_a | a \in \Phi, u, v \in R^* \rangle$. Then $\tilde{K} \subseteq K_2(\Phi, R) \cap \text{Cent}(St(\Phi, R))$.

**Definition.** $R$ is called universal for $\Phi$ if $K_2(\Phi, R) = \tilde{K}$.

Let $E_u(\Phi, R) = St(\Phi, R)/\tilde{K}$. Then the homomorphism $\theta$ induces a homomorphism $\tilde{\theta}$ of $E_u(\Phi, R)$ onto $E_i(\Phi, R)$. We see:

- "$R$ is universal for $\Phi$"
- $\Leftrightarrow$ "$\tilde{\theta}$ is an isomorphism"
- $\Rightarrow$ "$\theta$ is a central extension."

**Example 1** (cf. [20], [21], [22]). Let $k$ be a field.

1. $St(\Phi, k)$ is connected if $(\Phi, |k|) = (A_1, 2), (B_2, 2), (G_2, 2)$ and $(A_1, 3)$.
2. $k$ is universal for each $\Phi$.
3. $St(\Phi, k)$ is a universal covering of $E_i(\Phi, k)$ with a few exceptions.

2. The case of Laurent polynomial rings.

Let $k[T]$ be the ring of polynomials in $T$ with coefficients in a field $k$, and $\mathfrak{m}$ the maximal ideal of $k[T]$ generated by $T$. Let $k[T, T^{-1}]$ be the ring of Laurent polynomials in $T$ and $T^{-1}$ with coefficients in $k$. We identify $k[T]$ with a subring of $k[T, T^{-1}]$ naturally. Set

$$
U = \langle x_a(f), x_b(g) | a \in \Phi^+, f, g \in k[T], g \not\in \mathfrak{m} \rangle,
$$

$$
N = \langle w_a(T^m) | a \in \Phi, t \in k^*; m \in \mathbb{Z} \rangle,
$$

$$
H = \langle h_a(t) | a \in \Phi, t \in k^* \rangle,
$$

$$
B = \langle U, H \rangle
$$

as subgroups of $E(\Phi, k[T, T^{-1}])$, where $w_a(u) = x_a(u)\omega_a(-u^{-1})x_a(u)$ and $h_a(u) = w_a(u)$ $w_a(-1)$.

**Theorem 2** ([17]).

1. $B \cap N = H$.
2. $(E(\Phi, k[T, T^{-1}]), B, N)$ is a Tits system.
Coverings of generalized Chevalley groups

**Corollary 3.**

(1) The canonical homomorphism \( \phi: E_0(\Phi, k[T, T^{-1}]) \rightarrow E_0(\Phi, k[T, T^{-1}]) \) is a central extension.

(2) \( \ker \phi = \{ \prod h_n(t_i) \prod t_i^\beta a_i = 1 \text{ for all } \beta \in \Phi \} \), where \( \langle \beta, \alpha \rangle = 2(\beta, \alpha)/(\alpha, \alpha) \), and \( t_i \in k^* \).

We define the subgroups \( \tilde{U}, \tilde{N}, \tilde{H}, \tilde{B} \) of \( St(\Phi, k[T, T^{-1}]) \):

\[
\tilde{U} = \langle \tilde{x}_\alpha(f), \tilde{x}_\beta(g) | \alpha \in \Phi^+, \beta \in \Phi^- \rangle, \\
\tilde{N} = \langle \tilde{w}_n(t^m) | \alpha \in \Phi, t \in k^*, m \in \mathbb{Z} \rangle, \\
\tilde{H} = \langle \tilde{h}_n(t) | \alpha \in \Phi, t \in k^* \rangle \tilde{K}, \\
\tilde{B} = \langle \tilde{U}, \tilde{H} \rangle.
\]

We denote by \( U_u, N_u, H_u \) and \( B_u \) the canonical images of \( \tilde{U}, \tilde{N}, \tilde{H} \) and \( \tilde{B} \) in \( E_u(\Phi, k[T, T^{-1}]) \) respectively. Then \( (E_u(\Phi, k[T, T^{-1}]), B_u, N_u) \) and \( (St(\Phi, k[T, T^{-1}]), B, N) \) are Tits systems, which is established by using the same technique as in [17].

**Theorem 4.**

(1) \( G_1(\Phi, k[T]) \) is presented by the generators \( \tilde{x}_\alpha(f) \) and \( \tilde{w}_n(t) \) for all \( \alpha \in \Pi, \gamma \in \Phi^+, f \in k[T] \) and \( t \in k^* \), and the defining relations (R1)—(R9):

\[
\text{(R1)} \quad \tilde{x}_\alpha(f) \tilde{x}_\gamma(g) = \tilde{x}_\gamma(f + g), \\
\text{(R2)} \quad \tilde{w}_n(t)^{-1} = \tilde{w}_n(-t), \\
\text{(R3)} \quad \tilde{w}_n(t) \tilde{x}_{\alpha}(u) \tilde{w}_n(-t) = \tilde{x}_{\alpha}(-t^2 u^{-1}) \tilde{w}_n(t^2 u^{-1}) \tilde{w}_n(-t^2 u^{-1}), \\
\text{(R4)} \quad [\tilde{x}_\alpha(f), \tilde{x}_\beta(g)] = \prod \tilde{x}_{\gamma_{i+1}}(N_{\gamma_i, \gamma_{i+1}} f \gamma_{i+1}) \gamma_i, \\
\text{(R5)} \quad \tilde{h}_n(t) \tilde{h}_n(u) = \tilde{h}_n(tu), \\
\text{(R6)} \quad \tilde{w}_n(t)^{ \frac{q}{q} } \tilde{w}_n(u)^{ \frac{q}{q} } = \tilde{w}_n(u)^{ \frac{q}{q} } \tilde{w}_n(t)^{ \frac{q}{q} }, \\
\text{(R7)} \quad \tilde{w}_n(t) \tilde{w}_n(u)^{ \frac{q}{q} } = \tilde{x}_{\rho}(ct^{\frac{q}{q}} f), \\
\text{(R8)} \quad \tilde{h}_n(t) \tilde{h}_n(u)^{-1} = \tilde{x}_\alpha(t^2 f), \\
\text{(R9)} \quad \tilde{w}_n(t) \tilde{h}_n(u) \tilde{w}_n(-t) = \tilde{h}_n(u) \tilde{h}_n(u^{-c}, \rho)
\]

for all \( \alpha, \beta \in \Pi(\alpha \neq \beta), \gamma, \delta \in \Phi^+, \rho \in \Phi^+ - \{\alpha\}, f, g \in k[T] \) and \( t, u \in k^* \), where \( \tilde{h}_u(t) = \tilde{w}_u(t) \tilde{w}_u(-1) \), and \( N_{\gamma, \gamma_i, \gamma_j} \) and \( c \) are as in [20] or [22], and each side of the equation in (R6) is the product of \( q \) symbols, and \( q = 2, 3, 4 \) or 6 if \( (R\alpha + R\beta) \cap \Phi \) is of type \( A_1 \times A_1, A_2, B_3 \) or \( G_2 \) respectively, and \( \langle \gamma, \alpha \rangle = 2(\gamma, \alpha)/(\alpha, \alpha) \) and \( \rho' = \rho - \langle \rho, \alpha \rangle \alpha \).

(2) \( k[T] \) is universal for each root system \( \Phi \).

**Proof.** (1) One can get this presentation of \( G_1(\Phi, k[T]) \) by using the same argument as in [23], [24] and [25]. (2) It follows from (1) that \( k[T] \) is universal. (By using an amalgamated free product decomposition of \( G_1(\Phi, k[T]) \) which is
described in [26], Rehmann [19] has given a different proof of the statement (2) from ours.) q.e.d.

**Theorem 5.** $k[T, T^{-1}]$ is universal for each root system $\Phi$.

**Proof.** In the following commutative diagram:

\[
\begin{array}{c}
B_uN_uB_u = E_u(\Phi, k[T, T^{-1}]) \\
\xrightarrow{\tilde{\theta}} E_1(\Phi, k[T, T^{-1}]) = B_1N_1B_1 \\
E_u(\Phi, k[T]) \xrightarrow{\sim} E_1(\Phi, k[T]) \cong B_1
\end{array}
\]

we have $\text{Ker} \tilde{\theta} \subseteq B_u$. On the other hand, $B_u \simeq B_1$ by the universality of $k[T]$. Therefore $\tilde{\theta}$ is an isomorphism. q.e.d.

By taking $T = 1$, the sequence $0 \to k \to k[T, T^{-1}]$ splits, so $K_2(\Phi, k)$ is a direct summand of $K_2(\Phi, k[T, T^{-1}])$. Then:

**Theorem 6 ([2]).**

1. $K_2(A_1, k[T, T^{-1}]) = K_2(A_1, k) \oplus S$, where $S = \langle (T, t) \mid t \in k^* \rangle$ and $\alpha$ is a fixed root.
2. $S \simeq k^*$ if $k^2 = k$ (i.e. $k$ is a square root closed field).

**Corollary 7** (cf. [2], [12], [13]).

1. $K_2(\Phi, k[T, T^{-1}]) = K_2(\Phi, k) \oplus S$, where $S = \langle (T, t) \mid t \in k^* \rangle$ and $\alpha$ is a fixed long root.
2. $S$ is isomorphic to a factor group of $k^*$ if $\Phi \neq C_n$ ($n \geq 1$).
3. $S$ is isomorphic to a factor group of $k^*$ if $k^2 = k$.

**Proof.** (1) and (3) follow from Theorem 6. If $\Phi \neq C_n$ ($n \geq 1$), then $A_2$ can be embedded in the long roots of $\Phi$. By Matsumoto's theorem, one sees (2). q.e.d.

**Remark 8.** The statements of Theorem 5, Theorem 6 and Corollary 7 have been confirmed by Hurrelbrink [7] in the case when $\Phi \neq G_2$. He has directly calculated the relations of $G_2(\Phi, k[T, T^{-1}])$ of type $\Phi = A_1, A_3$, and $B_2$, and by using this has proved Theorem 5 for $\Phi \neq G_2$. Our proof of Theorem 5 is different from his, and contains the case of type $G_2$.

As an application of [20, (5.3) Theorem/Remarks] and Theorem 5, we can establish the following theorem.

**Theorem 9.** If $\text{char} k = 0$, then $\text{St}(\Phi, k[T, T^{-1}])$ is a universal covering of $E_0(\Phi, k[T, T^{-1}])$. 

An \( l \times l \) integral matrix \( C = (c_{ij}) \) is called a generalized Cartan matrix if (i) \( c_{ii} = 2 \), (ii) \( i \neq j \Rightarrow c_{ij} \leq 0 \), and (iii) \( c_{ij} = 0 \Leftrightarrow c_{ji} = 0 \). From now on, we suppose \( \text{char } k = 0 \). We denote by \( L_l = L_l(C) \) the Lie algebra over \( k \) generated by the \( 3l \) generators \( e_1, \ldots, e_l, h_1, \ldots, h_l, f_1, \ldots, f_l \) with the defining relations \([h_i, h_j] = 0, [e_i, f_j] = \delta_{ij} h_i, [h_i, e_j] = c_{ij} e_j, [h_i, f_j] = -c_{ji} f_j \) for all \( 1 \leq i, j \leq l \), and \((\text{ad } e_i)^{\delta_{ij} + 1} e_j = 0, (\text{ad } f_i)^{\delta_{ij} + 1} f_j = 0 \) for all \( 1 \leq i \neq j \leq l \). Then the generators \( e_i, e_i, h_i, \ldots, h_i, f_i, \ldots, f_i \) are linearly independent in \( L_l \). We view \( L_l \) as a \( \mathbb{Z}^l \)-graded Lie algebra defined by \( \deg(e_i) = (0, \ldots, 0, 1, 0, \ldots, 0) \), \( \deg(h_i) = (0, \ldots, 0, -1, 0, \ldots, 0) \), and \( \deg(f_i) = (0, \ldots, 0, 1, 0, \ldots, 0) \), where \( \pm 1 \) are in the \( i \)-th position. Then there is the maximal homogeneous ideal \( R_l = R_l(C) \) of \( L_l \) such that \( R_l \cap (\Sigma_{j=1}^l k h_j + \cdots + k h_l) = 0 \). Set \( L = L(C) = L_l/R_l \), called the Kac-Moody Lie algebra over \( k \) associated with a generalized Cartan matrix \( C \) (cf. [3],[5],[8],[10],[14]). The algebra \( L \) is also \( \mathbb{Z}^l \)-graded. For each \( l \)-tuple \( (n_1, \ldots, n_l) \in \mathbb{Z}^l \), we let \( L(n_1, \ldots, n_l) \) denote the homogeneous subspace of degree \( (n_1, \ldots, n_l) \) in \( L \). We identify \( e_i, h_i, f_i \) with their images in \( L \). Then:

**Proposition 10.**

1. \( L(n_1, \ldots, n_l) \) is the subspace of \( L \) spanned by the elements \([e_i, [e_i, \ldots, [e_i, \ldots \ldots, e_i] \ldots]] \) (resp. \([f_i, [f_i, \ldots, [f_i, \ldots \ldots, f_i] \ldots]] \)), where \( e_j \) (resp. \( f_j \)) occurs \( |n_j| \) times, if \( (n_1, \ldots, n_l) \) belongs to \( (\mathbb{Z})^l - \{ 0 \} \) (resp. \( (\mathbb{Z})^l - \{ 0 \} \)).
2. \( L(0, \ldots, 0) = kh_1 + \cdots + kh_l \).
3. \( L(n_1, \ldots, n_l) = 0 \) otherwise.

Put \( L_0 = L_0(C) = kh_1 + \cdots + kh_l \). For each \( i = 1, \ldots, l \), we define a degree derivation \( D_i \) on \( L \) such that \( D_i(x) = n_i x \) for all \( x \in L(n_1, \ldots, n_l) \). Set \( D_0 = k D_1 + \cdots + k D_l \), viewed as an abelian Lie algebra of dimension \( l \). For a subspace \( D \subseteq D_0 \), let \( L^* = L(C)^* = D \times L \) (semidirect product) and \((L_0)^* = D \times L_0 \) (direct product). For each \( j = 1, \ldots, l \), let \( \gamma_j \) be an element of \((L_0)^*)^* \), the dual of \((L_0)^*)^* \), such that \([h, e_j] = \gamma_j(h) e_j \) for all \( h \in (L_0)^* \). We note that \( \gamma_j(h_i) = c_{ij} \) for all \( i, j = 1, \ldots, l \). We will choose and fix a subspace \( D \) of \( D_0 \) such that \( \gamma_1, \ldots, \gamma_l \) are linearly independent in \((L_0)^*)^* \). This is possible, since \( \gamma_i(D_j) = \delta_{ij} \). Set \( L' = \{ x \in L | [h, x] = \gamma(h) x \} \) for all \( h \in (L_0)^* \) for each \( \gamma \in (L_0)^*)^* \). It is easily seen that \( L'^{\gamma_1 + \cdots + \gamma_l} = L(n_1, \ldots, n_l) \) for all \( (n_1, \ldots, n_l) \in \mathbb{Z}^l \). In particular, \( L' = k L_0, L' = k L_0 \) and \( L'^{\gamma_i} = k h_i \).

Let \( \Delta = \Delta(C) = \{ \gamma \in (L_0)^* | L' \neq 0 \} \), called the root system of \( L \). Set \( \Gamma = \sum_{i=1}^l \mathbb{Z}_{\gamma_i} \), a free \( \mathbb{Z} \)-submodule of \((L_0)^*)^* \). The Weyl group \( W = W(C) \) is defined to be the subgroup of \( GL((L_0)^*)^* \) generated by \( \omega_i \) for all \( i = 1, \ldots, l \), where \( \omega_i \) is an endomorphism of \((L_0)^*)^* \) such that \( \omega_i(\gamma) = -\gamma(h_i) \). Then \( \Delta \) and \( \Gamma \) are \( W \)-stable. Also \( W \) acts on \( L_0 \) naturally: \( \omega_i(h_j) = h_j - c_{ij} h_i \). Hence we see \( (\omega_i(\gamma) \omega_i \gamma) = \gamma(h) \) for
Let $F_0(C, k)$ be the subgroup of $\text{Aut}(L)$ generated by $\exp \text{ad} \, t e_i$ and $\exp \text{ad} \, t f_i$ for all $i \in k$ and $i = 1, \ldots, l$. Let $V$ be a standard $L^*$-module with a highest weight $\lambda_i \neq 0$ (cf. [5], [10]). We let $F_\nu(C, k)$ denote the subgroup of $GL(V)$ generated by $\exp \text{ad} \, t e_i$ and $\exp \text{ad} \, t f_i$ for all $i \in k$ and $i = 1, \ldots, l$. These groups $F_0(C, k)$ and $F_\nu(C, k)$ have Tits systems respectively (cf. [11], [16]). Then there is a homomorphism $\nu$ of $F_\nu(C, k)$ onto $F_0(C, k)$ such that $\nu(\exp \text{ad} \, t e_i) = \exp \text{ad} \, t e_i$ and $\nu(\exp \text{ad} \, t f_i) = \exp \text{ad} \, t f_i$ for all $i \in k$ and $i = 1, \ldots, l$ (cf. [11]), and $\nu$ is central (cf. [18]).

4. The affine case.

Let $\Phi$, $A$ and $\tilde{A}$ be as in §1. Then we can regard $L(A)$ as a subalgebra of $L(\tilde{A})$ naturally. We note that $R_{i}(A) = R_{i}(\tilde{A}) = 0$, and that $\Delta(A) = \Phi \cup \{0\}$ and $\Delta(\tilde{A}) = \Delta(A) \times Z$ (cf. [5], [9], [15]). Also we identify $W(A)$ with a subgroup of $W(\tilde{A})$. Therefore we have the following commutative diagram.

\[
W(A) \times L_0(A) \longrightarrow L_0(A) \\
\downarrow \\
W(\tilde{A}) \times L_0(\tilde{A}) \longrightarrow L_0(\tilde{A})
\]

We take an element $\sigma$ of $W(A)$ such that $\sigma(\alpha_i) = \alpha_{n+i}$. Put $h_n = \sigma(h_1)$ and $h_i = h_{n+i} - h_n$. Then $\gamma_1(h_0) = \gamma_0(h_1) = \langle \sigma^{-1} \gamma_1 \rangle(h_1) = \langle \sigma^{-1} \alpha, \alpha \rangle = \langle \alpha, \alpha_{n+1} \rangle = d,_{n+1}$ and $\gamma_1(h_2) = 0$. Therefore $\mathcal{Z} = k h_i$ is the center of $L(\tilde{A})$, and we have an exact sequence of Lie algebras over $k$ (cf. [5], [8], [15]):

\[
0 \longrightarrow \mathcal{Z} \longrightarrow L(\tilde{A}) \stackrel{\pi}{\longrightarrow} k[T, T^{-1}] \otimes L(A) \longrightarrow 0.
\]

Hence the map $\pi$ induces an isomorphism $\tilde{\pi}$ of $F_0(\tilde{A}, k)$ onto $E_0(\Phi, k[T, T^{-1}])$ such that

\[
\tilde{\pi}(\exp \text{ad} \, t e_i) = x_{n+i}(t) \quad \text{for all } 1 \leq i \leq n,
\]
\[
\tilde{\pi}(\exp \text{ad} \, t e_{n+i}) = x_{n+i+1}(tT),
\]
\[
\tilde{\pi}(\exp \text{ad} \, t f_i) = x_{-n+i}(t) \quad \text{for all } 1 \leq i \leq n,
\]
\[
\tilde{\pi}(\exp \text{ad} \, t f_{n+i}) = x_{-n+i+1}(tT^{-1}).
\]

Since $St(\Phi, k[T, T^{-1}])$ is a universal covering of $E_0(\Phi, k[T, T^{-1}])$ (cf. Theorem 9), there is a unique homomorphism, denoted by $\phi$, of $St(\Phi, k[T, T^{-1}])$ into $F_\nu(\tilde{A}, k)$ such that the following diagram is commutative.

\[
\begin{array}{ccc}
\phi \downarrow & & \downarrow \tilde{\pi} \\
F_{\nu}(\tilde{A}, k) & \longrightarrow & F_0(\tilde{A}, k) \\
\Phi \downarrow & & \downarrow \phi \\
St(\Phi, k[T, T^{-1}]) & \longrightarrow & E_0(\Phi, k[T, T^{-1}]) & \longrightarrow & E_0(\Phi, k[T, T^{-1}])
\end{array}
\]
Then, by the relation $h_n(t)x_n(a)h_n(t)^{-1}=x_n(t^a)$, we see
\[
\phi(x_n(a)) = \exp ae_i \quad \text{for all } 1 \leq i \leq n,
\]
\[
\phi(x_n(aT)) = \exp ae_{n+1},
\]
\[
\phi(x_{n+1}(a)) = \exp af_i \quad \text{for all } 1 \leq i \leq n,
\]
\[
\phi(x_{n+1}(aT)) = \exp af_{n+1},
\]
\[
\phi(x_n(aT^{-1})) = \exp af_{n+1},
\]
\[
\phi(x_{n+1}(aT^{-1})) = \exp af_{n+1},
\]
\[
\phi(x_{n+1}(aT)) = \exp af_{n+1},
\]
where $w_i(t) = (\exp t e_i)(\exp -t f_i)(\exp t e_i)$ and $h_i(t) = w_i(t)w_i(-1)$ for each $i = 1, 2, \ldots, n+1$, and $a \in k$ and $t \in k^*$. In particular, $\phi$ is an epimorphism. Thus:

**Theorem 11.** $St(\Phi, k[T, T^{-1}])$ is a universal covering of $F_V(\hat{A}, k)$.

Finally in this note, we will discuss the kernel of $\phi$. Since $Ker \phi \subseteq Ker (\theta \phi)$, an element $x$ of $Ker \phi$ can be written as $\prod_{i=1}^n h_{n+1}(t_i) \prod_{j=1}^{n+1} [T, c_j]_{m+1}^{x_{n+1}}$, where $t_i, a_{p, b, p, c_j} \in k^*$ and $p_j, s_j \in \mathbb{Z}_1$. Then $\phi(T, c_j)_{n+1} = h_{n+1}(c_j)h_1(c_j)^{-1}a^{-1}$. On each weight space $V_\mu$ of $V$ (cf. [5], [10]), $\phi(x) = \prod_{i=1}^n t_i^{(h_i)} \prod_{j=1}^{n+1} c_j^{(h_j)} = \prod_{i=1}^n t_i^{(h_i)} \prod_{j=1}^{n+1} c_j^{(h_j)} = \prod_{i=1}^n t_i^{(h_i)} \prod_{j=1}^{n+1} c_j^{(h_j)} = 1$ for all weight $\mu$. Therefore:

\[
\phi(x) = 1
\]
\[
\Leftrightarrow \prod_{i=1}^n t_i^{(h_i)} \prod_{j=1}^{n+1} c_j^{(h_j)} = 1 \quad \text{for all weight } \mu.
\]

Put $P = \langle \prod_{i=1}^n h_{n+1}(t_i) \prod_{j=1}^{n+1} [T, c_j]_{m+1}^{x_{n+1}} \prod_{i=1}^n t_i^{(h_i)} \prod_{j=1}^{n+1} c_j^{(h_j)} = 1 \quad \text{for all weight } \mu \rangle$ of $V$.

**Theorem 12.** $Ker \phi = K_2(\Phi, k) \oplus P$.

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**References**


Morita, J., Tits’ systems in Chevalley groups over Laurent polynomial rings, Tsukuba J. Math., (2) 3 (1979), 41-51.

———, Moody-Teo’s groups and Marcuson’s groups, preprint.


———, “Lectures on Chevalley groups,” Yale University Lecture notes, 1967/68.

Behr, H., Eine endliche Präsentation der symplektischen Gruppe \( Sp_4(Z) \), Math. Z., 141 (1975), 47-56.


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Note added in proof. Recently H. Garland [Publ. IHES 52 (1980), 181-312] has constructed a subgroup \( F_i \) of \( \text{Aut}(V) \) containing \( F_V(\hat{A}, k) \), and has shown that \( St(\Phi, k(T)) \) is a universal covering of \( F_i \), where \( k(T) \) is the \( T \)-adic completion of \( k[T, T^{-1}] \). Then the composite map \( St(\Phi, k(T, T^{-1})) \to St(\Phi, k(T)) \to F_i \) coincides with the covering map of \( F_V(\hat{A}, k) \)