NON-STANDARD REAL NUMBER SYSTEMS WITH REGULAR GAPS

By

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The purpose of this paper is to show that if an enlargement \(*M\) of the universe \(M\) is saturated, then the non-standard real number system \(*R\) has a regular gap and the uniform space \((*R, E[L(1)])\) is not complete.

Our notions and terminologies follow the usual use in the model theory. Let \(G=\langle G, +, \rangle\) be a first order structure which satisfies

(a) the axioms of ordered abelian groups,

(b) the axioms of dense linear order.

(i.e. \(\langle G, +, \rangle\) is an ordered abelian group and \(\langle G, \rangle\) is a densely ordered set.)

A Dedekind cut \((X, Y)\) in \(G\) is said to be a gap if \(\text{sup}(X) (\text{inf}(Y))\) does not exist. A gap \((X, Y)\) is said to be regular if, for all \(e\) in \(G_+ (=\{g \in G ; g > 0\})\), \(X+e \neq X\).

THEOREM. Suppose that \(G\) is saturated. Then, \(G\) has a regular gap. Moreover, \(G\) has \(2^\kappa\)-th regular gaps, where \(\kappa\) is the cardinality of \(G\).

PROOF. Since \(G\) is saturated, the coinitiality of \(G_+\) is \(\kappa\). Let \(\langle g_\alpha \mid \alpha < \kappa \rangle\) be an enumeration of \(G\) and let \(\langle e_\alpha \mid \alpha < \kappa \rangle\) be a strictly decreasing coinitial sequence in \(G_+\). By the induction on \(\alpha < \kappa\), we shall define a set \(\{I(x_u, y_u) ; u \in \alpha\} \) of open intervals in \(G\) such that

1. \(I(x_u, y_u) \neq \emptyset\) for all \(u\) in \(\alpha\),
2. \(y_u - x_u < e_\alpha\) for all \(u\) in \(\alpha\),
3. \(g_\alpha \not\in I(x_u, y_u)\) for all \(u\) in \(\alpha\),
4. \(I(x_u, y_u) \cap I(x_v, y_v) = \emptyset\) for all distinct elements \(u, v\) in \(\alpha\),
5. for all \(\beta < \alpha\), for all \(v \in \beta\) and for all \(u \in \alpha\), if \(v \subseteq u\), then \(I(x_v, y_v) \supseteq I(x_u, y_u)\).

The construction is as follows:

(Case 1) \(\alpha = 0\).

This case is obvious.

(Case 2) \(\alpha = \beta + 1\) for some \(\beta\).

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Suppose that \( \{I(x_v, y_v); v \in \mathfrak{B} \} \) has been defined and satisfies (1)\~(5). For each \( v \in \mathfrak{B} \), choose \( z_v, z'_v, w_v \) and \( w'_v \) in \( I(x_v, y_v) \) such that
\[
I(z_v, w_v) \neq \emptyset, \quad I(z'_v, w'_v) \neq \emptyset, \\
I(z_v, w_v) \cap I(z'_v, w'_v) = \emptyset, \\
w_v - z_v < e_a, \quad w'_v - z'_v < e_a, \\
g_v \notin I(x_v, w_v) \cup I(z'_v, w'_v).
\]
Set
\[
x_{\emptyset} = z_v, \\
y_{\emptyset} = w_v, \\
x_{1} = z'_v, \\
y_{1} = w'_v.
\]
Then, \( \{I(x_{\emptyset}, y_{\emptyset}); v \in \mathfrak{B} \} \) satisfies (1)\~(5).

(Case 3) \( \alpha \) is limit.

Suppose that, for all \( \beta < \alpha \), \( \{I(x_\beta, y_\beta); v \in \mathfrak{B} \} \) has been defined and satisfies (1)\~(5). Let \( u \) be in \( \alpha \). For each \( \beta < \alpha \), put
\[
x_\beta = x_{u\beta} \quad \text{and} \quad y_\beta = y_{u\beta}
\]
(where \( u\beta \) denotes the restriction of \( u \) to \( \beta \)).
The sequence \( \langle I(x_\beta, y_\beta) \mid \beta < \alpha \rangle \) satisfies that
\[
I(x_\beta, y_\beta) \neq \emptyset \quad \text{for all} \quad \beta < \alpha, \\
I(x_\beta, y_\beta) \subseteq I(x_\gamma, y_\gamma) \quad \text{for all} \quad \gamma < \beta < \alpha.
\]
Since \( G \) is saturated, \( \bigcap_{\beta < \alpha} I(x_\beta, y_\beta) \) contains elements \( x \) and \( y \) such that \( x < y \).
Since \( I(x, y) \subseteq \bigcap_{\beta < \alpha} I(x_\beta, y_\beta) \), we can choose \( x_u, y_u \) in \( I(x, y) \) such that
\[
x_u < y_u < x_u + e_a \quad \text{and} \quad g_u \notin I(x_u, y_u).
\]
Then, \( \{I(x_u, y_u); u \in \alpha \} \) satisfies (1)\~(5).

Now, \( \{I(x_u, y_u); u \in \bigcup_{\alpha \in \mathfrak{B}} \} \) is a set which satisfies (1)\~(5). For each \( f \) in \( \alpha \), define subsets \( X_f \) and \( Y_f \) of \( G \) by
\[
X_f = \{g \in G; \exists \alpha < \kappa (g < x_{f\alpha}) \}, \\
Y_f = \{g \in G; \exists \alpha < \kappa (y_{f\alpha} < g) \}.
\]
By (3) and (5), \((X_f, Y_f)\) is a cut in \(G\). By (4), if \(f, h\) are distinct elements in \(\kappa\), then \((X_f, Y_f) \neq (X_h, Y_h)\). To complete the proof of our theorem, it suffices to show that \((X_f, Y_f)\) is regular. Let \(e\) be any element in \(G_+\). Since \(\langle e_\alpha | \alpha < \kappa \rangle\) is coinitial in \(G_+\), there exists some \(\alpha < \kappa\) such that \(e_\alpha \leq e\). By (2),

\[ y_{f1a} < x_{f1a} + e_\alpha \leq x_{f1a} + e. \]

Since \(y_{f1a}\) is in \(Y_f\), \(x_{f1a} + e\) is in \(Y\). Thus \(X_f + e \neq X_f\).

Let \(R\) be the set of real numbers, let \(M\) be a universe with \(R \subseteq M\), let \(*M\) be an enlargement of \(M\) and let \(*R\) be the scope of \(R\). We shall regard \(*R\) as an ordered group \(<*R, +, <\). \(*R\) may be of the form \(<*R, *+, *<\). But we shall omit asterisks in \(*+\) and \(*<\), because there is no danger of confusion.

**Corollary 1.** Suppose that \(*M\) is saturated. Then, \(*R\) has a regular gap.

**Proof.** Since \(*M\) is saturated, \(*R\) is saturated. So, this follows from Theorem.

For each \(r\) in \(*R_+\), define \(E(r)\) by

\[ E(r) = \{(s, t) \in *R \times *R; |s-t| < r\}. \]

Define \(L(1)\) and \(E[L(1)]\) by

\[ L(1) = \{r \in *R; \forall r' \in R(r' < r)\}, \]

\[ E[L(1)] = \{E(r); r \in L(1)\}. \]

\(E[L(1)]\) is the base of some uniform topology on \(*R\). This uniform space is denoted by \((*R, E[L(1)])\) (see [6]). Define \(\overline{R}\) by

\[ \overline{R} = \{r \in *R; \exists r' \in R(|r| < r')\}. \]

\(\overline{R}\) is a convex subgroup of \(*R\). So, the quotient group \(*R/\overline{R}\) becomes an ordered group.

**Lemma.** \((*R, E[L(1)])\) is complete if and only if \(*R/\overline{R}\) does not have a regular gap.

**Proof.** It is easy from simple calculations.

**Corollary 2.** Suppose that \(*M\) is saturated. Then, \((*R, E[L(1)])\) is not complete.
PROOF. From Theorem and Lemma.

Assume GCH. There exists an enlargement $*M$ which is saturated (see [4, Proposition 5.1.5(ii)]). Therefore, from Corollaries 1 and 2, there exists an enlargement $*M$ such that

1. $*R$ has a regular gap,
2. $(*R, E[\mathcal{L}(1)])$ is not complete.

This is another proof of Theorems 4.5 and 4.2 in my paper [6].

References


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