A REMARK ON R. POL'S THEOREM CONCERNING A-WEAKLY INFINITE-DIMENSIONAL SPACES

By

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For notations and relevant definitions we refer to [1].

THEOREM (MA). There is no universal space in the class of all metrizable separable A-weakly infinite-dimensional spaces.

R. Pol proved this theorem in [1] under CH. The proof we shall give is similar with the one given in [1] but a little more direct.

Lemma 1. Let $S \subset \mathbb{I}^\omega$ be a countable union of zero-dimensional subsets. If $C \subset \mathbb{I}^\omega$ satisfies that for any open neighbourhood $U$ of $S \setminus C \cup U < \mathfrak{c}$, then $C \cup S$ is A-weakly infinite-dimensional.

The proof is parallel to the proof of Lemma 1 in [1], noting that in $\mathbb{I}^\omega$ every subset with cardinality less than $\mathfrak{c}$ is zero-dimensional.

Lemma 2 (MA). Let $\{G_\alpha : \alpha < \lambda\}$ be a family of open neighbourhoods of $\Sigma$ in $\mathbb{I}^\omega$ and $\lambda < \mathfrak{c}$, where $\Sigma = \{x \in \mathbb{I}^\omega : \text{all but finitely many coordinates of } x \text{ are equal to zero}\}$. Then there exist positive numbers $a_\alpha \in \mathbb{I}(i \in \omega)$ such that $\bigcup\{0, a_\alpha\} \subset \mathbb{I} \cap \{G_\alpha : \alpha < \lambda\}$. Therefore, if $E \subset \mathbb{I}^\omega$ can be embedded in an A-weakly infinite-dimensional space, then $\bigcap \{G_\alpha : \alpha < \lambda\} \setminus E \neq \emptyset$.

Proof. Let $\mathscr{B} = \{[0, 1/n] : n > 0\}$. We define $P = \{(a, b) : a \text{ is a finite sequence in } \mathscr{B} \text{ and } b \in [\lambda]^{<\omega}\}$ and for any $\langle a', b' \rangle$, $\langle a, b \rangle \in P$, where $a = \langle I_0, I_1, \ldots, I_n \rangle$ and $a' = \langle I_0', I_1', \ldots, I_n' \rangle$, $\langle a', b' \rangle \leq \langle a, b \rangle$ iff $b' \supseteq b$, $n \leq n'$, $I_i = I_i'$ for any $i \leq n$ and if $n < n'$, $\prod_{i \leq n'} \prod_{i > n'} I_i \subset \cap \{G_\alpha : \alpha \in b\}$. It is obvious that $\leq$ is a partial order on $P$. Since all of first components of elements of $P$ are countable, $P$ is $\text{ccc}$ (in fact $\sigma$-centred).

Let $D_\alpha = \{(a, b) \in P : \alpha \in b\}$ and $F_n = \{(a, b) : \text{the length of } a \text{ is larger than } n\}$. It is easily seen that $D_\alpha$ is dense in $P$ for any $\alpha < \lambda$. Now we want to

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show that $F_n$ is dense for any $n \in \omega$. Take any $(a, b) \in P$. If the length of $a$ is larger than $n$, then $(a, b) \in F_n$. So we suppose that $a = (I_0, I_1, \ldots, I_m)$, where $m < n$. Since $\bigcap \{G_\alpha : \alpha \in b\}$ is an open neighbourhood of $\Sigma$, we can find $a' = (I_0, \ldots, I_m, I_{m+1}, \ldots, I_n)$ such that $\prod_{i \in n} I_i \times \prod_{i > n} I_i \subset \bigcap \{G_\alpha : \alpha \in b\}$. Therefore, $(a', b) \subseteq (a, b)$ and $(a', b) \in F_n$.

By MA, we have a filter $G$ in $P$ such $G \cap D_\alpha \neq 0 \ G \cap F_n \neq 0$ for any $\alpha < \lambda$ and $n < \omega$. Let $\sigma \{ a : \text{there is a } (a, b) \in G \} = \{ I_n : n \in \omega \}$. Then $\prod_{n < \omega} I_n \subset \bigcap \{ G_\alpha : \alpha < \lambda \}$.

**Proof of Theorem.** Let $E \subset I^\omega$ be any $\mathcal{A}$-weakly infinite-dimensional space. Let $\{(H_\alpha, h_\alpha) : \alpha < \mathfrak{c}\}$ be the family of all pairs such that $H_\alpha$ is a $G_\delta$-set in $I^\mathfrak{c}$ containing $\Sigma$ and $h_\alpha : H_\alpha \to I^\omega$ is an embedding which maps $\Sigma$ onto a subset of $E$. Let $\{ G_\alpha : \alpha < \mathfrak{c} \}$ be all of the open sets which contain $\Sigma$. Take $x_\alpha \in \bigcap \{ G_\beta : \beta \leq \alpha \} \setminus h_\alpha^{-1}(E)$. Then by an argument paralleled to the one in the end of [1], we have $M = \Sigma \cup \{ x_\alpha : \alpha < \mathfrak{c} \}$ can not be embedded in $E$.

**Remark 3.** It is easily seen from the proof of Lemma 2 that the theorem is true under MA$_{\mathfrak{c}, \text{ centred}}$, i.e. $\mathfrak{p} = \mathfrak{c}$, which is strictly weaker than MA.

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**References**


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