COMPACTNESS CRITERIA FOR RIEMANNIAN MANIFOLDS
WITH COMPACT UNSTABLE MINIMAL HYPERSURFACES

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1. Introduction
In this paper, we shall prove the following Theorem.

THEOREM A. Let N be a complete Riemannian manifold with a compact embedded unstable minimal hypersurface M. Suppose that there exists a positive constant s_0 such that along each unit speed geodesic γ: [0, ∞) → N emanating from each point in the tubular neighborhood U_{s_0}(M) := \{q ∈ N; dist_N(q, M) < s_0\} the Ricci curvature satisfies

$$\liminf_{r→∞} \int_0^r Ric_N(dγ/dt, dγ/dt)dt ≥ 0.$$ 

Then N is compact.

The Myers' theorem [11] is one of the most well-known results relating the curvature and the topology of a complete Riemannian manifold N, which states that if the Ricci curvature has a positive lower bound then N is compact. In [1], Ambrose proved a generalization of Myers' theorem, that is, if there is a point q ∈ N such that along each unit speed geodesic γ: [0, ∞) → N emanating from q the Ricci curvature satisfies

$$\int_0^∞ Ric_N(dγ/dt, dγ/dt)dt = +∞$$

then N is compact. It should be pointed out that in this result the Ricci curvature is not required to be everywhere nonnegative. Further developments can be found in Galloway [9] and different sorts of extensions of Myers' theorem can be found in Avez [3], Calabi [5] and Shiohama [12].

Theorem A is an Ambrose-type theorem for Riemannian manifolds with compact embedded unstable hypersurfaces (see also Remark in section 3). It should be also pointed out that in Theorem A the existence of the global unit normal vector field on M is not required.

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§ 2. Definitions and formulas

Let $N=(N,g)$ be a complete Riemannian manifold of dimension $n \geq 2$ with a compact embedded hypersurface $M$. We choose a local orthonormal frame field $\{e_1, \ldots, e_n\}$ in $N$ such that, restricted to $M$, the vectors $\{e_1, \ldots, e_{n-1}\}$ are tangent to $M$. Let denote the Levi-Civita connection of $N$ by $\nabla$, the component normal to $M$ by $(\cdot)\nu$ and the restriction of $e_n$ to $M$ by $v$. The second fundamental form $A_M$ of $M$ is defined by

$$A_M(X, Y)\nu = (\nabla_X Y)^\perp,$$

where $X$ and $Y$ are local vector fields on $M$. $M$ is called minimal if $H_M = \text{Trace} A_M$ is identically zero.

We shall derive the equation $H_M = 0$ by another elegant way. For a smooth function $f \in C^\infty_0(\mathcal{D}(\nu))$ with compact support in $\mathcal{D}(\nu)$ and a small positive constant $\delta$, let $\{M(\varepsilon f ; \nu)\}_{\varepsilon \in (-\delta, \delta)}$ denote the one-parameter family of hypersurfaces $\{S(\varepsilon f ; \nu) \cup \{M - \mathcal{D}(\nu)\}\}_{\varepsilon \in (-\delta, \delta)}$, where $\mathcal{D}(\nu)$ is the domain of $\nu$ and $S(\varepsilon f ; \nu) = \{\exp_x \varepsilon f(x) \mu \in N ; x \in \mathcal{D}(\nu)\}$. We then get a local deformation $\{M(\varepsilon f ; \nu)\}_{\varepsilon \in (-\delta, \delta)}$ of $M$. Let $\mathcal{A}(\cdot)$ denote the $(n-1)$-dimensional area functional of hypersurfaces. Then $\mathcal{A}(M(\varepsilon f ; \nu))$ is class of $C^\infty$ with respect to $\varepsilon$ and

$$\frac{d}{d\varepsilon} \mathcal{A}(M(\varepsilon f ; \nu))|_{\varepsilon=0} = -\int_M f \cdot H_M dv_\varepsilon,$$

where $dv_\varepsilon$ is the induced volume element of $M$. If $M$ is a critical point of $\mathcal{A}$, then $H_M = 0$.

Suppose that $M$ is minimal. Then

1. $$\frac{d^2}{d\varepsilon^2} \mathcal{A}(M(\varepsilon f ; \nu))|_{\varepsilon=0} = \int_M \left[|\nabla^M f|^2 - (\text{Ric}(\nu, \nu) + |A_M|^2)f^2\right]dv_\varepsilon,$$

where $\nabla^M f = \sum_{i=1}^{n-1} e_i(f) \cdot e_i$ and $|A_M|^2 = \sum_{i=1}^{n-2} |[A_M(e_i, e_i)]|^2$. $M$ is called unstable if there exist a local unit normal vector field $\nu$ on $M$ and a smooth function $f \in C^\infty_0(\mathcal{D}(\nu))$ such that

$$\frac{d^2}{d\varepsilon^2} \mathcal{A}(M(\varepsilon f ; \nu))|_{\varepsilon=0} < 0.$$

For later references, we also give the second variational formula of arc length functional of rays with respect to special variations. Let $\gamma : [0, \infty) \to N$ be a ray satisfying $\gamma(0) \in M$ and $\text{dist}_N(M, \gamma(t)) = \text{dist}_N(\gamma(0), \gamma(t)) (\equiv t)$ for all $t \in [0, \infty)$. Let $\mathcal{L}(\cdot)$ denote the arc length functional. We note that for each $r > 0 \gamma|_{[s, r]}$
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is a critical point of $\mathcal{L}$. Choose a local orthonormal frame field $\{e_1, \ldots, e_n\}$ in $N$ around $\gamma(0)$ such that, restricted to $M$, the vectors $\{e_1, \ldots, e_{n-1}\}$ are tangent to $M$ and the vector $\nu = e_n|_M$ satisfies $\nu(\gamma(0)) = (d\gamma/dt)(0)$. Let $\gamma_{t, r} : [0, r] \times (-\delta, \delta) \to N$ be a variation of $\gamma_{[0, r]}$ satisfying $\gamma_{t, r}((0) \times (-\delta, \delta)) \subset M$, $\gamma_{t, r}((r) \times (-\delta, \delta)) = \gamma(r)$ and $\frac{\partial}{\partial t_{t, r}}(t, \varepsilon)|_{t=0} = \cos \frac{\pi t}{2r} \cdot e_i(t)$, where each $e_i(t)$ is the parallel translate vector of $e_i(\gamma(0))$ along $\gamma$. We then obtain (cf. [4, Chapter 11])

$$
\frac{d^2}{d\varepsilon^2} \sum_{i=1}^{n-1} \mathcal{L}(\gamma_{t, r}([0, r] \times \{\varepsilon\}))\bigg|_{\varepsilon=0} = (n-1)\pi^2/8r - \int_0^r \text{Ric}_N(d\gamma/dt, d\gamma/dt)(\cos \frac{\pi t}{2r})^2 dt - H_M(\gamma(0))
$$

where $H_M$ is the mean curvature of $M$ with respect to $\nu$.

§ 3. Proof of Theorem A

Theorem A is an immediate consequence of the following.

THEOREM B. Let $N = (N, g)$ be a complete Riemannian manifold with a compact embedded unstable minimal hypersurface $M$. Suppose that there exist positive constants $s_0$ and $\theta$ such that along each unit speed geodesic $\gamma : [0, \infty) \to N$ satisfying $\gamma(0) \in M$ and $|g((d\gamma/dt)(0), V)| \geq 1 - \theta$, the Ricci curvature satisfies

$$
\lim \inf_{r \to \infty} \int_0^r \text{Ric}_N(d\gamma/dt, d\gamma/dt) dt \geq 0
$$

for all $0 \leq s < s_0$, where $V$ is a unit vector normal to $M$ at $\gamma(0)$. Then $N$ is compact.

To prove Theorem B, we will suppose that $N$ is noncompact and, finally, lead a contradiction.

Since $N$ is noncompact, there exists a ray $\gamma : [0, \infty) \to N$ satisfying $\gamma(0) \in M$ and

$$
\text{dist}_N(M, \gamma(t)) = \text{dist}_N(\gamma(0), \gamma(t)) = t
$$

for all $t \geq 0$.

From the unstability of $M$, we will first construct $C^\alpha$-hypersurfaces $\{M(eu; \varpi)\}_{e \in (0, \sigma)}$ near $M$, which are smooth and have positive mean curvature around $\gamma \cap M(eu; \varpi)$.

**Lemma 1.** There exist a continuous nonnegative function $u \in C(M)$, a local unit normal vector field $\varpi$ on $M$ and a positive constant $\sigma$ such that
(i) \( \gamma(0) \in \partial(\bar{w}) = \{ x \in M; u(x) > 0 \} \),
(ii) \( u \) is smooth in \( \partial(\bar{w}) \),
(iii) \( M(\varepsilon u; \bar{w}) \subset U_{s_0}(M) \),
(iv) \( H_{M(\varepsilon u; \bar{w})} > 0 \) in \( \{ \exp_t \varepsilon N; x \in W, 0 \leq t < s_0 \} \)
for all \( 0 < \varepsilon < \sigma \), where \( W = \{ x \in M; u(x) > \frac{1}{2} u(\gamma(0)) \} \subset \partial(\bar{w}) \).

**PROOF.** From the unstability of \( M \), there exist a local unit normal vector field \( \bar{v} \) on \( M \) and a function \( f \in C^\infty(\partial(\bar{w})) \) such that

\[
\frac{d^2}{d\varepsilon^2} M(\varepsilon f; \bar{w}) \bigg|_{\varepsilon = 0} < 0.
\]

We may assume that the closure \( \overline{\partial(\bar{w})} \) is contained in a coordinate neighborhood of \( M \). Let \( \nu \) be a local unit normal vector field on \( M \) around \( \gamma(0) \) satisfying \( (\frac{d\gamma}{dt})(0) = \nu(\gamma(0)) \). Replacing \( \bar{v} \) by \( -\bar{v} \) if necessary, we can choose a local unit normal vector field \( \bar{v} \) on \( M \), which is an extension of \( \bar{v}, \nu \) and satisfies that \( \partial(\bar{w}) \) is connected with \( C^\infty \)-boundary \( \partial(\bar{w}) \).

Consider the functional

\[
I_\varepsilon(\phi) = \int_M \left[ |\nabla \phi|^2 - (\text{Ric}_N(\bar{w}, \bar{v}) + \lambda_M)^\phi \right] dv_x
\]
and define \( \lambda = \inf I_\varepsilon(\phi) \) for all \( \phi \in C^\infty(\partial(\bar{w})) \) satisfying \( \phi = 0 \) on \( M - \partial(\bar{w}) \) and \( \int_M \phi^2 dv_x = 1 \). From (1) and (5) we then obtain a continuous function \( u \in C(M) \) satisfying \( \lambda = I_\varepsilon(u) < 0 \), which \( u \) has the following properties (cf. [2], [7] and [8])

(6) \( u > 0 \) in \( \partial(\bar{w}) \) and \( u |_{\partial(\bar{w})} = 0 \),
(7) \( u \) is smooth in \( \partial(\bar{w}) \),
(8) \( Lu := -\Delta_M u - (\text{Ric}_N(\bar{w}, \bar{v}) + |A_M|\phi)u = \lambda u \) (\( < 0 \)) in \( \partial(\bar{w}) \),

where \( \Delta_M u = \sum_{i=1}^n g(e_i, \nabla e_i, \nabla u) \). In particular, the property (6) is an immediate consequence of Courant's nodal domain theorem for the linear elliptic operator of second order \( L \) (cf. [6, Chapter 1], [7, VI-§ 6]). From (6)-(8) and an easy calculation we obtain

\[
\frac{\partial}{\partial \varepsilon} H_{M(\varepsilon u; \bar{w})} \bigg|_{\varepsilon = 0} = \Delta_M u + (\text{Ric}_N(\bar{w}, \bar{v}) + |A_M|\phi)u = -\lambda u > 0 \quad \text{in} \quad \partial(\bar{w}).
\]

It follows from (6), (7) and (9) that there exists a positive constant \( \sigma \) such that for any \( 0 < \varepsilon < \sigma \) \( M(\varepsilon u; \bar{w}) \subset U_{s_0}(M) \) and \( H_{M(\varepsilon u; \bar{w})} = \frac{\partial}{\partial \varepsilon} H_{M(\varepsilon u; \bar{w})} \bigg|_{\varepsilon = 0} ds > 0 \) in \( \{ \exp_t \varepsilon N; x \in W, 0 \leq t < s_0 \} \). This completes the proof of Lemma 1.

**Lemma 2.** There exist positive constants \( \varepsilon_0(0 < \varepsilon < \sigma), t_0(0 < t_0 < s_0) \) and a unit
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Let \( f : [0, \infty) \to N \) such that

1. \( f(t) \in M(\varepsilon_0 u ; \bar{v}) \cap \{ \exp x = \exp t \bar{v} \in N ; x \in W, 0 \leq t < s_0 \} \),
2. \( f(0) \in W \subset \mathcal{D}(\bar{v}) \),
3. \( g((\text{d}f/\text{d}t)(0), \bar{v}(f(0))) \geq 1 - \theta \),
4. \( \text{dist}_N(M(\varepsilon_0 u ; \bar{v}), f(t)) = \text{dist}_N(f(t), f(t) = t - t_0 \text{ for all } t \geq t_0 \).

**Proof.** Take \( \varepsilon(0 < \varepsilon < \sigma) \) arbitrarily and fix it. For each \( i \in \mathbb{N} \), there exists a minimizing geodesic \( \gamma_{i, i} \), emanating from \( M(\varepsilon u ; \bar{v}) \), between \( M(\varepsilon u ; \bar{v}) \) and \( \gamma(i) \). Put \( \bar{W} = \{ x \in M ; u(x) \geq u(\gamma(0)) \} \subset W \subset \mathcal{D}(\bar{v}) \). Suppose that there exists \( j_i \in \mathbb{N} \) such that

\( \gamma_{i, j_i}(0) \notin M(\varepsilon u ; \bar{v}) \cap \{ \exp x = \exp t \bar{v} \in N ; x \in \bar{W}, 0 \leq t < s_0 \} \).

From (4), (10) and Lemma 1-(iii) we have

\[
\text{dist}_N(M, \gamma(j_i)) \leq \text{dist}_N(M, \gamma_{i, j_i}(0)) + \mathcal{L}(\gamma_{i, j_i})
\]

\[
< \mathcal{L}(\gamma_{i, j_i}) = \text{dist}_N(M, \gamma(j_i)).
\]

This is a contradiction. Then we obtain for all \( i \in \mathbb{N} \)

\( \gamma_{i, i}(0) \in M(\varepsilon u ; \bar{v}) \cap \{ \exp x = \exp t \bar{v} \in N ; x \in \bar{W}, 0 \leq t < s_0 \} \subset U_{\varepsilon_0}(M) \).

We also note that for each \( i \in \mathbb{N} \) the vector \( (d\gamma_{i, i})/dt(0) \) is perpendicular to \( TM(\varepsilon u ; \bar{v}) \) and

\( \gamma_{i, i} \cap M(\varepsilon u ; \bar{v}) = \{ \gamma_{i, i}(0) \} \).

Suppose that there exists \( j_i \in \mathbb{N} \) such that

\( g((d\gamma_{i, j_i})/dt(0), (d\exp t \bar{v})/dt(\gamma_{i, j_i}(0))) < 0 \).

From (11) and (12) it follows that \( c(0 < c < \mathcal{L}(\gamma_{i, j_i})) \) such that

\( \gamma_{i, j_i}(c) \in \bar{W} \cup \{ \exp x = \exp t \bar{v} \in N ; x \in \partial \bar{W}, 0 \leq t < \varepsilon u(\gamma(0)) \} \).

It then follows from (4), (11) and (13) that

\[
\text{dist}_N(M, \gamma(j_i)) \leq \text{dist}_N(M, \gamma_{i, j_i}(c)) + \mathcal{L}(\gamma_{i, j_i})\begin{pmatrix} 0 & \mathcal{L}(\gamma_{i, j_i}) \end{pmatrix}
\]

\[
< \text{dist}_N(M, \gamma_{i, j_i}(c)) + \mathcal{L}(\gamma_{i, j_i})
\]

\[
< \mathcal{L}(\gamma_{i, j_i}) = \text{dist}_N(M, \gamma(j_i)).
\]

This is a contradiction, too. Then we obtain for all \( i \in \mathbb{N} \)

\( g((d\gamma_{i, i})/dt(0), (d\exp t \bar{v})/dt(\gamma_{i, i}(0))) \geq 0 \).

Let \( v_i \in \{ v \in TM(\varepsilon u ; \bar{v}) \} \) be an accumulation point of the sequence
Let \( \gamma : [0, \infty) \to N \) be the geodesic such that \( \gamma(0) = \varphi(u) \) and \( (d\gamma/dt)(0) = u \), where \( \varphi : TN \to N \) is the bundle projection. Then \( \gamma \) is a ray satisfying
\[
(15) \quad \text{dist}_N(M(\varepsilon u; \mathcal{V}), \gamma(t)) = \text{dist}_N(\gamma(0), \gamma(t))
\]
for all \( t \geq 0 \). We say that \( \gamma \) is a limit ray of the sequence of minimizing geodesics \( \{\gamma_i(t)\}_{i \in \mathbb{N}} \). It then follows from (11) and (14) that
\[
(16) \quad \gamma_i(0) \in M(\varepsilon u; \mathcal{V}) \cap \{ \exp_\mathcal{V} t \in \mathcal{W} : 0 \leq t \leq s_i \},
\]
\[
(17) \quad g((d\gamma_i/dt)(0), (d(\exp_\mathcal{V})/dt)(\gamma_i(0))) \geq 0.
\]

Let \( \tilde{\gamma} \) be a limit ray of the sequence of rays \( \{\gamma_i(t)\}_{t \geq 0} \), where \( 1/i_0 < \sigma \). It then follows from (15)-(17) that
\[
(18) \quad \tilde{\gamma}(t) \subseteq \mathcal{W} \subseteq \mathcal{V}
\]
\[
(19) \quad g((d\tilde{\gamma}/dt)(0), \mathcal{V}(\tilde{\gamma}(0))) \geq 0,
\]
\[
(20) \quad \text{dist}_N(M, \tilde{\gamma}(t)) = \text{dist}_N(\tilde{\gamma}(0), \tilde{\gamma}(t))
\]
for all \( t \geq 0 \). Also from (19) and (20) \( (d\tilde{\gamma}/dt)(0) = \mathcal{V}(\tilde{\gamma}(0)) \) and then
\[
(21) \quad g((d\tilde{\gamma}/dt)(0), \mathcal{V}(\tilde{\gamma}(0))) = 1.
\]
By the construction of \( \tilde{\gamma} \), (18) and (21) there exists a positive constant \( \varepsilon_0(\varepsilon_0 = 1/i_0, i_0 \geq i_0) \) such that
\[
(22) \quad s_0 > t_0 := \inf\{ t > 0 ; \gamma_i^{-1}(t) \in \mathcal{W} \},
\]
\[
(23) \quad |g((d\gamma_i/dt)(t_0), \mathcal{V}(\gamma_i(t_0)))| \geq 1 - \theta,
\]
where \( \gamma_i^{-1}(t) = \exp_{\gamma_i(t_0)}(-t(\exp_{\gamma_i(t_0)}/(dt)(0))). \)

Let \( \tilde{\gamma} : [0, \infty) \to N \) be the geodesic such that
\[
\tilde{\gamma}(t) = \begin{cases} 
\gamma_i(t_0 - t) & \text{if } 0 \leq t \leq t_0 \\
\gamma_i(t_0) & \text{if } t \geq t_0.
\end{cases}
\]
It then follows from (15), (16), (22) and (23) that \( \tilde{\gamma} \) satisfies the properties (i)-(iv). This completes the proof of Lemma 2.

Let \( \{ \tilde{e}_1, \ldots, \tilde{e}_{n-1} \} \) be a local orthonormal frame field on \( M(\varepsilon_0 u; \mathcal{V}) \) around \( \tilde{\gamma}(t_0) \) and each \( \tilde{e}_i(t) \) be the parallel translate vector of \( \tilde{e}_i(\tilde{\gamma}(t_0)) \) along \( \tilde{\gamma} \) with the initial condition \( \tilde{e}_i(t_0) = \tilde{e}_i(\tilde{\gamma}(t_0)) \). Let \( \tilde{\gamma}_{i, r} : [0, r] \times (-\delta, \delta) \to N \) be a variation of \( \tilde{\gamma}_{t_0 + r} := \tilde{\gamma}(t_0 + r) \) satisfying \( \tilde{\gamma}_{i, r}(i, 0) \times (-\delta, \delta)) \subseteq M(\varepsilon_0 u; \mathcal{V}), \tilde{\gamma}_{i, r}(r) \times (-\delta, \delta) = \tilde{\gamma}(t_0 + r) \) and \( (d\tilde{\gamma}_{i, r}/dr)(t, \varepsilon) \big|_{\varepsilon = 0} = \cos \frac{\pi t}{2r} \cdot \tilde{e}_i(t_0 + t) \). From (2) we then obtain
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\begin{equation}
\left. \frac{d^2}{dt^2} \sum_{i=1}^{n-1} \mathcal{L}(\mathcal{F}(r, r([0, r] \times \{t\})) \right|_{t=0} = (n-1)\pi^2/(8r^2) \int_{t_0}^{t_0 + r_0} \text{Ric}_N(d\mathcal{F}/dt, d\mathcal{F}/dt)\left(\cos \frac{\pi(t-t_0)}{2r_0}\right)^2 dt - H_{M(t_0,r)}(\mathcal{F}(t_0)).
\end{equation}

It follows from (3), (24), Lemma 1, Lemma 2 and Lemma 3 below that there exists a large constant \( r_0 \) such that

\[(n-1)\pi^2/(8r_0) - \int_{t_0}^{t_0 + r_0} \text{Ric}_N(d\mathcal{F}/dt, d\mathcal{F}/dt)\left(\cos \frac{\pi(t-t_0)}{2r_0}\right)^2 dt - H_{M(t_0,r)}(\mathcal{F}(t_0)) < 0.\]

This contradicts that \( \mathcal{F}|_{[t_0, \infty)} \) is a ray. This completes the proof of Theorem B.

**Lemma 3.** For each constant \( K \)

\[
\liminf_{r \to \infty} \int_0^r \text{Ric}_N(d\mathcal{F}/dt, d\mathcal{F}/dt) dt \geq K
\]

implies

\[
\liminf_{r \to \infty} \int_0^r \text{Ric}_N(d\mathcal{F}/dt, d\mathcal{F}/dt)\left(\cos \frac{\pi t}{2r}\right)^2 dt \geq K.
\]

**Corollary.** Let \( N \) be a complete Riemannian manifold of nonnegative Ricci curvature with a compact embedded minimal hypersurface \( M \). Suppose that either

(i) \( M \) is unstable in \( N \) or
(ii) \( (N-M) \) is connected.

Then \( N \) is compact. In the case (ii) it is also established that \( (N-M) \) is isometric to a product Riemannian manifold \( M \times (0, l) \), where \( l \) is a suitable positive constant.

**Proof.** In the case (ii), Corollary was proved by Ichida [10].

**Remark.** Without the instability of \( M \) it follows immediately from (2) and Lemma 3 that

"Let \( N \) be a complete Riemannian manifold with a compact embedded minimal hypersurface \( M \). Suppose that along each unit speed geodesic \( \gamma : [0, \infty) \to N \) emanating perpendicularly from each point in \( M \) the Ricci curvature satisfies

\[
\liminf_{r \to \infty} \int_0^r \text{Ric}_N(d\gamma/dt, d\gamma/dt) dt > 0.
\]

Then \( N \) is compact."
References


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